Investment Tournaments: When Should a Rational Agent Put All Eggs in One Basket?*

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Abstract

In this paper we study "investment tournaments," a class of decision problems that involve gradual allocation of investment among several alternatives whose values are subject to exogenous shocks. The decision-maker's payoff is determined by the final values of the alternatives. An important example of career tournaments motivating our research is the career choice problem, since a person choosing a career often starts by investing in learning several professions. We show that in a broad range of cases it is optimal for the decision-maker in each time period to allocate all resources to the most promising alternative. We also show that in tournaments for a promotion the agents would rationally a higher effort in an early stage of the tournament in a bid to capture a larger share of employer's investment, such as mentoring.
Keywords: tournaments, investment, promotion

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1 Introduction

This paper introduces and studies a class of decision problems which we call *investment tournaments*. In an investment tournament, the decision-maker and nature choose actions across a number of periods. Particularly, in every period the decision-maker selects the level of investment in each of the several alternatives available to her, and the value of each alternative increases with the investment that it receives. The values of alternatives also change due to random shocks (actions of nature). The payoff to the decision-maker is determined by the final values of the alternatives.

Although investment tournaments have received little attention in the literature, they are ubiquitous, and have a variety of applications. An important example of an investment tournament providing the central motivation for this paper is a career choice problem. Indeed, consider a student deliberating whether to major in accounting or engineering. The student will, at first, take some courses in each field as an investment into both careers. From an ex-post perspective, courses in accounting may not be useful for someone who eventually chooses engineering, but the student would only select a major after trying several fields. As times goes by, new information and labor market shocks can change the career prospects across fields. In this context, the main question for a student choosing a career is how many courses to take in each discipline before making the final choice of a major.[1]

Similarly, the contests for promotion between the employees of a firm or an organization can also be regarded as investment tournaments. In the latter context, the firms and organizations make substantial investments in the human capital of their employees, in particular, by providing training, coaching and mentoring.

Another example of an investment tournament is the process of new product development. A firm or a government organization often develop several prototypes of a new product at the same time. Hence, investment dollars

[1] Our basic investment tournament model highlights the aspects of career choice stemming from the choice of investments into learning alternative professions, before the information regarding which one of them fits a person best, or has the highest value, is revealed. Obviously, the complexity of educational choices can not be fully captured by a simple investment tournament model. A person may take courses in various fields for reasons outside of our model, e.g. in order to satisfy intellectual curiosity or expand her/his social network. Yet, we believe that our model captures important aspects of students' career choices.

have to be allocated across alternative prototypes before their performance (i.e. the value) becomes known. The expected performance of a prototype is increasing in the amount of resources committed to it. If the final product combines the features of several prototypes, then the profits of the firm would depend on the realization of several alternatives. On the other hand, if the new product is based exclusively on the best prototype, then the firm's profits would depend only on the realization of the highest value alternative.[2]

In this paper, we develop a model of investment tournaments encompassing these and similar examples, and characterize the optimal investment strategy. Throughout, career choice problem remains our main motivating example, and therefore we will couch the discussion in terms of this problem.

The simplest model of an investment tournament has three periods. In the first and third periods the values of all alternatives increase as a result of random shocks, while in the second period the decision-maker chooses the level of investment into each alternative by allocating a fixed budget across them. The final value of each alternative is realized in the third period and is equal to the sum of the first and the third period shocks plus the investment received by this alternative in the second period. The payoff to the decision-maker is determined by the final values of the alternatives.

The class of investment tournaments analyzed below is far more general than this simple three period model. In Section 2 we consider investment tournaments with an arbitrary finite number of periods in which the decision-maker and the nature either alternate in taking actions or act simultaneously. The value function is given by the weighted sum of the final values of the alternatives. This multi-period model reflects that the decision-maker may learn how the value of each alternative changes over time and use this information to adjust the allocation of investment between the alternatives.

[2]Various aspects of R&D investment decisions that involve developing several prototypes of a new product are best studied by applying a combination of investment tournaments and optimal search methodology. In the context of optimal search literature, the amount of resources invested in each prototype is assumed to be exogenously fixed and all prototypes are assumed equally promising. As an example of the optimal search approach, Dahan and Mendelson (2001) study optimal prototyping strategy and R&D experimentation. They investigate the optimal number of prototypes and the optimal combination between parallel and sequential prototyping.

In contrast, in our investment tournament model a firm can adjust its investment level into each prototype depending on the preliminary (noisy) evaluations of the potential of each prototype. Thus, investment tournament model isolates an aspect of the optimal prototyping problem that has not been previously investigated.

For example, a student could learn that one major becomes more promising than the others due to a demand shift in the labor market, and modify her allocation of time to courses across different fields.

Proposition 1 characterizes the optimal investment strategy in such multi-period setting. At every decision node, it calls for investing all resources into the leading alternative with the highest current value. Thus, the optimal strategy magnifies the advantage of the leading alternative. However, due to random shocks, the leading alternative at one decision node need not remain in the lead at the next decision node. We can interpret Proposition 1 as saying that the optimal strategy is invariably to put all eggs in the "favorite basket," although the "favorite basket" may change over time.

Proposition 1 has a simple corollary showing that the strategy of investing only in one favorite alternative in each period remains optimal when the decision-maker's payoff depends only on the final value of the largest alternative. Proposition 2 establishes that our main result continues to hold if the payoff to the decision-maker is a sum of convex transforms of the final values of all the alternatives.

Section 2.1 generalizes the benchmark model of investment tournaments by allowing the returns to investment to be decreasing. We provide two types of results. First, if the aggregate investment cost function is decreasing in the total cost of investment, then our main result still holds i.e., it remains optimal to invest only in one alternative.

However, if the benefit function is concave in the values of alternatives, then several alternatives may receive positive investment at some decision nodes. Even though the optimal strategy in this case is less extreme than "putting all eggs in one basket," it is still optimal to significantly favor the leading alternative, even if its lead is very small.

These results provide insights regarding the optimal strategy for career choice. In particular, they explain why a student choosing between two majors may rationally devote substantially more effort to a field that seems slightly more promising. At the same time, the student should not completely disregard somewhat less promising fields. Arrival of new information regarding career prospects across the fields or shocks in the labor market can reverse the ranking of the fields and cause a student to dramatically change the amount of time (s)he invests in each field. Thus, a seemingly irrational jumping back and forth between majors may be consistent with expected utility maximization. Similarly, a firm working on several prototypes of a new product should always invest substantially more in the development of the prototype which, at the current moment, appears just slightly more promising than the other prototypes.

Section 3 considers another application of investment tournaments to labor economics. It builds upon the literature on incentive aspects of tournaments which was pioneered by Lazear and Rosen (1981) and was further developed by Bhattacharya and Guasch (1988), Bull et al. (1987), Ehrenberg and Bognannon(1990), Eriksson (1999), Ferrall (1996), Green and Stokey (1983), Nalebuff and Stiglitz (1983), Taylor (1995), Barut and Kovenock (1998), Krishna and Morgan (1998), Moldovanu and Sela (2001), and others.

In a traditional incentive tournament, the workers (who play the role of alternatives in our terminology) are competing for a prize, usually a promotion. This motivates them to exert costly effort because performance, and hence the probability of winning the tournament, (stochastically) increases with effort.

In our hybrid incentive/investment tournament framework of Section 3 we consider bilateral investment in the workers' human capital: both the firm and the worker invest in it over the course of a promotion contest. In particular, in law firms the investment into young workers' human capital takes the form of mentoring by senior partners. Mentoring is a scarce resource that can take the form of providing guidance and advice, being included in meetings with important clients or being assigned to more creative and complex projects.

A firm's profits at any given time depend primarily on the average performance of its workers. At the same time, in many contexts the firm-specific capital of a worker who wins the tournament has special significance. For instance, in an up or out tournament where the winners are promoted and the losers are laid off, the investments into firm-specific human capital of the losing contenders are wasted from ex-post perspective.[3] Therefore, the firm will be primarily interested in maximizing the human capital of the winner in the tournament.

Proposition 5 shows that, other things being equal, a worker would exert more effort in the early stages of an incentive tournament. By working harder and putting in longer hours early in her career, the worker tries to get ahead of her/his competitors at the beginning of the tournament. If she succeeds in

[3]Galanter and Palay (1991) and Rebitzer and Taylor (2007) provide an analysis of the organizational structure of law firms highlighting the role of up-or-out promotion contests, or tournaments.

this, she obtain a larger share of the firm's investment in its employees' human capital. This improves the worker's chances of winning the tournament and getting a promotion. Thus, the competition between the workers for their employer's investment in our model explains the phenomenon of a 'rat race' among young professionals.

It is worth noting that, starting from Akerlof (1976) (see also Landers et al. (1996) and Andersson (2002)), explanations of a rate race have typically relied on adverse selection and worker screening arguments. Our explanation of a rat race differs from the one that dominates in the literature. Notably, our theory of a rat race generates a testable prediction that the workers will put more effort early in their careers rather than later. In contrast, 'adverse selection' theories of a rat race imply that a typical worker will put in more than an efficient number of hours. The latter is hard to test empirically since the efficient number of hours depends on an unobservable worker type.

Several contributions in the literature on promotion tournaments, in particular (Barut and Kovenock (1998), Krishna and Morgan (1998), Moldovanu and Sela (2001)), examine the effect of the design of tournament prizes on the efforts of the contestants and provide recipes for the optimal design of such prizes. The focus of this paper is different. We concentrate on understanding the investment behavior by the other party in these contests- the decision-maker, who organizers the tournaments and benefits from the value generated by the contestants. Hence, this paper puts an emphasis not on the structure of prizes for the contestants, but on the value, or output, generated by the contestants for the firm. Another important feature, distinguishing our approach from most contributions in the literature on tournaments, is its dynamic nature, as we consider investment decisions made dynamically throughout multiple periods.

The rest of the paper is organized as follows. Section 2 introduces and solves the model of investment tournaments under different specifications of the decision-maker's value and cost functions. Section 3 combines investment tournament model with the model of incentive tournaments and applies the results to personnel economics. Section 4 concludes. The proofs are relegated to an Appendix.

2 Model and Main Results

In this section, we formulate and solve a model of career choice as an investment tournament. To start, suppose that there are T periods over which an individual decision-maker has to complete her or his career choice and investments in different professions. Specifically, in each period this individual has to choose how to allocate an amount B of resources (time, effort and money invested into career choice in each period) among N alternative professions. We will use the terms "profession" and "alternative" interchangeably in what follows.[4] Let b_{ti} denote the nonnegative amount of resources invested into profession/alternative i at period t. Then the individual's investment action in period t can be represented by a vector $\mathbf{b}_t = (b_{t1}, b_{t2}, ...b_{tN})$ lying in the feasible action space $A = \{\mathbf{b} \in R^N : b_i \geq 0,\ \sum_{i=1}^{N} b_i = B\}$.

