Technical Appendix to "Signalling, Screening and Costly Misrepresentation," Canadian Journal of Economics

Proofs of Theorems 5-6 and Lemmas 1-2, Section 3.4.

Raymond Deneckere, Department of Economics, University of Wisconsin-Madison
Sergei Severinov,[1] Vancouver School of Economics, The University of British Columbia,

Proof of Theorem 5: In the optimal mechanism the principal selects "quantity" and transfer functions, $q(\cdot)$ and $t(\cdot)$, and a vector of signals $\mathbf{m}^n(\cdot)$ to solve:

$$\max_{q(\theta),t(\theta),\mathbf{m}^n(\theta)} \int_{\underline{\theta}}^{\overline{\theta}} (v(q(\theta)) - t(\theta))f(\theta)d\theta$$

subject to the following incentive and the individual rationality constraints for all θ and θ':

$$t(\theta) - h(q(\theta),\theta) - C^n(\mathbf{m}^n(\theta),\theta) \geq t(\theta') - h(q(\theta'),\theta) - C^n(\mathbf{m}^n(\theta'),\theta), \quad (42)$$

$$U(\theta) \equiv t(\theta) - h(q(\theta),\theta) - C^n(\mathbf{m}^n(\theta),\theta) \geq 0. \quad (43)$$

Using (43) to substitute $t(\theta)$ from the objective, and replacing the incentive constraints (42) by the first-order condition associated with the agent's utility maximization, yields the following "relaxed" problem:

$$\max_{q(\theta),\mathbf{m}^n(\theta),U(\theta)} \int_{\underline{\theta}}^{\overline{\theta}} \{v(q(\theta)) - h(q(\theta),\theta) - C^n(\mathbf{m}^n(\theta),\theta) - U(\theta)\} f(\theta)d\theta, \quad (44)$$

subject to individual rationality constraint (43) and the first-order condition:

$$U'(\theta) = -h_\theta(q(\theta),\theta) - C^n_\theta(\mathbf{m}^n(\theta),\theta). \quad (45)$$

The proof of the Theorem proceeds as follows. First, we obtain the solution to the relaxed program. Then we will verify that the solution to the relaxed program satisfies the incentive constraints (42) and hence also solves the unrelaxed problem.

To solve the relaxed problem, define the Hamiltonian:

$$H = (v(q) - h(q,\theta) - C^n(\mathbf{m}^n,\theta) - U) f(\theta) - \sigma (h_\theta(q,\theta) + C^n_\theta(\mathbf{m}^n,\theta)) + \rho U \quad (46)$$

Maximizing (46) w.r.t. q and $\mathbf{m}^n$ yields the first order conditions:

$$\{v_q(q) - h_q(q,\theta)\}f(\theta) - \sigma h_{q\theta}(q,\theta) \leq 0 \ (= 0, \text{ if } q > 0) \quad (47)$$

$$\frac{\partial C^n}{\partial m_i}(\mathbf{m}^n,\theta)f(\theta) + \sigma \frac{\partial^2 C^n}{\partial m_i \partial \theta}(\mathbf{m}^n,\theta) = 0 \quad (48)$$

1 Corresponding Author; Email: sseverinov@gmail.com.

The costate equation is

$$\sigma'(\theta) = f(\theta) - \rho(\theta). \tag{49}$$

Furthermore, the solution has to satisfy complementary slackness conditions

$$\rho(\theta)U(\theta) = 0,\ \rho(\theta) \geq 0, \text{ and } U(\theta) \geq 0. \tag{50}$$

We also have the following transversality conditions: $\sigma(\underline{\theta})U(\underline{\theta}) = 0$, $\sigma(\bar{\theta})U(\bar{\theta}) = 0$, $\sigma(\underline{\theta}) \leq 0$ and $\sigma(\bar{\theta}) \geq 0$.

The rest of the proof proceeds through a number of Claims.

Claim 1. *For $\theta \in [\underline{\theta}, \overline{\theta}]$ and $\sigma \geq 0$ let $q(\sigma, \theta)$ and $\mathbf{m}^n(\sigma, \theta)$ maximize the Hamiltonian H in (46) w.r.t. q and $\mathbf{m}^n$, respectively. Then $q(\sigma, \theta)$ is decreasing in σ, strictly so whenever $q(\sigma, \theta) > 0$, and $\mathbf{m}^n(\sigma, \theta)$ is strictly increasing in σ*, while $\mathbf{m}^n(\sigma, \theta) \geq \gamma^n(\theta)$ with strict inequality when $\sigma > 0$.

By definition, $q(\sigma, \theta)$ is the solution in q to

$$\max_{q \geq 0} \left\{ v(q) - h(q, \theta) - \frac{\sigma}{f(\theta)} h_\theta(q, \theta) \right\}, \tag{51}$$

and $\mathbf{m}^n(\sigma, \theta)$ is the solution in $\mathbf{m}^n$ to

$$\min_{\mathbf{m}^n \in \mathbb{R}^n} \left\{ C^n(\mathbf{m}^n, \theta) + \frac{\sigma}{f(\theta)} C^n_\theta(\mathbf{m}^n, \theta) \right\}. \tag{52}$$

The existence of $q(\sigma, \theta)$ and $\mathbf{m}^n(\sigma, \theta)$ is guaranteed by the Weierstrass theorem because, respectively: (i) $q(\sigma, \theta)$ belongs to $[0, q^{FB}(\theta)]$; (ii) the value of (52) goes to ∞ as $||\mathbf{m}^n|| \to \infty$.[2]

Next, observe that the cross partial of the objective (51) in (q, σ) is equal to $-\frac{1}{f(\theta)} h_{q\theta}(q, \theta) < 0$. Hence it has strictly decreasing differences in (q, σ), and so $q(\sigma, \theta)$ must be decreasing in σ. Similarly, $\mathbf{m}^n(\sigma, \theta)$ is increasing in σ. Indeed, by Assumption 4(iii) the cross-partial of the objective in (52) w.r.t. $\mathbf{m}^n$ and θ equals $\frac{1}{f(\theta)} \frac{\partial^2 C^n}{\partial m_i \partial \theta}(\mathbf{m}^n, \theta) < 0$, so this objective has decreasing differences in $(\mathbf{m}^n, \sigma)$. Furthermore the objective in (52) is submodular in $\mathbf{m}^n$ because $\frac{\partial^2 C^n}{\partial m_i \partial m_j} + \frac{\sigma}{f(\theta)} \frac{\partial^3 C^n}{\partial m_i \partial m_j \partial \theta} < 0$ for all $j \neq i$. This inequality follows from Assumption 4 (iii) when $\frac{\partial^3 C^n}{\partial m_i \partial m_j \partial \theta} \leq 0$; and from Assumption 4 (ii) and the fact that $0 \leq \sigma \leq F(\theta)$ when $\frac{\partial^3 C^n}{\partial m_i \partial m_j \partial \theta} > 0$.