In each period the nature draws a random shock to every alternative/profession. The shock to the value of alternative i in period t is denoted by s_{ti}. It is drawn from an atomless distribution $F(\cdot)$ with support $[0, \bar{s})$. The shocks are independent across alternatives and across time. Thus, the action of nature at time t is denoted by vector $\mathbf{s}_t = (s_{t1}, s_{t2}, ...s_{tN})$. These shocks could represent changes in expected labor market prospects, or earnings, in a given career. The lowest value of the shock is normalized to 0, which can be replaced with any other lowest threshold, as long as the threshold is the same across all alternatives.

Here we assume that the decision-maker and nature act simultaneously, although the situation would be identical if they took alternating turns, with the decision-maker being the first to move in each period. The final value of alternative/profession i at the terminal node is the sum of all shocks and all investments into it and is given by $V_i = \sum_{t=1}^{T} s_{ti} + \sum_{t=1}^{T} b_{ti}$.

Let the history at time t be denoted by $\mathbf{h}_t = (\mathbf{s}_1, \mathbf{b}_1, \mathbf{s}_2, \mathbf{b}_2, ..., \mathbf{s}_{t-1}, \mathbf{b}_{t-1})$. The payoff-relevant information contained in history $\mathbf{h}_t$ can be summarized by a vector of the current values of alternatives/professions $\mathbf{V}_t = \mathbf{V}_t(\mathbf{h}_t) = (V_{t1}, V_{t2}..., V_{tN})$, where $V_{ti} = \sum_{\tau=1}^{t-1} s_{\tau,i} + \sum_{\tau=1}^{t-1} b_{\tau,i}$.[5] We will say that alternative i is a *favorite* in period t if $V_{ti} \geq V_{tj}$ for all $j \in \{1, ..., n\}$. The winner in this career choice problem is the alternative/profession with the highest

[4] For simplicity of exposition, we assume that the amount of resources available for investment is the same in each period. The results and proofs do not change qualitatively if the investment budget was changing across periods deterministically or randomly.

[5] The value at the terminal node is $V_i \equiv V_{(T+1)i}$.

value at the end of period T, i.e. a favorite at the terminal node. [6]

Let us also define the extended value of an alternative/profession i at time t, $\widetilde{V}_{ti}$, as its value at the end of period t after the decision-maker has allocated period-t budget but not including period-t shock, i.e. $\widetilde{V}_{ti} \equiv V_{ti} + b_{ti}$. Thus, the values of alternatives at period t contain all relevant information about the history prior to t, and extended values of alternatives at period t contain all relevant information regarding the history prior to period t and the decision-maker's action at t. If the decision-maker follows some strategy σ, her expected payoff and extended expected payoff at time t can be expressed as functions of σ and the current values of alternatives, $\Pi(V_{t1}, ..., V_{tN}, \sigma)$ and $\widetilde{\Pi}(\widetilde{V}_{t1}, ..., \widetilde{V}_{tN}, \sigma)$, respectively.

One of the central results of this paper shows that, under a broad range of conditions, it is optimal for an individual making her career choice (or involved in some other investment tournament) to invest all her resources in each period into one favorite alternative. Let us illustrate this result with a simple example.[7] Suppose that there are only two alternatives/professions. The decision-maker's objective is $\max\{V_1^f, V_2^f\}$, where V_i^f is the final value of alternative $i \in \{1, 2\}$.

Consider that there is only one period left to make investments. The current values of alternatives 1 and 2 are V_1 and V_2, respectively, with $V_1 > V_2$. The investment budget is B. If in the last period the individual makes an investment x in the first alternative and an investment $B - x$ in the second alternative, then her expected payoff equals

$$E_{s_1,s_2} \max\{V_1 + x + s_1, V_2 + B - x + s_2\},$$

where s_1 and s_2 are the expected values of the shocks to alternatives 1 and 2 in the last period. Let us show that it is optimal to invest the whole budget B into alternative 1, the favorite. If the individual reallocates her budget increasing the investment into alternative 1 by 1 cent, then her payoff increases by 1 cent if $V_1 + x + s_1 > V_2 + B - x + s_2$ and decreases by 1 cent if $V_1 + x + s_1 > V_2 + B - x + s_2$. Hence, after this reallocation the decision-maker's expected payoff changes by

$$2E_{s_1,s_2} Prob.[V_1 + x + s_1 > V_2 + B - x + s_2] - 1. \qquad (1)$$

[6] If there is more than one favorite alternative at the terminal node, then an arbitrary tie-breaking rule can be used to determine the winner.

[7]We thank the anonymous referee for suggesting this example.

Because s_1 and s_2 are identically distributed, the expression in (1) is positive if $x > \frac{V_2 - V_1 + B}{2}$ and is negative if $x < \frac{V_2 - V_1 + B}{2}$. So, starting from x s.t. $x > \frac{V_2 - V_1 + B}{2}$, the decision-maker can increase her expected payoff by raising her investment into alternative 1 all the way up to B. On the other hand, starting from $x < \frac{V_2 - V_1 + B}{2}$, the decision-maker can increase her expected payoff by raising her investment into alternative 2 all the way up to B. This implies that all budget must be invested in a single alternative. The symmetry of the shocks and the fact that $V_1 > V_2$ then imply that the expected payoff from investing all budget into alternative 1, the favorite, is greater than the expected payoff from investing all budget into alternative 2.

Generalizing this simple example, we will first study a T-period career choice problem in which the decision-maker's payoff is equal to a weighted sum of the terminal values of all alternatives/professions, with higher-ranked alternatives assigned higher weights. We will show that in this case the decision-maker's optimal strategy is to invest all resources into a favorite alternative/profession in every period. Indeed, we have:

Proposition 1 *Suppose that the decision-maker's payoff is given by* $\sum_{k=1}^{N} \lambda_k V_{r(k)}$ *where* $\lambda_1 \geq \lambda_2 \geq \lambda_3 ... \geq \lambda_N \geq 0$, *and* $r(k)$ *denotes the alternative with the* k*-th highest final value, while* $V_{r(k)}$ *denotes the final value of the alternative ranked* k*-th at the end of the tournament.*

The following strategy is optimal in the investment tournament: in period $t \in \{1, ..., T\}$ *the decision-maker allocates all investment resources to an alternative that is a favorite in this period. If there is more than one favorite alternative in period* t*, then all resources in period* t *are allocated to one of the favorites. The remaining* $N - 1$ *alternatives receive zero amount of investment in period* t*.*

Note that any investment strategy is optimal if $\lambda_1 = \lambda_2 = \lambda_3 ... = \lambda_N$. However, if at least one inequality is strict, then the optimal strategy calls for investing all resources into a favorite alternative.

This formulation of the decision-maker's value function with $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_N$ is appropriate in a promotion tournament of a firm which invests in the human capital of its employees, when the promotion is based on the employees' past performance, and the employees are given increased responsibilities depending on their performance in the tournament.

An important special case of this set-up is where $\lambda_1 = 1$, $\lambda_2 = ... = \lambda_N = 0$. The following corollary of Proposition 1 applies in this case.

Corollary 1 *Suppose that the decision-maker's payoff is equal to the value of the winning alternative* $\max\{V_1, ..., V_N\}$*. Then the decision-maker's optimal strategy in period* $t \in \{1, ..., T\}$ *is to invest all resources into one favorite alternative.*

The Corollary is important since there is a number of environments in which the decision-maker cares only about the realization of the highest value alternative. For example, this is so in the promotion tournament with up-or-out rule, under which only the winner gets promoted and stays with the firm. Similarly, this case applies when a student is choosing a major, or a firm is selecting a new product among several prototypes, or a person is choosing a business or life partner among several candidates. Also, this generalizes the auction environment introduced in Schwarz and Sonin (2001).

The case considered in Corollary 1 is related to the optimal search problem, as both are maximal problems. However, neither problem is a special case of the other. Optimal search literature studies the optimal strategy for investment in information acquisition, when there is a cost of obtaining information about a particular alternative. In contrast, in investment tournament problem the information about the value of each alternative is available at no cost to the decision-maker, but the issue is how to allocate the investment between alternatives.

On the other hand, when the promotion is merely a prize and does not entail increased responsibility, all workers stay with the firm whether promoted or not (as in medical doctors' practices), and a worker's value reflects how well she has learned her trade or profession, then each worker's human capital contributes to the firm's profits in the same way. So, it is more appropriate to model the firm's final payoff as $\sum_{k=1}^{N} \mu(V_k)$ where $\mu(\cdot)$ is increasing and convex. The latter assumptions on $\mu(\cdot)$ reflect the accelerating nature of learning one's profession.

Surprisingly, even in this case it is optimal for the firm, as the decision-maker, to invest all resources in any period in the favorite alternative (a worker with the highest human capital) in that period. Indeed, we have:

Proposition 2 *Suppose that the decision-maker's payoff is equal to* $\sum_{k=1}^{N} \mu(V_k)$ *where* $\mu'(\cdot) > 0$ *and* $\mu''(\cdot) > 0$*. Then the decision-maker's optimal strategy is to invest all resources in period* $t \in \{1, ..., T\}$ *into one favorite alternative at period* t*.*

Note that Proposition 2 is not a special case of Proposition 1, because here the value of an alternative does not depend on its rank.

The results of Propositions 1 and 2 hold if the decision-maker and nature take turns making their moves. The proofs remains virtually unchanged.

2.1 Extensions: Decreasing Returns and Risk-Aversion.

So far, we have assumed that the decision-maker has a fixed investment budget B in each period. In this section, we consider two different settings. At first, we will consider the situation in which the decision-maker can choose to invest any amount in every period, but the returns to investment decrease in the total amount of investment into all alternatives. Specifically, suppose that the decision-maker's action space in period t is given by

$$A_t = \{\mathbf{b}_t : b_{ti} \geq 0 \text{ for all } i = 1...N\}.$$

The cost of investment in period t is measured by an increasing and convex function $\sum_{t=1}^{T} C(\sum_{i=1}^{N} b_{ti})$, reflecting that the returns to investment are decreasing in the total amount of investment in a given period. The decision-maker's payoff is equal to the final value of the winning alternative.