Finally, recall that $C_{m_i}(\mathbf{m}^n(\sigma, \theta), \theta) < 0$ if $m^n_i(\theta) < \gamma^n_i(\theta)$, and $\frac{\partial C^n(\mathbf{m}^n, \theta)}{\partial m_i \partial \theta} < 0$. So, if $\mathbf{m}^n(\sigma, \theta)$ is such that $m^n_i(\theta) \leq \gamma^n_i(\theta)$ then $\frac{\partial C^n(\mathbf{m}^n, \theta) + \frac{\sigma}{f(\theta)} C^n_\theta(\mathbf{m}^n, \theta)}{\partial m_i} = C^n_{m_i}(\mathbf{m}^n, \theta) + \frac{\sigma}{f(\theta)} C^n_{m_i \theta}(\mathbf{m}^n, \theta) \leq 0$ with strict inequality either if $m^n_i(\theta) < \gamma^n_i(\theta)$ or if $m^n_i(\theta) = \gamma^n_i(\theta)$ and $\sigma > 0$. Therefore, we must have $m^n_i(\theta) \geq \gamma^n_i(\theta)$ for all i, with strict inequality when $\sigma > 0$.

2 Henceforth, we will also assume that $q(\sigma, \theta)$ and $\mathbf{m}^n(\sigma, \theta)$ are unique. This can always be guaranteed by assuming that the objective function in (51) is quasiconcave in q, and that the objective function in (52) is quasiconvex in $\mathbf{m}^n$.

Claim 2. *Let* $U'(\sigma,\theta) = -h_\theta(q(\sigma,\theta),\theta) - C^n_\theta(m^n(\sigma,\theta),\theta)$*. Then for all* $\theta \in [\underline{\theta},\overline{\theta}]$ *there exists a unique* $\breve{\sigma}(\theta)$ *such that* $U'(\sigma,\theta) < 0$ *if* $\sigma < \breve{\sigma}(\theta)$ *and* $U'(\sigma,\theta) > 0$ *if* $\sigma > \breve{\sigma}(\theta)$*. Furthermore,* $\breve{\sigma}(\overline{\theta}) = 0$ *and* $\breve{\sigma}(\theta) > 0$ *for all* $\theta < \overline{\theta}$*.*

Since $h_{q\theta} > 0$ and $\frac{\partial^2 C^n}{\partial m_i \partial\theta} < 0$ for all i, it follows from Claim 1 that $U'(\sigma,\theta)$ is strictly increasing in σ. Hence there is at most one value σ such that $U'(\sigma,\theta) = 0$. To establish that such value exists, let us show that $U'(\sigma,\theta) \leq 0$ when $\sigma = 0$, and $U'(\sigma,\theta) > 0$ when σ is sufficiently large.

First, consider $\sigma = 0$. It follows from (51) and (52) that $q(0,\theta) = q^{FB}(\theta) \geq 0$ and $\mathbf{m}^n(0,\theta) = \gamma^n(\theta)$ for all θ. Since $C^n_\theta(\gamma(\theta),\theta) = 0$, we have $U'(0,\theta) = -h_\theta(q_{FB}(\theta),\theta) \leq 0$. This inequality is strict for all $\theta < \overline{\theta}$ because $q_{FB}(\theta) > 0$ for all such θ. It follows that $\breve{\sigma}(\theta) > 0$ for all $\theta < \overline{\theta}$. Furthermore, since $q_{FB}(\overline{\theta}) = 0$ we have $U'(0,\overline{\theta}) = -h_\theta(0,\overline{\theta}) = 0$, so $\breve{\sigma}(\overline{\theta}) = 0$.

Next, let us show that $U'(\sigma,\theta) > 0$ for all $\sigma > \overline{\sigma}(\theta)$, where $\overline{\sigma}(\theta) = \frac{v'(0) - h_q(0,\theta)}{\min_{q\in[0,q^{FB}(\theta)]} h_{q\theta}(q,\theta)} f(\theta)$. To this end, we first claim that $q(\sigma,\theta) = 0$ for all $\sigma \geq \overline{\sigma}(\theta)$. Indeed, suppose to the contrary that $q(\sigma,\theta) > 0$ for some $\sigma \geq \overline{\sigma}(\theta)$. Since $q(\sigma,\theta)$ satisfies (47), we have:

$$v_q(q(\sigma,\theta)) - h_q(q(\sigma,\theta),\theta) = \frac{\sigma}{f(\theta)} h_{q\theta}(q(\sigma,\theta),\theta)$$
$$\geq \frac{\overline{\sigma}}{f(\theta)} h_{q\theta}(q(\sigma,\theta),\theta) \geq v'(0) - h_q(0,\theta). \tag{53}$$

But (53) contradicts the assumption that $v_q(q) - h_q(q,\theta)$ is strictly decreasing in q.

Since $q(\sigma,\theta) = 0$ for all $\sigma \geq \overline{\sigma}(\theta)$, it follows that $U'(\sigma,\theta) = -C^n_\theta(\mathbf{m}^n(\sigma,\theta),\theta)$ for all $\sigma \geq \overline{\sigma}(\theta)$. By Claim 1, $\mathbf{m}^n(\sigma,\theta) > \gamma^n(\theta)$ whenever $\sigma > \overline{\sigma}(\theta)$, because $\overline{\sigma}(\theta) \geq 0$. Since $C^n_\theta(\gamma^n(\theta),\theta) = 0$ and $\frac{\partial^2 C^n}{\partial\theta\partial m_i} < 0$, it then follows that $C^n_\theta(\mathbf{m}^n(\sigma,\theta),\theta) < 0$, thereby establishing that $U'(\sigma,\theta) > 0$ for all $\sigma > \overline{\sigma}(\theta)$. Combining this with $U'(0,\theta) \leq 0$ and the uniqueness of $\breve{\sigma}(\theta)$ s.t. $U'(\breve{\sigma}(\theta),\theta) = 0$ implies that $U'(\sigma,\theta) < 0$ if $\sigma < \breve{\sigma}(\theta)$, and $U'(\sigma,\theta) > 0$ if $\sigma > \breve{\sigma}(\theta)$.