This model is natural when the investment involves the decision-maker's time or effort, since the returns to time, or effort, are typically decreasing in the total amount of it. For example, consider a student deliberating the choice between accounting and engineering majors. One extra course in accounting brings a student one course closer to completing the accounting major, regardless of whether it is a second or fifth course in accounting. That is, an increase in the value of the accounting alternative from taking one course in accounting is the same, regardless of the number of accounting courses that a student takes in a given semester. At the same time, the student's aggregate effort cost of attaining a certain performance level in a given semester typically depends on the total number of courses that the student takes in this semester, rather than on the distribution of courses by field.

Generalizing the proof of Proposition 1, we can show that in this case there exists an optimal strategy $b_{ti}(V_{t1}...V_{tN})$ that depends only on the current values of the alternatives. The following Proposition shows that it remains optimal in any time period to allocate all resources, or effort, to the favorite alternative.

Proposition 3 *Suppose that the action space at each decision node is*

$$A = \{\mathbf{b} \in R^N : b_{ti} \geq 0\}$$

and the decision-maker's payoff is given by

$$\max\{V_1, V_2...V_N\} - \sum_{t=1}^{T} C(B_t),$$

where $C'(\cdot) > 0$ and $C''(\cdot) > 0$ and $B_t = \sum_{i=1}^{N} b_{ti}$. Then,
(i) an optimal strategy requires that at each decision node only one favorite alternative receives positive investment;
(ii) the amount of optimal investment in period t is increasing in the value of the favorite alternative at t.

This proposition confirms the robustness of our main result establishing the optimality of investing only in one favorite alternative in every time period. However, it does not imply, for example, that an individual would not switch from one field to another in her career choice, or a student will never switch between majors. We could observe complete switching between fields or professions if the job-market situation changes and some professions becomes more attractive than others. The latter change would constitute an act of nature in our model.

It is also interesting to explore to what extent our results survive under different specification of the decision-maker's value function. To explore this case, suppose that the decision-maker's payoff is given by $\sum_{i=1}^{n} \lambda_k \mu(V_{r(k)})$, where $\mu(.)$ is a concave function satisfying $\mu'(.) > 0$ and $\mu''(.) < 0$. (Recall that $r(k)$ is the index of the alternative with the k-th highest rank among the terminal alternative values, and $\lambda_i > \lambda_{i+1}$ for all $i \in \{1, ..., N-1\}$).

In this case, the decision-maker is risk-averse. It is intuitive that a risk-averse decision-maker would make positive investments in more than one alternative in order to mitigate the risk and smooth her payoff. However, she would still prefer to invest a larger share of her investment budget in the favorite alternative(s) since it (they) are more likely to have higher value at the end of the tournament. Furthermore, the intuition suggests that when risk-aversion is weak, the strategy of investing all resources into one alternative should still remain optimal. For a more precise understanding of the optimal investment strategy, let us suppose that $\mu(.)$ is a quadratic function. Then we have:

Proposition 4 *Suppose that the decision-maker's value function at the terminal node is given by $\sum_{i=1}^{n} \lambda_k \mu(V_{r(k)})$ where $\mu(V) = aV - c\frac{V^2}{2}$ for some*

$a > 0$, $c > 0$, and $r(k)$ is the index of the alternative which is ranked k-th. Then the decision-maker's optimal strategy has the following properties:
(i) If $\lambda_1 = \ldots = \lambda_n$, then the unique optimal strategy is to invest equal amounts in all alternatives in period 1. The optimal strategy in period t is to invest in such a way that maximizes the current value of the lowest-ranked alternative.
(ii) If $\frac{a}{c}$ is sufficiently large, and per-period budget B and the upper bound on the support of the distribution of shocks $\bar{s}$ are sufficiently small so that $\frac{a}{c} > n((B + \bar{s})T + 1)$, then the optimal strategy of the decision-maker is to invest all budget in a favorite alternative.

The Proposition illustrates the effect of the decision-maker's risk-aversion. In particular, it indicates that the effect of risk-aversion is weaker when the decision-maker cares much more about the value of the winning alternative than the other alternatives. In contrast, the strategy of maximizing the current value of the lowest-rank alternative allows to minimize the differences between the values of alternatives. This strategy is optimal for a risk-averse decision-maker when all alternatives have equal weight at the end of the tournament. Due to technical complexity, we have limited our analysis to the quadratic case. However, we believe that these results generalize to other payoff functions.

3 Bilateral Investment in Promotion Tournaments

In this section we consider tournaments in which competing alternatives are themselves players in the tournament and can take actions affecting their values. To emphasize the active role played by the alternatives, we will from now on refer to them as "contenders." In contrast to the previous sections, the motivation for studying such tournaments comes not from the career choice problem, but from competition for a promotion within an organization.

As an example, consider competition for a promotion among associates in a law partnership or a consulting firm. The decision-maker is a senior partner or a management committee of the firm. She selects the levels of investment into contenders' firm-specific human capital. This may involve dividing scarce mentoring resources among contenders, assigning them to more or less high-profile projects, etc. At the same time, the contenders also choose the amount of effort or investment in their own human capital to increase their own value. For simplicity, we assume that the tournament is up-or-out, and so the investments into firm-specific human capital of associates who are not promoted are wasted from the ex-post perspective.[8] However, as we have shown, the results of the previous sections hold under a range of specifications and value functions, as long as the investment in the winning alternative is at least as useful as the investment into a losing alternative. The results that we present below extend in a similar way.

The goal of this section is to characterize the outcome of a tournament where the investment incentives of the decision-maker and the contenders interact. We model this situation as follows. In every period starting from $t = 1$ the contenders select nonnegative efforts which they invest in acquiring firm-specific human capital. Contender i's effort in period t is denoted by e_{ti}. Her cost of effort e_{ti} in period t is given by $g(e_{ti}) > 0$, where $g'(e_{ti}) > 0$, $g''(e_{ti}) > 0$ for all $e_{ti} > 0$. Also, in every period starting from $t = 1$ the decision-maker selects the levels of investment into each contender. Her action space is $A^d = \{\mathbf{b}_t \in R^N : b_{ti} \geq 0 \quad \sum_{i=1}^{N} b_{ti} = B\}$, where b_{ti} denotes the decision-maker's investment in contender i in period t.

We maintain the assumption that in each period the nature independently draws a random shock to the value of each contender from an atomless distribution $F(\cdot)$ over nonnegative support. A random shock to the value of contender i in period t is denoted by s_{ti}.

Nature takes an action in period zero. In subsequent periods the nature, the decision-maker and the contenders move simultaneously. Thus, the value of contender i in period $t \in \{1, ..., T\}$, V_i^t, is a sum of her/his own investments, the investments by the decision-maker and random shocks up to period t. That is, $V_i^t = \sum_{t'=0}^{t} s_{t'i} + \sum_{t'=1}^{t} b_{t'i} + \sum_{t'=1}^{t} e_{t'i}$. The terminal value of contender i in the last period T is equal to $V_i = V_i^T$. The history of the game after period $t \in \{1, ..., T\}$ is summarized by the vector of the contenders' values $(V_1^t, ..., V_N^t)$.

The contender with the highest final value wins the tournament. The decision-maker's payoff is equal to $\max\{V_1, ...V_N\}$ - the value of the tournament winner. The payoff of contender i, who loses the tournament, is equal to the negative of the sum of the costs of effort that (s)he has invested across all time periods, i.e. $-\sum_{t=1}^{T} g(e_{ti})$. If contender i wins the tournament her/his

[8] Galanter and Palay (1991) provide a detailed account of the role of tournaments in large law firms in the U.S. On this topic, see also Rebitzer and Taylor (2007).

payoff is $R - \sum_{t=1}^{T} g(e_{ti})$, where R can be interpreted as the rent associated with winning a promotion.

Thus, a promotion tournament of this section combines an investment tournament introduced in the previous section, where the only players are the decision-maker and the nature, with the elements of an incentive tournament where the players are the contenders and the nature (see Lazear and Rosen (1981)).

The information structure of this tournament is as follows. In any period the decision-maker observes the value of each contender in the previous period. The contenders do not observe random shocks or the decision-maker's investment allocation. So, in every period each contender's information set contains only her efforts in the previous periods. Technically, our observability assumption allows us to avoid dealing with complex dynamic game effects that would arise if a contender could condition her strategy on the full history of events.

The assumption that the agents cannot observe the value of the firm-specific capital invested in them by the employer is plausible when the employer's investment takes the form of task allocation, and the employees do not know which tasks allow them to develop firm specific human capital and hence constitute an investment by the employer, and which tasks do not enrich their firm-specific human capital.

Further, the key aspect of our observability structure is that a contender cannot observe her relative ranking among her peers. This is consistent with reality in a number of situations. In such environments, even if the contenders do observe investments in their own firm specific human capital, they may learn very little about their position in the tournament. For example, in professional services firms junior associates may not be aware of the details of the mentoring programs provided to other associates. An academic department may keep confidential the details of the research support provided to different junior members. If a contender has a diffuse prior regarding the distribution from which mentoring resources are drawn, the investments into individual contenders become entirely uninformative.

Under the diffuse prior assumption, it is possible to relax the restriction on observability by assuming that a contender can observe the investment that she receives from the decision-maker, but observes neither the decision-maker's budget, nor the investments received by the other contenders. At the same time, allowing for strategies contingent on the full history would significantly complicate the analysis and most likely generate a multiplicity of equilibria.

The following result characterizes symmetric Nash equilibria in this promotion tournament game. The symmetry restriction only requires that all alternatives are treated symmetrically by the decision-maker.

Proposition 5 *Every symmetric pure strategy Nash equilibrium of a promotion tournament has the following properties:*
(i) in every period, the decision-maker chooses to invest all resources into one of the favorite contenders;
(ii) the effort of each contender is decreasing over time.

Note that the investment strategy of the decision-maker is qualitatively similar to her strategy in the tournament without contenders' investment. This is so because the contenders cannot condition their behavior on the decision-maker's actions.

On the other hand, the contenders do have an incentive to influence the actions of the decision-maker. So, the contenders invest more effort into improving their position at the early stages of the tournament, because early effort can attract investments from the decision-maker who would put a promising employee on a "fast track" for a promotion. Consequently, all contenders will put forth more effort at the early stages of the tournament in order to become a leader.

Note that we have assumed that contender i's effort cost is separable across periods. If her cost of effort was equal to $Q(\sum_{t=1}^{T} e_{ti})$, where $Q'(\cdot) > 0$, $Q''(\cdot) > 0$, then the decision-maker would also invest all resources into the favorite contender in every period, and each contender would invest all effort in the first period. This is so because the contenders receive no additional information after the first period. Consequently, each contender would shift all her effort to the first period in a bid to receive more mentoring (investment) from the decision-maker.

The results of this section explain the phenomenon of a "rat race" which young professionals often have to endure and "fast track" promotion schemes frequently used by the employers. In our model, a rat race emerges as an outcome of a competition between the employees for the employer's investment and mentoring. This competition motivates the employees to overwork in the initial stages of their careers. More specifically, an employer's optimal strategy of investing all resources in one favorite employee in every period effectively puts this employee on a fast track for a promotion. So, a higher effort by an employee early in her career increases her chances of becoming an early favorite and thereby obtaining the benefit of the employer's investment in her human capital. Plainly, our results suggest that first-year graduate students would work harder than second-year students, and first-year associates in a law firm would put in longer hours than later-year associates. Anecdotal evidence regarding career development appears to confirm that.

It is worth noting that our explanation of a rat race differs from those in the existing literature. Most existing contributions explain a rat race via adverse selection motives. This explanation was first suggested by Akerlof (1976). Developing this approach, Landers et. al (1996) provide a model of a 'rat race' in law and other professional services firms. In their model, the employees differ in their willingness to work, and the firm wants to retain as partners only those types who are more willing to work long hours. In equilibrium, the hours of work serve as a screening device. The employees who are willing to worker longer hours have to overwork substantially to prevent imitation by the types with lower willingness to work. Landers et. al (1996) also provide supporting evidence that the associates in law firms log in an inefficiently high number of hours.

Exploiting a related, signalling, motivation, Holmstrom (1999) shows that a worker would exert a relatively high effort early in her career in order to raise her productivity and thereby to signal her ability. Andersson (2002) also focuses on signalling ability as an explanation of the rate race. He considers a framework in which the wage history of a worker is observable by a hiring firm, but the ability and the employment contract of a job applicant are not observable. Then the first-period wage serves as a signal of ability for the second-period employer, inducing the worker to put in an efficiently large amount of effort in the first period.

These settings are different from the one explored in our paper. Notably, our explanation of a rat race does not rely on asymmetric information about the employees' productivity or willingness to work. A rat race arises in our model as a consequence of complementarity between the employees' and the managers' investments in the employees' human capital.

It is important to note the differences between the conclusions of our theory of a rate race, on the one hand, and the adverse selection theory of it, on the other hand. Indeed, our theory generates a testable prediction that the workers will put in longer hours in the initial stages of their careers. In contrast, the adverse selection theory of a rat race predicts that the young workers will put an inefficiently long number of hours. However, efficiency is hard to estimate and measure since it is determined by unobservable workers' productivity types.

4 Conclusions

In this paper we have analyzed investment tournaments that arise naturally in career choice promotion and other contexts, such as product design. We have shown that in every period the decision-maker will optimally allocate all her investment resources to a single alternative, or a contender. This result holds robustly under a variety of specifications and assumptions.

In many tournaments, a contender with a small lead tends to enjoy substantially better chances of winning the tournament than other contenders. This paper provides an explanation for why an early leader may be favored by a fully rational decision-maker.

Applications of investment tournaments are not limited to career choice, promotion and product design contexts, and extend beyond purely economic domain. Indeed, our investment tournament model is appropriate for modeling various decisions that involve a choice between several alternatives. For instance, investment tournaments may help to explain why people tend to date one person at a time. Dating amounts to spending time with a potential partner. This can be viewed as an investment into the relationship-specific value of a particular match. By Proposition 5, even if it is highly uncertain which partner will be ultimately preferred, it is still optimal to invest disproportionately into the most promising alternative. Proposition 5 predicts that the effort invested into a relationship by competing contenders is largest at the early stages of the relationship.

Of course, the investment tournament model does not reflect the full complexity of career choice or the dating problem. In fact, the limitations of the investment tournament model suggest several directions for future research. In particular, we would like to extend the investment tournament model to two-sided matching in which a participant on one side of the market can make investments into individual contenders on the other side of the market, in particular, in dating and marriage.

Also, the investment tournament model does not address search aspects arising in career choice, dating, etc. In these and in a number of other contexts optimal search and investment tournament approaches are complementary, as each approach highlights certain important aspects of choice. In particular, the optimal search approach focuses on information acquisition assuming away the possibility of investment into improving the quality of a match. In contrast, the investment tournament model focuses on the investments into relationships. Combining search and matching models with investment tournament models would open interesting directions in research. We intend to pursue these directions in future work.

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A Appendix

A.1 Notation and Definitions.

We start with a list of notation and definitions used in the proofs below.

Markov strategy: a strategy is Markov if the action in any period t depends only on the vector of current values of alternatives $\mathbf{V}_t$. For a Markov strategy we can write $\sigma(\mathbf{V}_t)$ instead of $\sigma(\mathbf{h}_t)$. So, when considering a Markov strategy we will, with a slight abuse of notation, refer to the vector of the values of alternatives $\mathbf{V}_t$ as history.
Extended history: $\widetilde{\mathbf{h}}_t = (\mathbf{h}_t, \mathbf{b}_t) = (\mathbf{s}_1, \mathbf{b}_1, \mathbf{s}_2, \mathbf{b}_2, ...\mathbf{s}_{t-1}, \mathbf{b}_{t-1}, \mathbf{b}_t)$
Extended value: $\widetilde{V}_{ti} = V_{ti} + b_{ti}$; $\widetilde{\mathbf{V}}_t = \widetilde{\mathbf{V}}_t(\widetilde{\mathbf{h}}_t) = (\widetilde{V}_{t1}, \widetilde{V}_{t2}...\widetilde{V}_{tN})$
Extended favorite: we will say that an alternative i is an extended favorite at time t if $\widetilde{V}_{ti} \geq \widetilde{V}_{tj}$ for any $j \in \{1, ..., N\}$.
Allocated investment: $b_{ti}(\mathbf{h}_t, \sigma)$ (or $b_{ti}(\mathbf{V}_t, \sigma)$) is the investment allocated to alternative i by a pure strategy σ, conditional on history $\mathbf{h}_t$ (or $\mathbf{V}_t$).
Expected payoff: $\Pi(\mathbf{h}_t, \sigma)$ (or $\Pi(\mathbf{V}_t, \sigma)$) is the expected payoff from strategy σ conditional on $\mathbf{h}_t$, (or $\mathbf{V}_t$). $\widetilde{\Pi}(\widetilde{\mathbf{h}}_t, \sigma)$ (or $\widetilde{\Pi}(\widetilde{\mathbf{V}}_t, \sigma)$) is the expected payoff from strategy σ conditional on $\widetilde{\mathbf{h}}_t$ (or $\widetilde{\mathbf{V}}_t$).
Hybrid history: $f(\mathbf{h}_t, \mathbf{h}'_\tau) \equiv (\mathbf{s}_1, \mathbf{b}_1, ...\mathbf{s}_{t-1}, \mathbf{b}_{t-1}, \mathbf{s}'_t, \mathbf{b}'_t, ...\mathbf{s}'_{\tau-1}, \mathbf{b}'_{\tau-1})$ where $\mathbf{h}_t = (\mathbf{s}_1, \mathbf{b}_1, ...\mathbf{s}_{t-1}, \mathbf{b}_{t-1})$ and $\mathbf{h}'_\tau = (\mathbf{s}'_1, \mathbf{b}'_1, ...\mathbf{s}'_{\tau-1}, \mathbf{b}'_{\tau-1})$ and $\tau \geq t$.
Continuation strategy: For a Markov strategy $\sigma(.)$, *continuation strategy in period* τ satisfies $\sigma(\mathbf{h}'_\tau) = \sigma(f(\mathbf{h}_t, \mathbf{h}'_\tau))$.
Equivalence: $\mathbf{h}_t$ is equivalent to $\mathbf{h}'_t$ if and only if $\mathbf{V}_t(\mathbf{h}_t) = \mathbf{V}_t(\mathbf{h}'_t)$
Probability of winning: $P_{ti}(\mathbf{V}_t, \sigma)$ represents the probability that alternative i wins the tournament conditional on $\mathbf{V}_t$ and Markov strategy σ; $\mathbf{P}_t =$

$(P_{t1}...P_{tN})$.
$\widetilde{P}_{ti}(\widetilde{\mathbf{V}}_t, \sigma)$ represents the probability that alternative i wins the tournament conditional on $\widetilde{\mathbf{V}}_t$ and Markov strategy σ;
$\widetilde{\mathbf{P}}_t = (\widetilde{P}_{t1}...\widetilde{P}_{tN})$.
Modified value: $\mathbf{V}_{t|i}(\delta) = (V_{t1}, ..., V'_{ti} = V_{ti}+\delta, ...V_{tN})$ and $\widetilde{\mathbf{V}}_{t|i}(\delta) = (\widetilde{V}_{t1}, ..., \widetilde{V}'_{ti} = \widetilde{V}_{ti} + \delta, ...\widetilde{V}_{tN})$.

Proof of Proposition 1.

First, we prove the following Lemma which allows us to focus on Markov strategies.

Lemma 1 *There exists a Markov optimal strategy σ.*

Proof. The proof is by backwards induction. Consider the last period T. An optimal strategy in this period, σ^*_T, prescribes such allocation of budget B between N alternatives that maximizes the expected value of the objective. That is, σ^*_T is a solution to the following problem:

$$\max_{(b_{T1}\geq 0,...,b_{TN}\geq 0:\sum_{i=1,...,n} b_{Ti}=B)} E_{F\times...\times F} \max\{V_{T1} + b_{T1} + s_{T1}, ..., V_{TN} + b_{TN} + s_{TN}\}. \tag{2}$$

Note that the expectation is taken with respect to the vector $(s_{T1}, ..., s_{TN})$. Since the objective is continuous in $(b_{T1}, ..., b_{TN})$ and the feasible domain of $(b_{T1}, ..., b_{TN})$ is compact, this maximization problem has a solution - an optimal strategy σ^*_T. Clearly, this solution depends only on $(V_{T1}, ..., V_{TN})$, i.e. σ^*_T is Markov.

By Berge's maximum Theorem, the value of (2) is a continuous function of $(V_{T1}, ..., V_{TN})$ which we denote by $W^T(V_{T1}, ..., V_{TN})$.