Claim 3. $U(F(\theta),\theta)$ *has a unique minimizer* $\widehat{\theta}_n \in (0,\theta^*)$*. Moreover,* $U'(F(\theta),\theta) < 0$ *for* $\theta < \widehat{\theta}_n$ *and* $U'(F(\theta),\theta) > 0$ *for* $\theta > \widehat{\theta}_n$*.*

It is easy to see that $U^n_{SB}(\theta) \equiv U(F(\theta),\theta)$ is strictly quasiconvex. So it has a unique global and local minimizer which we take to be $\widehat{\theta}_n$. So $U'(F(\theta),\theta) < 0$ for all $\theta \in [\underline{\theta},\widehat{\theta}_n)$ and $U'(F(\theta),\theta) > 0$ for all $\theta \in [\widehat{\theta}_n,\overline{\theta}]$.

We will establish that $U'(F(\underline{\theta}),\underline{\theta}) < 0$ and $U'(F(\theta^*),\theta^*) > 0$, implying that $\widehat{\theta}_n \in (0,\theta^*)$. By Claim 2, $\breve{\sigma}(\underline{\theta}) > 0$ and hence $U'(F(\underline{\theta}),\underline{\theta}) = U'(0,\underline{\theta}) < 0$. Now, $q(F(\theta^*),\theta^*) = q^{SB}(\theta^*) = 0$ so that $h_\theta(q(F(\theta^*),\theta^*),\theta^*) = 0$. Also, since $F(\theta^*) > 0$, by Claim 1 $\mathbf{m}^n(F(\theta^*),\theta^*) > \gamma^n(\theta^*)$ and so $C^n_\theta(\mathbf{m}^n(F(\theta^*),\theta^*),\theta^*) < 0$. Thus $U'(F(\theta^*),\theta^*) > 0$.

Claim 4. *On the interval* $[\underline{\theta},\widehat{\theta}_n)$ *the solution to the relaxed program is such that* $q(\theta) = q^{SB}(\theta)$*,* $m^n(\theta) = m^n(F(\theta),\theta)$ and $U'(\theta) < 0$.

By Claim 3 it suffices to prove that $\sigma(\theta) = F(\theta)$ for all $\theta \leq \widehat{\theta}_n$. Note that $\sigma(\theta) \leq F(\theta)$ by the transversality condition $\sigma(\underline{\theta}) \leq 0$, and the fact that $\sigma'(\theta) \leq f(\theta)$, which is implied by the costate equation (49) and the condition $\rho(\theta) \geq 0$.

Now recall from the proof of Claim 2 that $U(\sigma,\theta)$ is strictly increasing in σ. It follows from $\sigma(\theta) \leq F(\theta)$ that $U'(\theta) = U'(\sigma(\theta),\theta) \leq U'(F(\theta),\theta) < 0$ for all $\theta \leq \widehat{\theta}_n$, where the final inequality holds by Claim 3. Since $U(\widehat{\theta}_n) \geq 0$, we therefore have $U(\theta) > 0$ for all $\theta < \widehat{\theta}_n$. The transversality condition then implies that $\sigma(\underline{\theta}) = 0$, and the complementary slackness condition (50) then implies that $\rho(\theta) = 0$ so that $\sigma'(\theta) = f(\theta)$ for all $\theta < \widehat{\theta}_n$. This establishes that $\sigma(\theta) = F(\theta)$ for all $\theta < \widehat{\theta}_n$, and hence by continuity also at $\widehat{\theta}_n$.

Claim 5. $U(\widehat{\theta}_n) = 0$, *and* $\sigma(\theta) < F(\theta)$ *for all* $\theta > \widehat{\theta}_n$.

Since $\sigma(\widehat{\theta}_n) = F(\widehat{\theta}_n)$ by Claim 4 and $\sigma'(\theta) \leq f(\theta)$ for all θ by (49), it follows that $\sigma(\theta) \leq F(\theta)$ for all $\theta > \widehat{\theta}_n$.

Let $\theta' = \max\{\theta \geq \widehat{\theta}_n : \sigma(\theta) = F(\theta)\}$, and suppose that contrary to this Claim $\theta' > \widehat{\theta}_n$. Let us show that in this $\theta' = \overline{\theta}$ then. Indeed, since $\sigma(\theta') = F(\theta')$, we must have $\sigma(\theta) = F(\theta)$ for all $\theta \in (\widehat{\theta}_n, \theta']$. It then follows from Claim 3 that $U'(\theta) = U'(\sigma(\theta),\theta) = U'(F(\theta),\theta) > 0$ on $(\widehat{\theta}_n, \theta']$, implying that $U(\theta') > 0$. Thus if $\theta' < \overline{\theta}$, there exists a right neighborhood V of θ' on which the individual rationality constraint $U(\theta) \geq 0$ is not binding. It then follows from the complementary slackness condition (50) that on this neighborhood we have $\rho(\theta) = 0$, and hence by the costate equation (49) that $\sigma'(\theta) = f(\theta)$. Thus $\sigma(\theta) = F(\theta)$ for all $\theta \in V$, contradicting the definition of θ', thereby establishing that $\theta' = \overline{\theta}$.

Next, let us show that $\breve{\sigma}(\theta) < F(\theta)$ for all $\theta > \widehat{\theta}_n$. Indeed, since $\sigma(\theta) = F(\theta)$ for all $\theta > \widehat{\theta}_n$, Claim 3 yields $U'(\theta) = U'(F(\theta),\theta) > 0$ for all $\theta > \widehat{\theta}_n$. Because $U'(\sigma,\theta)$ is strictly increasing in σ this implies that $\breve{\sigma}(\theta) < F(\theta)$ for all $\theta > \widehat{\theta}_n$.

Since $q(\theta) = q(\sigma(\theta),\theta) = q(F(\theta),\theta)$ and $\mathbf{m}^n(\theta) = \mathbf{m}^n(\sigma(\theta),\theta) = \mathbf{m}^n(F(\theta),\theta)$, and since $0 < \breve{\sigma}(\theta) < F(\theta)$ for all $\theta > \widehat{\theta}_n$, it follows from Claim 1 that $q(\theta) < q(\breve{\sigma}(\theta),\theta) < q^{FB}(\theta)$ and $\gamma^n(\theta) < \mathbf{m}^n(\breve{\sigma}(\theta),\theta) < \mathbf{m}^n(F(\theta),\theta) = \mathbf{m}^n(\theta)$ for all $\theta \in (\widehat{\theta}_n, \overline{\theta})$. But then the value of the relaxed program can be strictly increased by setting $U(\theta) = 0$ and assigning $(q(\breve{\sigma}(\theta),\theta), \mathbf{m}^n(\breve{\sigma}(\theta),\theta))$ on the interval $[\widehat{\theta}_n, \overline{\theta}]$, as follows from the fact that:

$$\begin{aligned} &v(q(\breve{\sigma}(\theta),\theta)) - h(q(\breve{\sigma}(\theta),\theta),\theta) - C^n(\mathbf{m}^n(\breve{\sigma}(\theta),\theta),\theta) > \\ &v(q(\theta)) - h(q(\theta),\theta) - C^n(\mathbf{m}^n(\theta),\theta). \end{aligned} \tag{54}$$

This contradiction establishes that $\sigma(\theta) < F(\theta)$ for all $\theta > \widehat{\theta}_n$.

It follows that there exists a decreasing sequence $\{\theta_\ell\} \subset (\widehat{\theta}_n, \overline{\theta})$ converging to $\widehat{\theta}_n$, such that $\rho(\theta_\ell) > 0$ and hence $U(\theta_\ell) = 0$. So, by continuity of $U(\cdot)$ we have $U(\widehat{\theta}_n) = 0$.

Claim 6. *$U(\theta) = 0$ for all $\theta > \widehat{\theta}_n$ if and only if $\breve{\sigma}'(\theta) \le f(\theta)$ for all $\theta > \widehat{\theta}_n$.*

If $U(\theta) = 0$ for all $\theta > \widehat{\theta}_n$, then $\sigma(\theta) = \breve{\sigma}(\theta)$ and hence $\sigma'(\theta) = \breve{\sigma}'(\theta)$ for all $\theta > \widehat{\theta}_n$. Since $\rho(\theta) \ge 0$, by the costate equation we have $\breve{\sigma}'(\theta) = \sigma'(\theta) = f(\theta) - \rho(\theta) \le f(\theta)$ for all $\theta > \widehat{\theta}_n$.

Conversely, suppose that $\breve{\sigma}'(\theta) \le f(\theta)$ for all $\theta > \widehat{\theta}_n$, and that contrary to the statement of the claim there exists some $\theta \ge \widehat{\theta}_n$ such that $U(\theta) > 0$. Let $\theta_1 = \inf\{\theta \ge \widehat{\theta}_n : U(\theta) > 0\}$.

We now claim that $\sigma(\theta_1) = \breve{\sigma}(\theta_1)$. If $\theta_1 > \widehat{\theta}_n$ this is immediate, since we then have $U(\theta) = 0$ and hence $\sigma(\theta) = \breve{\sigma}(\theta)$ for all $\theta \in [\widehat{\theta}_n, \theta_1]$. Now if $\theta_1 = \widehat{\theta}_n$, it follows from Claim 5 that for every $\varepsilon > 0$ there exists $(\theta', \theta'') \subset (\widehat{\theta}_n, \widehat{\theta}_n + \varepsilon)$ on which $U(\theta) = 0$ and hence $\sigma(\theta) = \breve{\sigma}(\theta)$. The continuity of the functions $\sigma(\cdot)$ and $\breve{\sigma}(\cdot)$ then implies that $\breve{\sigma}(\widehat{\theta}_n) = \sigma(\widehat{\theta}_n)$.

Next, we establish that $\sigma(\theta) > \breve{\sigma}(\theta)$ for all $\theta > \theta_1$. Indeed, $\sigma(\theta) \ge \breve{\sigma}(\theta)$ for all $\theta > \theta_1$, since on any interval on which $U(t) > 0$ we have $\sigma'(t) = f(t) \ge \breve{\sigma}'(t)$, and on any interval on which $U(t) = 0$ we have $\sigma(t) = \breve{\sigma}(t)$. But we cannot have $\sigma(\theta) = \breve{\sigma}(\theta)$ for any $\theta > \theta_1$, as this would imply that $\sigma'(t) = f(t) = \breve{\sigma}'(t)$ for all $t \in (\theta_1, \theta)$, hence that $\sigma(t) = \breve{\sigma}(t)$ for all $t \in [\theta_1, \theta]$. But then we have $U(t) = U(\theta_1)$ for all $t \in [\theta_1, \theta]$, contradicting the definition of θ_1.

It follows from the fact that $\sigma(\theta) > \breve{\sigma}(\theta)$ for all $\theta > \theta_1$ that $U'(\theta) = U'(\sigma(\theta), \theta) > U'(\breve{\sigma}(\theta), \theta) = 0$, and hence that $U(\theta)$ is strictly increasing for all $\theta > \theta_1$. As in the proof of Claim 5, we can then show that the value of the relaxed program can be improved by setting $U(\theta) = 0$ and assigning $(q(\breve{\sigma}(\theta), \theta), \mathbf{m}^n(\breve{\sigma}(\theta), \theta))$ for all $\theta > \theta_1$. This contradiction establishes that $U(\theta) = 0$ for all $\theta > \widehat{\theta}_n$.

Claim 7. $\mathbf{m}^n(\underline{\theta}) = \gamma^n(\underline{\theta})$, $\mathbf{m}^n(\overline{\theta}) = \gamma^n(\overline{\theta})$ and $\mathbf{m}^n(\theta) > \gamma^n(\theta)$ for all $\theta \in (\underline{\theta}, \overline{\theta})$. Furthermore, $q(\theta) > 0$ *for all* $\theta \in [\underline{\theta}, \overline{\theta})$.

By Claim 4 $\sigma(\theta) = F(\theta)$ for all $\theta \le \widehat{\theta}_n$, and by Claim 6 $\sigma(\theta) = \breve{\sigma}(\theta)$ for all $\theta \ge \widehat{\theta}_n$. Also, by Claim 2, $\breve{\sigma}(\overline{\theta}) = 0$ and $\breve{\sigma}(\theta) > 0$ for all $\theta < \overline{\theta}$. So, $\sigma(\underline{\theta}) = \sigma(\overline{\theta}) = 0$, and $\sigma(\theta) > 0$ for all $\theta \in (\underline{\theta}, \overline{\theta})$. Then from Claim 1 it follows that $\mathbf{m}^n(\theta) > \gamma^n(\theta)$ for all $\theta \in (\underline{\theta}, \overline{\theta})$. Further, from (52) it follows that $\mathbf{m}^n(0, \theta) = \gamma^n(\theta)$. Hence, $\mathbf{m}^n(\underline{\theta}) = \gamma^n(\underline{\theta})$, $\mathbf{m}^n(\overline{\theta}) = \gamma^n(\overline{\theta})$.