Proceeding to period $T-1$, we can use a similar method to show that the optimal strategy for this period, σ^*_{T-1}, is Markov. Indeed, σ^*_{T-1} prescribes an allocation of budget B between N alternatives, $(b_{(T-1)1},...,b_{(T-1)N})$, to maximize the expected value of the following objective:

$$E_{F\times...\times F} W^T(V_{(T-1)1} + b_{(T-1)1} + s_{(T-1)1}, ..., V_{(T-1)N} + b_{(T-1)N} + s_{(T-1)N})$$

This objective is continuous in $(b_{(T-1)1}, ..., b_{(T-1)N})$ and the feasible domain of $(b_{(T-1)1}, ..., b_{(T-1)N})$ is compact, so the maximization problem has a solution - an optimal strategy σ^*_{T-1} which depends only on $(V_{(T-1)1}, ..., V_{(T-1)N})$.

Thus, σ^*_{T-1} is Markov. Proceeding backwards through all periods to the start of the game, we establish that the optimal strategy $(\sigma^*_1, ..., \sigma^*_T)$ is Markov. ■

With Lemma 1 in hand, we proceed to prove the Proposition in two steps. Step 1 shows that in every period, an optimal strategy requires all investment to be allocated to one alternative. Step 2 shows that the alternative that receives all investment in some period t is a favorite in that period.

Step 1. Let σ^* be an optimal Markov strategy. Recall that $\mathbf{V_t}= (V_{t1}, ..., V_{tN})$ ($\widetilde{\mathbf{V}}_t= (\widetilde{V}_{t1}, ..., \widetilde{V}_{tN})$) stands for the vector of values of alternatives (extended alternatives) at period t, and V_i stands for the terminal value of alternative i at the end of the tournament. Note that $V_i = \widetilde{V}_{it} + s_{it} + \sum_{\tau=t+1}^{T}(b_{i\tau} + s_{i\tau})$.

Given the information available at period t and given the decision-maker's Markov strategy σ^*, let $\eta_{ti}(\widetilde{\mathbf{V}}_t, \sigma^*, \mathbf{s_t}, ..., \mathbf{s_T}) \equiv \mathbf{s_{it}} + \sum_{\tau=\mathbf{t+1}}^{\mathbf{T}}(\mathbf{b_{i\tau}} + \mathbf{s_{i\tau}})$. That is, $\eta_{ti}(\widetilde{\mathbf{V}}_t, \sigma^*, \mathbf{s_t}, ..., \mathbf{s_T})$ is the change in the value of alternative i between period t after investment b_{ti} has been made and its terminal value, given that the decision-maker uses the strategy σ^* and the profile of random shocks in periods t,...,T is given by $(\mathbf{s_t}, ..., \mathbf{s_T})$. As a Markov strategy, σ^* depends on $\widetilde{\mathbf{V}}_\tau$ for all $\tau \in \{t, ..., T\}$ or, equivalently, on the vector $\widetilde{\mathbf{V}}_t$ of extended alternatives in period t and the profile of random shocks $(\mathbf{s_t}, ..., \mathbf{s_T})$. For brevity, we will sometimes write $\eta_{ti}(\widetilde{\mathbf{V}}_t, \sigma^*)$ omitting the dependence on $(\mathbf{s_t}, ..., \mathbf{s_T})$, but this dependence is implicitly understood. So, the terminal value of alternative i can be written as $V_i(\widetilde{\mathbf{V}}_t, \sigma^*, \mathbf{s_t}, ..., \mathbf{s_T})= \widetilde{V}_{ti} + \eta_{ti}(\widetilde{\mathbf{V}}_t, \sigma^*, \mathbf{s_t}, ..., \mathbf{s_T})$.

Further, let $R_{i,r}(\widetilde{\mathbf{V}}_t, \sigma^*, \mathbf{s_t}, ..., \mathbf{s_T})$ denote the rank function which is equal to 1 if the terminal value of alternative i, V_i, is r-th highest among the terminal values of all N alternatives $(V_1, ..., V_N)$, and is equal to zero otherwise. Given σ^*, $R_{i,r}(.)$ depends only on the vector of values of extended alternatives $\widetilde{\mathbf{V}}_t$ at time t and the profile of random shocks in periods t,..., T.

Suppose that at period t, σ^* prescribes positive investments b_{tj} and b_{tk} into alternatives j and k. Let $\delta \in (-\min\{b_{tk}, b_{tj}\}, \min\{b_{tk}, b_{tj}\})$ be a small reallocation of investment between alternatives j and k on top of what is prescribed by strategy σ^*. We will make this reallocation without changing the future allocation rule, so we still use $\widetilde{\mathbf{V}}_t$ as an argument of $\eta_{ti}(.)$ and $R_{i,r}(.)$ for all i and $r \in \{1, ..., N\}$. That is, after this modification the strategy σ^* prescribes the same actions in periods $t+1$,...,T as without this modification.

Define the vector of perturbed terminal values $\mathbf{V}^{f(j(\delta),k(-\delta))}(\widetilde{\mathbf{V}}_t, \sigma^*, \mathbf{s}_t, ..., \mathbf{s}_T)$

as follows:

$$\mathbf{V}^{f(j(\delta),k(-\delta))}(\widetilde{\mathbf{V}}_t,\sigma^*,\mathbf{s}_t,...,\mathbf{s}_T)\equiv$$
$$(V_1(\widetilde{\mathbf{V}}_t,\sigma^*,\mathbf{s}_t,...,\mathbf{s}_T),...,V_j(\widetilde{\mathbf{V}}_t,\sigma^*,\mathbf{s}_t,...,\mathbf{s}_T)+\delta,...,V_k(\widetilde{\mathbf{V}}_t,\sigma^*,\mathbf{s}_t,...,\mathbf{s}_T)-\delta,...,V_N(\widetilde{\mathbf{V}}_t,\sigma^*,\mathbf{s}_t,...,\mathbf{s}_T)). \tag{3}$$

That is, all entries of the vector $\mathbf{V}^{f(j(\delta),k(-\delta))}(\widetilde{\mathbf{V}}_t,\sigma^*,\mathbf{s}_t,...,\mathbf{s}_T)$, except the j-th and k-th, are the same as in the vector $\mathbf{V}(\widetilde{\mathbf{V}}_t,\sigma^*,\mathbf{s}_t,...,\mathbf{s}_T)$, while the j-th (k-th) entry of $\mathbf{V}^{f(j(\delta),k(-\delta))}(\widetilde{\mathbf{V}}_t,\sigma^*,\mathbf{s}_t,...,\mathbf{s}_T)$ is equal to the j-th (k-th) entry of $\mathbf{V}(\widetilde{\mathbf{V}}_t,\sigma^*,\mathbf{s}_t,...,\mathbf{s}_T)$ plus (minus) δ.

Then for any $\delta\in(-\min\{b_{tk},b_{tj}\},\min\{b_{tk},b_{tj}\})$, the decision-maker's expected value function in period t is equal to

$$\begin{aligned}
&\mathbb{E}_{(\mathbf{s}_t,...,\mathbf{s}_T)}\sum_{i=1,...,N,i\notin\{j,k\}}\ \sum_{r=1,...,N}\lambda_r R_{i,r}(\mathbf{V}^{f(j(\delta),k(-\delta))}(\widetilde{\mathbf{V}}_t,\sigma^*,\mathbf{s}_t,...,\mathbf{s}_T))V_i(\tilde{\mathbf{V}}_t,\sigma^*,\mathbf{s}_t,...,\mathbf{s}_T)\\
&+\mathbb{E}_{(\mathbf{s}_t,...,\mathbf{s}_T)}\sum_{r=1,...,N}\lambda_r R_{j,r}(\mathbf{V}^{f(j(\delta),k(-\delta))}(\widetilde{\mathbf{V}}_t,\sigma^*,\mathbf{s}_t,...,\mathbf{s}_T))(V_j(\tilde{\mathbf{V}}_t,\sigma^*,\mathbf{s}_t,...,\mathbf{s}_T)+\delta)\\
&+\mathbb{E}_{(\mathbf{s}_t,...,\mathbf{s}_T)}\sum_{r=1,...,N}\lambda_r R_{k,r}(\mathbf{V}^{f(j(\delta),k(-\delta))}(\widetilde{\mathbf{V}}_t,\sigma^*,\mathbf{s}_t,...,\mathbf{s}_T))(V_k(\tilde{\mathbf{V}}_t,\sigma^*,\mathbf{s}_t,...,\mathbf{s}_T)-\delta)
\end{aligned}\tag{4}$$

Consider (4) as a function of δ. Since $b_{tj}>0$ and $b_{tk}>0$ are optimal investments, $\delta=0$ is an interior optimum of (4). Therefore, the first derivative of (4) with respect to δ must be equal to zero at $\delta=0$ while its second derivative must be nonnegative. In the rest of the proof, we show that this is not the case, thereby establishing a contradiction with our original hypothesis that $b_{tj}>0$ and $b_{tk}>0$. Indeed, the first derivative of (4) with respect to δ is equal to:

$$\begin{aligned}
&\mathbb{E}_{(\mathbf{s}_t,...,\mathbf{s}_T)}\sum_{r=1,...,N}\lambda_r R_{j,r}(\mathbf{V}^{f(j(\delta),k(-\delta))}(\widetilde{\mathbf{V}}_t,\sigma^*,\mathbf{s}_t,...,\mathbf{s}_T))\\
&-\mathbb{E}_{(\mathbf{s}_t,...,\mathbf{s}_T)}\sum_{r=1,...,N}\lambda_r R_{k,r}(\mathbf{V}^{f(j(\delta),k(-\delta))}(\widetilde{\mathbf{V}}_t,\sigma^*,\mathbf{s}_t,...,\mathbf{s}_T))\\
&+\mathbb{E}_{(\mathbf{s}_t,...,\mathbf{s}_T)}\left(\sum_{i=1}^{N}\sum_{r=1,...,N}\lambda_r\frac{\partial R_{i,r}(\mathbf{V}^{f(j(\delta),k(-\delta))}(\widetilde{\mathbf{V}}_t,\sigma^*,\mathbf{s}_t,...,\mathbf{s}_T))}{\partial\mathbf{V}_j}\mathbf{V}_i^{f(j(\delta),k(-\delta))}\right)\\
&-\mathbb{E}_{(\mathbf{s}_t,...,\mathbf{s}_T)}\left(\sum_{i=1}^{N}\sum_{r=1,...,N}\lambda_r\frac{\partial R_{i,r}(\mathbf{V}^{f(j(\delta),k(-\delta))}(\widetilde{\mathbf{V}}_t,\sigma^*,\mathbf{s}_t,...,\mathbf{s}_T))}{\partial\mathbf{V}_k}\mathbf{V}_i^{f(j(\delta),k(-\delta))}\right)
\end{aligned}\tag{5}$$