To establish that $q(\theta) > 0$ for all $\theta < \overline{\theta}$, note that $q(\theta) = q^{SB}(\theta) > 0$ for all $\theta \in [\underline{\theta}, \widehat{\theta}_n)$. Now, if $q(\theta) = 0$ for some $\theta \in [\widehat{\theta}_n, \overline{\theta})$, then since $h_\theta(0, \theta) = 0$ equation (45) implies that $0 = U'(\theta) = C^n_\theta(\mathbf{m}^n(\theta), \theta)$. But from $\mathbf{m}^n(\theta) > \gamma^n(\theta)$, $C^n_\theta(\gamma^n(\theta), \theta) = 0$ and $C^n_{m_i} < 0$ for all i it follows that $C^n_\theta(\mathbf{m}^n(\theta), \theta) < 0$, a contradiction. Thus $q(\theta) > 0$ for all $\theta \in [\widehat{\theta}_n, \overline{\theta})$.

Claim 8. *Global incentive compatibility of the solution to the relaxed program.*

It remains to show that this solution satisfies incentive constraints (42) i.e., for any pair of types (θ, θ') we have:

$$U(\theta) - U(\theta') + h(q(\theta'), \theta) + C^n(\mathbf{m}^n(\theta'), \theta) - h(q(\theta'), \theta') - C^n(\mathbf{m}^n(\theta'), \theta') \geq 0 \tag{55}$$

First, suppose that $\theta' \in [\widehat{\theta}, \overline{\theta}]$ i.e., $U(\theta') = 0$. We will consider the case $\theta' > \theta$. The proof for the case $\theta' < \theta$ is similar. Then we have:

$$\begin{aligned} &U(\theta) - U(\theta') + h(q(\theta'), \theta) + C^n(\mathbf{m}^n(\theta'), \theta) - h(q(\theta'), \theta') - C^n(\mathbf{m}^n(\theta'), \theta') = \\ &U(\theta) - U(\theta') - \int_{\theta}^{\theta'} h_\theta(q(\theta'), s) + C^n_\theta(\mathbf{m}^n(\theta'), s) ds > \\ &- \int_{\theta}^{\theta'} h_\theta(q(\theta'), \theta') + C^n_\theta(\mathbf{m}^n(\theta'), \theta') ds = 0 \end{aligned} \tag{56}$$

The first inequality holds because $U(\theta) \geq 0 = U(\theta')$ The last equality holds because $\theta' \in [\widehat{\theta}_n, \overline{\theta}]$ and so $U'(\theta') = -h_\theta(q(\theta'), \theta') - C^n_\theta(\mathbf{m}^n(\theta'), \theta') = 0$, establishing the incentive compatibility of our mechanism for this case.

Next, suppose that $\theta, \theta' \in [\underline{\theta}, \widehat{\theta}_n]$. Over this region, the solution is $\{q^{SB}(\theta), \mathbf{m}^n(F(\theta), \theta), F(\theta)\}$, and incentive compatibility holds if $q^{SB}(\theta)$ is decreasing in θ, and $\mathbf{m}^n(F(\theta), \theta)$ is increasing in θ (Guesnerie and Laffont, 1984, Theorem 2). That $q^{SB}(\theta)$ is decreasing in θ follows from Assumption 4(i). Next, as a maximizer of the Hamiltonian H, $\mathbf{m}^n(F(\theta), \theta)$ minimizes $C^n(\mathbf{m}^n, \theta) + \frac{F(\theta)}{f(\theta)} C^n_\theta(\mathbf{m}^n, \theta)$. By Assumption 4(ii), this objective has strictly increasing differences in $(\mathbf{m}^n, \theta)$, and is supermodular in $\mathbf{m}^n$. Therefore, $\mathbf{m}^n(F(\theta), \theta)$ is increasing in θ, establishing incentive compatibility in this case.

Finally, let us show that incentive constraints hold for any pair (θ, θ') such that $\theta \in (\hat{\theta}, 1]$ and $\theta' \in [\underline{\theta}, \hat{\theta}]$. Let us rewrite the left-hand side of (55) as follows:

$$\begin{aligned} &U(\theta) - U(\theta') + h(q(\theta'), \theta) + C^n(\mathbf{m}^n(\theta'), \theta) - h(q(\theta'), \theta') - C^n(\mathbf{m}^n(\theta'), \theta') = \\ &U(\theta) - U(\widehat{\theta}) + \Big(h(q(\theta'), \theta) + C^n(\mathbf{m}^n(\theta'), \theta) - h(q(\theta'), \widehat{\theta}) - C^n(\mathbf{m}^n(\theta'), \widehat{\theta})\Big) + \\ &- U(\theta') + U(\widehat{\theta}) + h(q(\theta'), \widehat{\theta}) + C^n(\mathbf{m}^n(\theta'), \widehat{\theta}) - h(q(\theta'), \theta') - C^n(\mathbf{m}^n(\theta'), \theta'). \end{aligned} \tag{57}$$

To confirm that the incentive constraint between θ and θ' holds, we need to show that the expression in (57) is nonnegative. To this end, we will establish separately that both the second line and the third line in (57) are nonnegative. Start with the second line. We have $U(\theta) = U(\widehat{\theta}) = 0$. Further, consider the expression in brackets in the second line. We have:

$$
\begin{aligned}
&h(q(\theta'),\theta) + C^n(\mathbf{m}^n(\theta'),\theta) - h(q(\theta'),\widehat{\theta}) - C^n(\mathbf{m}^n(\theta'),\widehat{\theta}) = \\
&\int_{\widehat{\theta}}^{\theta} h_\theta(q(\theta'),s) + C^n_\theta(\mathbf{m}^n(\theta'),s)ds \geq \\
&\int_{\widehat{\theta}}^{\theta} h_\theta(q(\theta'),\widehat{\theta}_n) + C^n_\theta(\mathbf{m}^n(\theta'),\widehat{\theta}_n)ds > \int_{\widehat{\theta}}^{\theta} h_\theta(q(\widehat{\theta}),\widehat{\theta}) + C^n_\theta(\mathbf{m}^n(\widehat{\theta}),\widehat{\theta}_n)ds = 0.
\end{aligned} \tag{58}
$$

The first inequality in (58) holds because $h_{\theta\theta} \geq 0$ and $C^n_{\theta\theta} > 0$. The second inequality holds because $q(\theta') > q(\widehat{\theta})$ and $\mathbf{m}^n(\theta') < \mathbf{m}^n(\widehat{\theta})$ (as established above in this Claim), while $h_{\theta q} > 0$ and $C^n(\mathbf{m}^n,\theta)$ has strictly decreasing differences in $(\mathbf{m}^n,\theta)$. The last equality holds because by Claim 8 $h_\theta(q(\widehat{\theta}),\widehat{\theta}) + C^n_\theta(\mathbf{m}^n(\widehat{\theta}),\widehat{\theta}) = 0$. So, the second line in (57) is nonnegative.

Now consider the third line in (57). We have:

$$
\begin{aligned}
&-U(\theta') + U(\widehat{\theta}) + h(q(\theta'),\widehat{\theta}) + C^n(\mathbf{m}^n(\theta'),\widehat{\theta}) - h(q(\theta'),\theta') - C^n(\mathbf{m}^n(\theta'),\theta') \\
&= -\int_{\theta'}^{\widehat{\theta}} h_\theta(q(s),s) + C^n_\theta(\mathbf{m}^n(s),s)ds + \int_{\theta'}^{\widehat{\theta}_n} h_\theta(q(\theta'),s) + C^n_\theta(\mathbf{m}^n(\theta'),s)ds \\
&= \int_{\theta'}^{\widehat{\theta}} (h_\theta(q(\theta'),s) - h_\theta(q(s),s)) + (C^n_\theta(\mathbf{m}^n(\theta'),s) - C^n_\theta(\mathbf{m}^n(s),s))\, ds > 0
\end{aligned} \tag{59}
$$

The first equality holds because $U'(\theta) = -h_\theta(q(\theta),\theta) - C^n_\theta(\mathbf{m}^n(\theta),\theta)$. The inequality holds because, as shown in this Claim, $q(\theta') > q(s)$ and $\mathbf{m}^n(\theta') < \mathbf{m}^n(s)$ for all $s \in (\theta',\widehat{\theta}_n]$, while $h_{\theta q} > 0$ and $C^n(\mathbf{m}^n,\theta)$ has decreasing differences in $(\mathbf{m}^n,\theta)$. So, (57) is nonnegative. *Q.E.D.*

Proof of Lemma 1: Let us show that $\breve{\sigma}'(\theta) \leq f(\theta)$. By definition, $\breve{\sigma}(\theta)$ satisfies $U'(\breve{\sigma}(\theta),\theta) = 0$, and so $U'_\sigma\breve{\sigma}'(\theta) + U'_\theta = 0$. Because $U'_\sigma > 0$, $\breve{\sigma}'(\theta) \leq f(\theta)$ if and only if

$$
U'_\sigma(\breve{\sigma}(\theta),\theta)f(\theta) + U'_\theta(\breve{\sigma}(\theta),\theta) \geq 0. \tag{60}
$$

Recall that $U'(\sigma,\theta) = -h_\theta(q(\sigma,\theta),\theta) - C^n_\theta(\mathbf{m}^n(\sigma,\theta),\theta)$. Hence, $U'_\sigma = -h_{q\theta}q_\sigma - C^n_{\theta\mathbf{m}}\mathbf{m}^n_\sigma$ and $U'_\theta = h_{q\theta}q_\theta - h_{\theta\theta} - C^n_{\theta\mathbf{m}}\mathbf{m}^n_\theta - C^n_{\theta\theta}$. Thus (60) is equivalent to

$$
-h_{q\theta}[q_\sigma f + q_\theta] - h_{\theta\theta} - C^n_{\theta\mathbf{m}}[\mathbf{m}^n_\sigma f + \mathbf{m}^n_\theta] - C^n_{\theta\theta} \geq 0. \tag{61}
$$

So it is sufficient to verify that the assumptions of the Lemma guarantee that the inequality (61) holds. In fact, we will show that $h_{q\theta}[q_\sigma f + q_\theta] + h_{\theta\theta} \leq 0$ and $C^n_{\theta\mathbf{m}}[\mathbf{m}^n_\sigma f + \mathbf{m}^n_\theta] + C^n_{\theta\theta} \leq 0$.

To this effect, let us first calculate q_σ and q_θ. As a maximizer of (51), $q(\sigma,\theta)$ satisfies the first-order condition $v'(q) - h_q(q,\theta) - \frac{\sigma}{f(\theta)} h_{q\theta}(q,\theta) = 0$, from which it follows that:

$$\left(v''(q) - h_{qq} - \frac{\sigma}{f} h_{qq\theta}\right) q_\sigma f - h_{q\theta} = 0,$$
$$\left(v''(q) - h_{qq} - \frac{\sigma}{f} h_{qq\theta}\right) q_\theta - h_{q\theta}\left(1 - \frac{\sigma f'}{f^2}\right) - \frac{\sigma}{f} h_{q\theta\theta} = 0.$$

Because $q(\sigma,\theta)$ is a maximizer, the second order condition $v''(q) - h_{qq} - \frac{\sigma}{f} h_{qq\theta} \leq 0$ holds. Consequently, the inequality $h_{q\theta}[q_\sigma f + q_\theta] + h_{\theta\theta} \leq 0$ is equivalent to

$$h_{q\theta}\left[h_{q\theta}\left(2 - \frac{\sigma f'}{f^2}\right) + \frac{\sigma}{f} h_{q\theta\theta}\right] + h_{\theta\theta}[v''(q) - h_{qq} - \frac{\sigma}{f} h_{qq\theta}] \geq 0. \tag{62}$$

The second term in (62) is positive because $h_{\theta\theta} \leq 0$ by assumption of the Lemma.

Next, $1 - \frac{\sigma f'}{f^2} > 0$. This is immediate if $f' \leq 0$. If $f' > 0$ then, since $\frac{F(\theta)}{f(\theta)}$ is increasing by assumption, it follows that $1 - \frac{\sigma f'}{f^2} \geq 1 - \frac{F f'}{f^2} = 1 - \left(\frac{F}{f}\right)' > 0$. So the assumption that $h_{q\theta} \geq 0$ and $h_{q\theta\theta} \geq 0$ also guarantees that the first term in (62) is positive.

Similar steps establish that $C^n_{\theta\mathbf{m}}[\mathbf{m}^n_\sigma f + \mathbf{m}^n_\theta] + C^n_{\theta\theta} \leq 0$ under the conditions of the Lemma.