Observe that the third and fourth terms in (5) are equal to zero. Indeed, for all $r \in \{1, ..., N\}$, we have:

$$\mathbb{E}_{(\mathbf{s}_t,...,\mathbf{s}_T)} \left(\sum_{i=1}^{N} \frac{\partial R_{i,r}(\mathbf{V}^{f(j(\delta),k(-\delta))}(\widetilde{\mathbf{V}}_t, \sigma^*, \mathbf{s}_t, ..., \mathbf{s}_T))}{\partial \mathbf{V}_j} \mathbf{V}_i^{f(j(\delta),k(-\delta))} \right) = 0 \qquad (6)$$

To see why (6) holds, note that $\sum_{i=1}^{N} R_{i,r}(\mathbf{V}) \equiv 1$ for all r and all $\mathbf{V}$, because some alternative is always ranked r-th. So, $\frac{\partial R_{j,r}(\mathbf{V}^{f(j(\delta),k(-\delta))}(\widetilde{\mathbf{V}}_t,\sigma^*,\mathbf{s}_t,...,\mathbf{s}_T))}{\partial \mathbf{V}_j} + \sum_{i' \in \{1,...,N\},\ i' \neq j} \frac{\partial R_{i',r}(\mathbf{V}^{f(j(\delta),k(-\delta))}(\widetilde{\mathbf{V}}_t,\sigma^*,\mathbf{s}_t,...,\mathbf{s}_T))}{\partial \mathbf{V}_j} = 0$. Furthermore, if $\frac{\partial R_{i',r}(\mathbf{V}^{f(j(\delta),k(-\delta))}(\widetilde{\mathbf{V}}_t,\sigma^*,\mathbf{s}_t,...,\mathbf{s}_T))}{\partial \mathbf{V}_j} \neq 0$ for some $i' \in \{1, ..., N\}$, $i' \neq j$ and some $(\widetilde{\mathbf{V}}_t, \sigma^*, \mathbf{s}_t, ..., \mathbf{s}_T)$, then i' and j must be the favorite (highest value) alternatives at the terminal period T, and so $V_{i'}(\widetilde{\mathbf{V}}_t, \sigma^*, \mathbf{s}_t, ..., \mathbf{s}_T)) = V_j(\widetilde{\mathbf{V}}_t, \sigma^*, \mathbf{s}_t, ..., \mathbf{s}_T)) + \delta$. This establishes that the third term in (5) is equal to zero. An identical argument establishes that the fourth term in (5) is also equal to zero.

Thus, we conclude that the derivative of (4) with respect to δ is equal to the first two terms of (5) which can be rearranged as follows:

$$\mathbb{E}_{(\mathbf{s}_t,...,\mathbf{s}_T)} \left(\sum_{r=1,...,N} (\lambda_r - \lambda_{r+1}) \sum_{m=1}^{r} R_{j,m}(\mathbf{V}^{f(j(\delta),k(-\delta))}(\widetilde{\mathbf{V}}_t, \sigma^*, \mathbf{s}_t, ..., \mathbf{s}_T)) \right) -$$
$$\mathbb{E}_{(\mathbf{s}_t,...,\mathbf{s}_T)} \left(\sum_{r=1,...,N} (\lambda_r - \lambda_{r+1}) \sum_{m=1}^{r} R_{k,m}(\mathbf{V}^{f(j(\delta),k(-\delta))}(\widetilde{\mathbf{V}}_t, \sigma^*, \mathbf{s}_t, ..., \mathbf{s}_T)) \right) \qquad (7)$$

where by convention we set $\lambda_{N+1} = 0$.

Since $\lambda_r - \lambda_{r+1} \geq 0$ for all $r \in \{1, ..., N\}$, in order to establish that the second derivative of (4) is increasing in δ and hence to complete the proof, it suffices to show that for all $r \in \{1, ..., N\}$,
$\mathbb{E}_{(\mathbf{s}_t,...,\mathbf{s}_T)} \sum_{m=1}^{r} R_{j,m}(\mathbf{V}^{f(j(\delta),k(-\delta))}(\widetilde{\mathbf{V}}_t, \sigma^*, \mathbf{s}_t, ..., \mathbf{s}_T))$ is increasing in δ and
$\mathbb{E}_{(\mathbf{s}_t,...,\mathbf{s}_T)} \sum_{m=1}^{r} R_{k,m}(\mathbf{V}^{f(j(\delta),k(-\delta))}(\widetilde{\mathbf{V}}_t, \sigma^*, \mathbf{s}_t, ..., \mathbf{s}_T))$ is decreasing in δ.

To see the former, let $\mathbf{V}_{(r)}^{f(j(\delta),k(-\delta))}$ denote the r-th highest entry in the vector of alternatives $\mathbf{V}^{f(j(\delta),k(-\delta))}$. Then,

$$\mathbb{E}_{(\mathbf{s}_t,...,\mathbf{s}_T)} \sum_{m=1}^{r} R_{j,m}(\mathbf{V}^{f(j(\delta),k(-\delta))}(\widetilde{\mathbf{V}}_t, \sigma^*, \mathbf{s}_t, ..., \mathbf{s}_T))$$
$$= Prob_{(\mathbf{s}_t,...,\mathbf{s}_T)}(\mathbf{V}_j^{f(j(\delta),k(-\delta))} \geq \mathbf{V}_{(r)}^{f(j(\delta),k(-\delta))} | \widetilde{\mathbf{V}}_t, \sigma^*, \delta) \qquad (8)$$

where $Prob_{(\mathbf{s}_t,...,\mathbf{s}_T)}(\mathbf{V}_j^{f(j(\delta),k(-\delta))} \geq \mathbf{V}_{(r)}^{f(j(\delta),k(-\delta))}|\widetilde{\mathbf{V}}_t,\sigma^*,\delta)$ is the probability that alternative j is at least the r-th highest in the vector $\mathbf{V}^{f(j(\delta),k(-\delta))}$. This probability is non-decreasing in δ because an increase in δ raises $\mathbf{V}_j^{f(j(\delta),k(-\delta))}$, the value of alternative j, lowers $\mathbf{V}_k^{f(j(\delta),k(-\delta))}$, the value of alternative k, and leaves the values of all other alternatives unchanged. By a similar argument,
$\mathbb{E}_{(\mathbf{s}_t,...,\mathbf{s}_T)} \sum_{m=1}^{r} R_{k,m}(\mathbf{V}^{f(j(\delta),k(-\delta))}(\widetilde{\mathbf{V}}_t,\sigma^*,\mathbf{s}_t,...,\mathbf{s}_T))$ is decreasing in δ. So, the value of (7) is increasing in δ.

Next, we will show that the alternative that receives all investment in any period t must be a favorite. That is, if alternative i' receives all investment in period t, then it must be that $i' \in \arg\max_{i\in\{1,...,N\}} V_{ti}$. The proof is by contradiction. So suppose not, i.e. the vector of optimal investments in period t, $\mathbf{b}_t^* = (b_{t1}^*,...,b_{tN}^*)$, is such that $b_{ti'}^* = B$, $b_{ti}^* = 0$ for $i \neq i'$ and $i' \notin \arg\max_{i\in\{1,...,N\}} V_{ti}$.

There are two cases to consider.

Case 1. $V_{tj'} - V_{ti'} < B$ for some $j' \in \arg\max_{i\in\{1,...,N\}} V_{ti}$.

In this case, we have $B > \widetilde{V}_{ti'} - \widetilde{V}_{tj'} = (V_{ti'} + b_{ti'}^*) - (V_{tj'} + b_{tj'}^*) > 0$. Since the optimal strategy σ^* is Markov and the shocks are identically distributed, the same expected payoff can be attained by making the following alternative investment decisions $\hat{\mathbf{b}}_t$ in period t: $\hat{b}_{ti'} = V_{tj'} - V_{ti'}$ and $\hat{b}_{tj'} = B - \hat{b}_{ti'}$, $\hat{b}_{ti} = 0$ for all $i \notin \{i', j'\}$. That is, we have $\widetilde{\Pi}(V_t + \mathbf{b}_t^*, \sigma^*) = \widetilde{\Pi}(V_t + \hat{\mathbf{b}}_t, \sigma^*)$. But the argument in the first part of the proof establishes that investment $\hat{\mathbf{b}}_t$ is strictly suboptimal, so $\mathbf{b}_t^*$ cannot be an optimal investment allocation either.

Case 2. $V_{tj'} - V_{ti'} \geq B$ for some $j' \in \arg\max_{i\in\{1,...,N\}} V_{ti}$. In this case, $\widetilde{V}_{ti'} - \widetilde{V}_{tj'} = (V_{ti'} + b_{ti'}^*) - (V_{tj'} + b_{tj'}^*) \leq 0$.

Then, pick some $\delta \in (0, B)$ and consider the following feasible allocation of investments in period t: in period t alternative j' receives an investment δ, and alternative i' receives investment $B - \delta$. Further, suppose that in all subsequent periods the decision-maker continues to use the same optimal strategy σ^* ignoring this reallocation of investment i.e. acting as if investments $\mathbf{b}_t^*$ have been made in period t.

Then, using the same computation as in Part 1 of the proof and differentiating with respect to δ, we conclude that the derivative of the expected payoff with respect to δ, at $\delta = 0$, is equal to (7) with $j = j'$ and $k = i'$. The result of Step 1 implies that the probability distribution of $V_{j'}$ first-order stochastically dominates the probability distribution of $V_{i'}$. So (7) has a strictly positive value for $j = j'$ and $k = i'$. That is, this reallocation of investment from i' to j' strictly increases the expected payoff to the decision-maker. Hence, $\mathbf{b}_t^*$ is suboptimal. We conclude that the optimal strategy requires allocating all investment to a favorite alternative in every period.

Proof of Proposition 2.

First, the same proof as in Lemma 1 can be used to show that there exists a Markov optimal strategy. So, let us focus on such strategies.

We proceed further arguing by contradiction as in the proof of Proposition 1. Let us use the same notation and the same sequence of steps as in Proposition 1 and only explain the steps that require a modification.

Specifically, suppose that an optimal strategy in period t prescribes positive investments b_{tj} and b_{tk} into alternatives j and k, and consider a small perturbation $\delta \in (-\min\{b_{tk}, b_{tj}\}, \min\{b_{tk}, b_{tj}\})$ reallocating investments between alternatives j and k on top of what is prescribed by the optimal strategy. This reallocation is made without changing the future allocation rule. Then the decision-maker's objective is given by the following counterpart of expression (4):

$$\begin{aligned}&\mathbb{E}_{(\mathbf{s}_t,\ldots,\mathbf{s}_T)} \sum_{i=1,\ldots,N, i\notin\{j,k\}} \mu(V_i(\tilde{\mathbf{V}}_t, \sigma^*, \mathbf{s}_t, \ldots, \mathbf{s}_T)) \\ &+\mathbb{E}_{(\mathbf{s}_t,\ldots,\mathbf{s}_T)} \Big(\mu(V_j(\tilde{\mathbf{V}}_t, \sigma^*, \mathbf{s}_t, \ldots, \mathbf{s}_T) + \delta) + \mu(V_k(\tilde{\mathbf{V}}_t, \sigma^*, \mathbf{s}_t, \ldots, \mathbf{s}_T) - \delta)\Big) \end{aligned} \tag{9}$$

The first-order derivative of (9) with respect to δ is equal to:

$$\mathbb{E}_{(\mathbf{s}_t,\ldots,\mathbf{s}_T)} \Big(\mu'(V_j(\tilde{\mathbf{V}}_t, \sigma^*, \mathbf{s}_t, \ldots, \mathbf{s}_T) + \delta) - \mu'(V_k(\tilde{\mathbf{V}}_t, \sigma^*, \mathbf{s}_t, \ldots, \mathbf{s}_T) - \delta)\Big) \tag{10}$$

while the second-order derivative of (9) is given by:

$$\mathbb{E}_{(\mathbf{s}_t,\ldots,\mathbf{s}_T)} \Big(\mu''(V_j(\tilde{\mathbf{V}}_t, \sigma^*, \mathbf{s}_t, \ldots, \mathbf{s}_T) + \delta) + \mu''(V_k(\tilde{\mathbf{V}}_t, \sigma^*, \mathbf{s}_t, \ldots, \mathbf{s}_T) - \delta)\Big) > 0 \tag{11}$$

Since (11) is positive, $\delta = 0$ cannot be an optimal choice. So, the decision-maker will never make positive investments in two different alternatives in any period t.

To show that an optimal strategy requires allocating all investment to a favorite alternative in every period, apply the same argument that establishes a similar assertion in Proposition 1, with the only difference that in the current case the proof has to refer to (10), instead of (7). *Q.E.D.*

Proof of Proposition 3

The proof of part (i) of the Proposition follows the same lines as its counterpart in Proposition 1. That it, suppose that in period t the optimal strategy prescribes positive investments b_{tj} and b_{tk} into alternatives j and k. On top of these investments, consider a small reallocation of investment $\delta \in (-\min\{b_{tk}, b_{tj}\}, \min\{b_{tk}, b_{tj}\})$ from alternative k to alternative j in period t, without changing either the total amount of investment in period t or the future allocation. That is, after this reallocation our strategy prescribes the same actions in periods $t+1$,...,T, as without this reallocation. Since $\delta = 0$ is optimal by assumption, the first-order condition (5) must still hold, and the second-order derivative of the objective, i.e. the derivative of (5) with respect to δ, must be nonpositive. Repeating the same steps as in the proof of Proposition 1, we establish a contradiction by showing that this second-order derivative is, in fact, strictly positive.

Let us now prove part (ii) of the Proposition. Suppose that i is the favorite alternative that receives all investment under the optimal strategy in period t. Then, we have:

$$\frac{\partial \widetilde{\Pi}(\widetilde{V}_{t1}...\widetilde{V}_{tN}, \sigma)}{\partial \widetilde{V}_{ti}} = \frac{dC(B_t)}{dB_t} \tag{12}$$

Differentiating the decision-maker's value function, we obtain:

$$\frac{\partial \widetilde{\Pi}(\widetilde{V}_{t1}...\widetilde{V}_{tN}, \sigma)}{\partial \widetilde{V}_{ti}} = \widetilde{P}_{ti}(\widetilde{\mathbf{V}}_t, \sigma) \tag{13}$$

where, as defined above, $\widetilde{P}_{ti}(\widetilde{\mathbf{V}}_t, \sigma)$ stands for the probability that alternative i wins the tournament conditional on period t information that includes $\widetilde{\mathbf{V}}_t$ and strategy σ. Combining equations (12) and (13) yields:

$$\widetilde{P}_{ti}(\widetilde{\mathbf{V}}_t, \sigma) = \frac{dC(B_t)}{dB_t} \tag{14}$$

Further, since in each period the decision-maker invests all resources into a favorite alternative, the probability that the favorite alternative i wins increases in its value, i.e.

$$\frac{d\widetilde{P}_{ti}(\widetilde{\mathbf{V}}_t, \sigma)}{d\widetilde{V}_{ti}} > 0 \tag{15}$$

Note that $C''(\cdot) > 0$. So, as the value of the favorite alternative $\widetilde{\mathbf{V}}_{it}$ increases, equation (14) will continue to hold only if B_t also increases in $\widetilde{\mathbf{V}}_{it}$. *Q.E.D.*

Proof of Proposition 4.

Applying the method used in the proof of Proposition 1, let $\hat{\sigma}$ be an equilibrium strategy. Recall that $\widetilde{\mathbf{V}}_t$ stands for the vector of values of extended alternatives at time period t, and b_{tj} stands for an equilibrium investment into alternative j at time period t. Further, $R_{i,r}(\widetilde{\mathbf{V}}_t, \hat{\sigma}, \mathbf{s_t}, ..., \mathbf{s_T})$ is the probability that alternative i has rank r (i.e. that the terminal value of alternative i, V_i, is the r-th highest among the terminal values of all N alternatives $(V_1, ..., V_N)$). The expected value of alternative j, given the information at time t, is equal to:

$$\mathbb{E}_{(\hat{\sigma},\mathbf{s}_t,...,\mathbf{s}_T)}\mu(V_j) = a\mathbb{E}_{(\hat{\sigma},\mathbf{s}_t,...,\mathbf{s}_T)}V_j - \frac{c}{2}\mathbb{E}_{(\hat{\sigma},\mathbf{s}_t,...,\mathbf{s}_T)}\left(V_j^2\right) \tag{16}$$

When $\lambda_1 = ... = \lambda_n = \bar{\lambda}$, the decision-maker's objective is to maximize:

$$\sum_{j=1,...,N}\left(a\bar{\lambda}\mathbb{E}_{(\hat{\sigma},\mathbf{s}_t,...,\mathbf{s}_T)}V_j - \frac{c}{2}\left(\mathbb{E}_{(\hat{\sigma},\mathbf{s}_t,...,\mathbf{s}_T)}V_j^2\right)\right) =$$
$$a\bar{\lambda}\left(\sum_{j=1,...,N}V_{tj} + (T-t)\times B\right) - \bar{\lambda}\frac{c}{2}\left(\mathbb{E}_{(\hat{\sigma},\mathbf{s}_t,...,\mathbf{s}_T)}\sum_{j=1,...,N}V_j^2\right)$$

Since the first term on the second line of the previous expression does not depend on the strategy $\hat{\sigma}$ from period $t+1$ on, consider only the second term. Recalling that s_{it} are i.i.d. across alternatives and periods, let $Var(s)$ stand for the variance of s_{it} for any i and t. Then we have:

$$\begin{aligned}&-\mathbb{E}_{(\hat{\sigma},\mathbf{s}_t,...,\mathbf{s}_T)}\sum_{j=1,...,N}V_j^2 = -\mathbb{E}_{(\hat{\sigma},\mathbf{s}_t,...,\mathbf{s}_T)}\left(\sum_{j=1,...,N}V_j\right)^2 + 2\sum_{i,j\in\{1,...,N\}\ i\neq j}\mathbb{E}_{(\hat{\sigma},\mathbf{s}_t,...,\mathbf{s}_T)}V_jV_i = \\ &-(T-t)\times N\times Var.(s) + \left(\sum_{j=1,...,N}V_{tj} + (T-t)\times B\right)^2 + 2\sum_{i,j\in\{1,...,N\}\ i\neq j}\mathbb{E}_{(\hat{\sigma},\mathbf{s}_t,...,\mathbf{s}_T)}V_jV_i\end{aligned} \tag{17}$$

The only term in (17) that depends on the decision-maker's strategy from $t+1$ onwards is

$$\sum_{i,j\in\{1,...,N\},\ i\neq j}\mathbb{E}_{(\hat{\sigma},\mathbf{s}_t,...,\mathbf{s}_T)}V_jV_i. \tag{18}$$

Thus, the optimal strategy $\hat{\sigma}$ should maximize (18).

Let us start from the last period T. The derivative of (18) with respect to b_{Ti} is equal to: $\mathbb{E}_{(\hat{\sigma},\mathbf{s}_T)} \sum_{j\in\{1,...,N\},\, i\neq j} V_j = \sum_{j\in\{1,...,N\}\; i\neq j} V_{Tj}+b_{Tj}$. Using this derivative, we conclude that $b_{Ti} > 0$ and $b_{Tj} > 0$ if and only if the following first-order condition holds:

$$V_{Tj} + b_{Tj} = V_{Ti} + b_{Ti} \tag{19}$$

However, if (19) cannot hold for all pairs of alternatives (i, j) i.e. if $V_{Tj} > \sum_{i':i'\neq j,\, V_{Ti'}<V_{Tj}} \frac{V_{Ti'}+B}{\#\{i':i'\neq j,\, V_{Ti'}<V_{Tj}\}}$, then it is optimal to set $b_{Tj} = 0$.

This implies that the optimal strategy in period T it to allocate the budget B to attain the following objective:

$$\max \min_{i'\in\{1,...,N\}} V_{Ti'} + b_{Ti'}.$$

To show that the same strategy is optimal in any period $t \in \{1, ..., T-1\}$, proceed by induction and suppose that such strategy is optimal in all periods starting from some $t + 1$. Then, $\mathbb{E}_{(\hat{\sigma},\mathbf{s}_{t+1},...,\mathbf{s}_T)}V_{(t+1)i} > (=)\mathbb{E}_{(\hat{\sigma},\mathbf{s}_{t+1},...,\mathbf{s}_T)}V_{(t+1)j}$, implies that

$$\mathbb{E}_{(\hat{\sigma},\mathbf{s}_{t+1},...,\mathbf{s}_T)}V_i > (=)\mathbb{E}_{(\hat{\sigma},\mathbf{s}_{t+1},...,\mathbf{s}_T)}V_j.$$

Therefore, given the strategy $\hat{\sigma}$ and the fact that s_{ti} an s_{tj} are distributed identically and independently, $V_{ti}+b_{ti} > (=)V_{tj}+b_{tj}$ implies that $\mathbb{E}_{(\hat{\sigma},\mathbf{s}_{t+1},...,\mathbf{s}_T)}V_{(t+1)i} > (=)\mathbb{E}_{(\hat{\sigma},\mathbf{s}_{t+1},...,\mathbf{s}_T)}V_{(t+1)j}$.