Proof of Theorem 6: First, we claim that there exists K such that $\sigma(\theta,n) \leq \frac{K}{n}$. Since $\frac{\partial C^n}{\partial m_i}(\gamma_i(\theta), \mathbf{m}^n_{-i}, \theta) = 0$, it follows from the mean value Theorem that $\frac{\partial C^n}{\partial m_i}(m_i, \mathbf{m}^n_{-i}, \theta) = \frac{\partial^2 C^n}{\partial m_i^2}(\bar{m}_i, \mathbf{m}^n_{-i}, \theta)(m_i - \gamma_i(\theta))$ for some $\bar{m}_i \in (m_i, \gamma_i)$. From (48) we then have:

$$m_i(\theta) - \gamma_i(\theta) = -\sigma(\theta,n)\frac{\frac{\partial^2 C^n}{\partial\theta\partial m_i}(\mathbf{m}^n(\theta),\theta)}{\frac{\partial^2 C^n}{\partial m_i^2}(\bar{m}_i, \mathbf{m}^n_{-i}(\theta),\theta)}. \tag{63}$$

Using the fact that $C^n_\theta(\gamma^n(\theta),\theta) = 0$, and applying the mean value Theorem once more yields:

$$C^n_\theta(\mathbf{m}^n(\theta),\theta) = \sum_{i=1}^n \frac{\partial^2 C^n}{\partial\theta\partial m_i}(\bar{\mathbf{m}}^n,\theta)(m_i(\theta) - \gamma_i(\theta)), \tag{64}$$

where $\bar{\mathbf{m}}^n = \gamma^n(\theta) + \varepsilon(\theta)(\mathbf{m}^n(\theta) - \gamma^n(\theta))$, for some $\varepsilon(\theta) \in (0,1)$. Recall that by Claim 2 of Theorem 5 $\sigma(\theta,n) \geq 0$. Using the assumption that $0 \leq \frac{\partial^2 C^n}{\partial m_i^2} \leq \bar{v}$, $\left|\frac{\partial^2 C^n}{\partial\theta\partial m_i}\right| \geq \underline{v} > 0$, (63) and (64) then yield:

$$C^n_\theta(\mathbf{m}^n(\theta),\theta) \leq -n\sigma(\theta,n)\frac{\underline{v}^2}{\bar{v}}. \tag{65}$$

Next, $h_\theta(q(\theta),\theta) \leq h_\theta(q^{FB}(\theta),\theta) \leq q^{FB}(\underline{\theta}) \max h_{q\theta}$, where the first inequality follows from $h_{q\theta} > 0$, and the second inequality holds because $h_\theta(0,\theta) = 0$ and $q^{FB}(\theta)$ is decreasing in θ. Since $U'(\theta) \leq 0$ on the interval $[\underline{\theta},\overline{\theta}]$, equation (45) implies

$$0 \geq U'(\theta) = -h_\theta(q(\theta),\theta) - C^n_\theta(\mathbf{m}^n(\theta),\theta) \geq -q^{FB}(\underline{\theta}) \max h_{q\theta} + n\sigma(\theta,n)\frac{\underline{v}^2}{\bar{v}}.$$

Setting $K = \frac{\bar{v}q^{FB}(\underline{\theta})}{\underline{v}^2} \max h_{q\theta}$ then establishes the claim.

By (63), $m_i(\theta) - \gamma_i(\theta) \leq \sigma(\theta,n)\frac{\bar{v}}{\underline{v}} \leq \frac{K}{n}\frac{\bar{v}}{\underline{v}}$. Furthermore, by (47) we have

$$\begin{aligned}&\sigma(\theta,n)h_{q\theta}(q(\theta),\theta) = \{v_q(q(\theta)) - h_q(q(\theta),\theta)\}f(\theta) = \\ &f(\theta)(v_{qq}(q_1(\theta)) - h_{qq}(q_1(\theta),\theta)(q(\theta) - q^{FB}(\theta)),\end{aligned} \tag{66}$$

for some $q_1(\theta) \in (q(\theta), q^{FB}(\theta))$. Hence $q^{FB}(\theta) - q(\theta) \leq \sigma(\theta,n)M \leq \frac{KM}{n}$, where $M = \frac{\max h_{q\theta}}{\min f(\theta)\times \min|v_{qq}-h_{qq}|}$, and so $q(\theta) \to q^{FB}(\theta)$ and $m_i(\theta) \to \gamma_i(\theta)$ uniformly in θ.

Next, recall that $q^{SB}(\hat{\theta}(n)) = q(\hat{\theta}(n))$ for all n. Hence,

$$q^{FB}(\hat{\theta}(n)) - q^{SB}(\hat{\theta}(n)) = \frac{F(\hat{\theta}(n))}{f(\hat{\theta}(n))}\frac{h_{q\theta}}{|v_{qq}-h_{qq}|}(q_2(\hat{\theta}(n)),\hat{\theta}(n) \leq \frac{KM}{n},$$

for some $q_2(\hat{\theta}(n)) \in (q^{SB}(\hat{\theta}(n)), q^{FB}(\hat{\theta}(n))$. Since $F(\hat{\theta}(n)) \geq \left(\hat{\theta}(n) - \underline{\theta}\right) \min_{\theta\in[\underline{\theta},\hat{\theta}(n)]} f(\theta)$, we have $\hat{\theta}(n) - \underline{\theta} \leq \frac{KMP}{n}$, where $P = \frac{\max f(\theta)}{\min f(\theta)} \frac{\min|v_{qq}-hqq|}{\max h_{q\theta}}$. Thus $\hat{\theta}(n) \to \underline{\theta}$.

Since $U(\theta)$ is decreasing on $[\underline{\theta},\widehat{\theta}_n(n)]$ and $U(\theta) = 0$ for all $\theta \in [\widehat{\theta}_n(n),\overline{\theta}]$, we have:

$$U(\underline{\theta}) \leq (\widehat{\theta}_n(n) - \underline{\theta}) \max_{\theta\in[\underline{\theta},\widehat{\theta}_n(n)]} |U'(\theta)| \leq (\widehat{\theta}_n(n) - \underline{\theta}) \max_{\theta\in[\underline{\theta},\widehat{\theta}_n(n)]} h_\theta(q(\theta),\theta) \leq \frac{KMPQ}{n},$$

where $Q = \max_{\theta\in[\underline{\theta},\widehat{\theta}_n(n)]} h_{q\theta}(q(\theta),\theta)$. Hence $U(\theta) \to 0$ uniformly in θ.

Finally, since $C^n(\gamma^n(\theta),\theta) = 0$ and $C^n_\theta(\gamma^n(\theta),\theta) = 0$, Taylor series expansion yields:

$$C^n(\mathbf{m}^n(\theta),\theta) = \sum_{i=1}^n \frac{(m_i - \gamma_i(\theta))^2}{2}\frac{\partial^2 C^n}{\partial m_i^2}(\underline{\mathbf{m}}^n,\theta) \leq \frac{\bar{v}}{2}\sum_{i=1}^n (m_i - \gamma_i(\theta))^2 \leq \frac{1}{n}\frac{K^2\bar{v}^3}{\underline{v}^2}, \tag{67}$$

where $\underline{\mathbf{m}}^n \in (\mathbf{m}^n(\theta),\gamma^n(\theta))$. Thus $C^n(\mathbf{m}^n(\theta),\theta) \to 0$, uniformly in θ. *Q.E.D.*

Proof of Lemma 2: Let us first show that $W(n)$ is strictly concave. Since the optimal mechanism is unique, $W(n)$ is continuously differentiable and by Seierstad and Sydsaeter (1999, p. 217), $\frac{dW(n)}{dn} = \int_{\underline{\theta}}^{\bar{\theta}} \frac{\partial H}{\partial n}(q,m,U,\sigma,n,\theta)d\theta$.

By the first-order condition (48), we have $\frac{c_{m\theta}(m_i^n(\theta),\theta)}{c_m(m_i^n(\theta),\theta)} = \frac{c_{m\theta}(m_j^n(\theta),\theta)}{c_m(m_j^n(\theta),\theta)}$ for all $i,j \in \{1,...,n\}$. So, the assumption that $c_{mm}c_{m\theta} - c_{mm\theta}c_m < 0$ implies that the optimal $m_i^n(\theta)$ is unique and is the same for all i, so we can drop the subscript i and denote the optimal message by $m^n(\theta)$.

Using (46) yields $\frac{\partial H}{\partial n} = -c(m^n(\theta),\theta)f(\theta) - \sigma(\theta)c_\theta(m^n(\theta),\theta)$. Substituting for $\sigma(\theta)$ from (48) we then obtain $\frac{\partial H}{\partial n} = \left(\frac{c_\theta c_m}{c_{m\theta}} - c\right)f$, so we have:

$$\frac{dW(n)}{dn} = \int_{\underline{\theta}}^{\bar{\theta}} \left(\frac{c_\theta(m^n(\theta),\theta)c_m(m^n(\theta),\theta)}{c_{m\theta}(m^n(\theta),\theta)} - c(m^n(\theta),\theta)\right) f(\theta)d\theta. \tag{68}$$

Thus, $W(n)$ is concave in n if $\left(\frac{c_\theta c_m}{c_{m\theta}} - c\right)$ decreases in n. Note that

$$\frac{d}{dn}\left(\frac{c_\theta c_m}{c_{m\theta}} - c\right) = \frac{c_\theta}{c_{m\theta}^2}\left(c_{mm}c_{m\theta} - c_{mm\theta}c_m\right)\frac{\partial m^n}{\partial n}. \tag{69}$$

By the first-order condition (48) and Claim 7 in Theorem 5, $m^n(\theta)$ satisfies $c_m(m^n(\theta),\theta)f(\theta) + c_{m\theta}(m^n(\theta),\theta)F(\theta) = 0$, and so $\frac{\partial m^n(\theta)}{\partial n} = 0$ for $\theta \in [\underline{\theta}, \widehat{\theta}_n)$.

Also, $h_\theta(q(\theta),\theta) + nc_\theta(m^n(\theta),\theta) = 0$ for all $\theta \in [\widehat{\theta}_n, \overline{\theta}]$. This equality can hold only if $\frac{\partial m^n(\theta)}{\partial n} < 0$ for $\theta \in [\widehat{\theta}_n, \overline{\theta}]$ because $h_{q\theta} > 0$ and $c_{m\theta} < 0$. Finally, by Theorem 5 $m^n(\theta) > \gamma(\theta)$, so $c_\theta < 0$. It then follows from the assumption $c_{mm}c_{m\theta} - c_{mm\theta}c_m < 0$ that (69) is strictly negative on $[\underline{\theta}, \widehat{\theta}_n)$ and equal to zero on $[\widehat{\theta}_n, \overline{\theta}]$. So $W(n)$ is strictly concave.

Now let $\eta(m,\theta) = \frac{c_\theta c_m}{c_{m\theta}} - c$. Observe that $\eta(\gamma(\theta),\theta) = 0$ since $c(\gamma(\theta),\theta) = 0$ and $c_\theta(\gamma(\theta),\theta) = 0$ for every θ. Furthermore, since $c_\theta(m,\theta) < 0$ for $m > \gamma(\theta)$ and $c_{mm}c_{m\theta} - c_{mm\theta}c_m < 0$, we have $\eta_m(m,\theta) > 0$ and hence $\eta(m,\theta) > 0$ for $m > \gamma(\theta)$. Since $m^n(\theta) > \gamma(\theta)$, it follows that the integrand in (68) is strictly positive for all θ, so $W(n)$ is strictly increasing in n.

To see that $\underline{K} > 0$, note that $\phi(m,\theta) = \frac{c_\theta c_m}{c_{m\theta}c} - 1$ is continuous in m with $\phi(m,\theta) = \frac{\eta(m,\theta)}{c(m,\theta)} > 0$ for all $m > \gamma(\theta)$. Applying l'Hospital's rule yields:

$$\lim_{m\to\gamma(\theta)} \phi(m,\theta) = \lim_{m\to\gamma(\theta)} \frac{c_{m\theta}c_m + c_\theta c_{mm}}{c_{m\theta}c_m + c_{mm\theta}c} = \lim_{m\to\gamma(\theta)} \frac{c_{m\theta} + \frac{c_\theta}{c_m}c_{mm}}{c_{m\theta} + \frac{c}{c_m}c_{mm\theta}} = 2,$$

where the final inequality holds because by l'Hospital's rule we have $\lim_{m\to\gamma(\theta)} \frac{c_\theta}{c_m} = \frac{c_{m\theta}}{c_{mm}}$ and $\lim_{m\to\gamma(\theta)} \frac{c}{c_m} = \lim_{m\to\gamma(\theta)} \frac{c_m}{c_{mm}} = 0$.

Thus $\phi(m,\theta)$ is continuous at $m = \gamma(\theta)$, and satisfies $\phi(\gamma(\theta),\theta) = 2$. It follows that $\underline{K}$, the minimum of $\phi(m,\theta)$ over the set D, is strictly positive.

Finally, it follows from Theorem 5 that $m^n(\theta) \in [\gamma(\theta), \gamma(\overline{\theta})]$. The definition of $\underline{K}$ and $\overline{K}$ then imply that $\underline{K}c \le \frac{c_m c_\theta}{c_{m\theta}} - c \le \overline{K}c$ for all θ, and so

$$0 < \underline{K}\int_{\underline{\theta}}^{\bar{\theta}} c(m(\theta),\theta)f(\theta)d\theta \le \frac{dW(n)}{dn} = G \le \overline{K}\int_{\underline{\theta}}^{\bar{\theta}} c(m(\theta),\theta)f(\theta)d\theta.$$

Q.E.D.