We will use this property to establish the optimality of allocating investments in period t to maximize

$$\min_{i'\in\{1,...,N\}} V_{ti'} + b_{ti'}. \tag{20}$$

To show this, suppose that it is optimal to make strictly positive investments into alternatives i and j in period t. As in the proof of Proposition 1 consider a small reallocation of investment $\delta \geq 0$ from j to i in period t. Recall that this reallocation is done in such a way that the investment strategy in the subsequent periods remains unchanged. Then, we have $\frac{\partial V_i}{\partial \delta} = 1 = -\frac{\partial V_j}{\partial \delta}$. Since b_{ti} and b_{tj} are strictly positive, the following first-order condition (obtained by differentiating (18)) has to hold.

$$\sum_{i',j'\in\{1,...,N\},\, i'\neq j'} \frac{\partial \mathbb{E}_{(\hat{\sigma},\mathbf{s}_t,...,\mathbf{s}_T)} V_{j'}V_{i'}}{\partial \delta}_{|\delta=0} = \mathbb{E}_{(\hat{\sigma},\mathbf{s}_t,...,\mathbf{s}_T)}V_j - \mathbb{E}_{(\hat{\sigma},\mathbf{s}_t,...,\mathbf{s}_T)}V_i = 0 \tag{21}$$

But as we have shown before, the last equality in (21)) holds only if $V_{ti}+b_{ti} = V_{tj} + b_{tj}$.

On the other hand, if the value of (21) is positive, i.e. if $V_{ti} + b_{ti} < V_{tj} + b_{tj}$, then it is optimal to set $\delta > 0$, and hence b_{tj} must be equal to zero. Alternatively, if the value of (21) is negative, i.e. if $V_{ti} + b_{ti} > V_{tj} + b_{tj}$, it is optimal to set $\delta < 0$, and hence b_{ti} must be equal to zero. So, a positive investment is always made into the lowest value alternative at t and, when two alternatives i and j receive positive investments, then $V_{ti}+b_{ti} = V_{tj}+b_{tj}$. Consequently, the unique optimal strategy is to maximize (20).

Part (ii). Now suppose that $\lambda_1 > 0$ and $\lambda_2 = ... = \lambda_n = 0$. The proof of this part follows the proof of Proposition 1. So suppose that the equilibrium strategy $\hat{\sigma}$ prescribes positive investments b_{tj} and b_{tk} into alternatives j and k in period t. Let $\delta \in (-\min\{b_{tk}, b_{tj}\}, \min\{b_{tk}, b_{tj}\})$ be a small reallocation of investment from alternative j into alternative k on top of what is prescribed by $\hat{\sigma}$. This reallocation is done without changing the future allocation rule. That is, after this reallocation the strategy $\hat{\sigma}$ prescribes the same actions in periods $t+1$,...,T as without it.

Recall that $R_{j,1}(.)$ is the rank function equal to 1 if alternative j is the winner at the terminal node, and equal to zero otherwise. The vector of perturbed terminal values $\mathbf{V}^{f(j(\delta),k(-\delta))}(\widetilde{\mathbf{V}}_t, \sigma^*, \mathbf{s}_t, ..., \mathbf{s}_T)$ is defined in (3). All entries of the vector $\mathbf{V}^{f(j(\delta),k(-\delta))}(\widetilde{\mathbf{V}}_t, \sigma^*, \mathbf{s}_t, ..., \mathbf{s}_T)$, except the j-th and k-th, are the same as in the vector $\mathbf{V}(\widetilde{\mathbf{V}}_t, \sigma^*, \mathbf{s}_t, ..., \mathbf{s}_T)$, while the j-th (k-th) entry of $\mathbf{V}^{f(j(\delta),k(-\delta))}(\widetilde{\mathbf{V}}_t, \sigma^*, \mathbf{s}_t, ..., \mathbf{s}_T)$ is equal to the j-th (k-th) entry of $\mathbf{V}(\widetilde{\mathbf{V}}_t, \sigma^*, \mathbf{s}_t, ..., \mathbf{s}_T)$ plus (minus) δ. Dropping the argument $(\widetilde{\mathbf{V}}_t, \hat{\sigma}, \mathbf{s}_t, ..., \mathbf{s}_T))$ of $\mathbf{V}^{f(j(\delta),k(-\delta))}$ and $V_i(.)$, for any $\delta \in (-\min\{b_{tk}, b_{tj}\}, \min\{b_{tk}, b_{tj}\})$, the decision-maker's expected value function in period t is equal to:

$$
\begin{aligned}
&\mathbb{E}_{(\mathbf{s}_t,...,\mathbf{s}_T)} \sum_{i=1,...,N, i\notin\{j,k\}} \lambda_1 R_{i,1}(\mathbf{V}^{f(j(\delta),k(-\delta))} \left(aV_i - \frac{c}{2}V_i^2 \right) \\
&+\mathbb{E}_{(\mathbf{s}_t,...,\mathbf{s}_T)} \lambda_1 R_{j,1}(\mathbf{V}^{f(j(\delta),k(-\delta))})(a(V_j+\delta) - \frac{c}{2}(V_j+\delta)^2) \\
&+\mathbb{E}_{(\mathbf{s}_t,...,\mathbf{s}_T)} \lambda_1 R_{k,r}(\mathbf{V}^{f(j(\delta),k(-\delta))})(a(V_k-\delta) - \frac{c}{2}(V_k-\delta)^2) \qquad (22)
\end{aligned}
$$

Consider (22) as a function of δ. Since $b_{tj} > 0$ and $b_{tk} > 0$ are optimal investments, $\delta = 0$ is an interior optimum of (22). Therefore, the first derivative of (22) with respect to δ must be equal to zero at $\delta = 0$ while its second derivative must be nonnegative. In the rest of the proof, we show that this is not the case, thereby establishing a contradiction with our original hypothesis that $b_{tj} > 0$ and $b_{tk} > 0$. The first derivative of (22) with respect to δ

can be written as:

$$
\begin{aligned}
&\mathbb{E}_{(\mathbf{s}_t,\ldots,\mathbf{s}_T)}\lambda_1\left(R_{j,1}(\mathbf{V}^{f(j(\delta),k(-\delta))})\,(a-c(V_j+\delta))-R_{k,1}(\mathbf{V}^{f(j(\delta),k(-\delta))})\,(a-c(V_k-\delta))\right)\\
&+\mathbb{E}_{(\mathbf{s}_t,\ldots,\mathbf{s}_T)}\left(\sum_{i=1}^{N}\lambda_1\frac{\partial R_{i,1}(\mathbf{V}^{f(j(\delta),k(-\delta))})}{\partial \mathbf{V}_j}\left(aV_i-\frac{c}{2}V_i^2\right)\right)\\
&\quad-\mathbb{E}_{(\mathbf{s}_t,\ldots,\mathbf{s}_T)}\left(\sum_{i=1}^{N}\lambda_1\frac{\partial R_{i,1}(\mathbf{V}^{f(j(\delta),k(-\delta))})}{\partial V_k}\left(aV_i-\frac{c}{2}V_i^2\right)\right)
\end{aligned}
\tag{23}
$$

The second and the third lines in (23) are equal to zero. To see this note that $\sum_{i=1}^{N} R_{i,1}(\mathbf{V}) \equiv 1$ for all $\mathbf{V}$, because some alternative is always ranked 1-st. So, $\sum_{i\in\{1,\ldots,N\}}\frac{\partial R_{i,1}(\mathbf{V}^{f(j(\delta),k(-\delta))})}{\partial \mathbf{V}_j}=0$. Furthermore, if $\frac{\partial R_{i,1}(\mathbf{V}^{f(j(\delta),k(-\delta)})}{\partial \mathbf{V}_j}\neq 0$ for some $i\in\{1,\ldots,N\}$, $i\neq j$, then i and j must be the favorite (highest value) alternatives at the terminal period T and so $V_i=V_j+\ \delta$. This establishes that the third term in (23) is equal to zero. An identical argument establishes that the fourth term in (23) is also equal to zero.

Thus, we conclude that the derivative of (22) with respect to δ is equal to:

$$
\mathbb{E}_{(\mathbf{s}_t,\ldots,\mathbf{s}_T)}\lambda_1\left(R_{j,1}(\mathbf{V}^{f(j(\delta),k(-\delta))})\,(a-c(V_j+\delta))-R_{k,1}(\mathbf{V}^{f(j(\delta),k(-\delta))})\,(a-c(V_k-\delta))\right)
\tag{24}
$$

The second derivative of (24) is obtained by differentiating (24) with respect to δ. It is equal to:

$$
\begin{aligned}
&\mathbb{E}_{(\mathbf{s}_t,\ldots,\mathbf{s}_T)}-\lambda_1 c\left(R_{j,1}(\mathbf{V}^{f(j(\delta),k(-\delta))})+R_{k,1}(\mathbf{V}^{f(j(\delta),k(-\delta))})\right)\\
&+\lambda_1\left(\frac{\partial R_{j,1}(\mathbf{V}^{f(j(\delta),k(-\delta))})}{\partial\delta}\,(a-c(V_j+\delta))\right)-\left(\frac{\partial R_{k,1}(\mathbf{V}^{f(j(\delta),k(-\delta))})}{\partial\delta}\,(a-c(V_k-\delta))\right)
\end{aligned}
\tag{25}
$$

Note that $ER_{j,1}(\mathbf{V}^{f(j(\delta),k(-\delta))})\leq 1$ and $ER_{k,1}(\mathbf{V}^{f(j(\delta),k(-\delta))})\leq 1$ by definition. Also, $ER_{j,1}(\mathbf{V}^{f(j(\delta),k(-\delta))})>0$ and $ER_{k,1}(\mathbf{V}^{f(j(\delta),k(-\delta))})>0$, for otherwise it cannot be optimal to set $b_{jt}>0$ and $b_{jk}>0$. Note that $\partial R_{j,1}(\mathbf{V}^{f(j(\delta),k(-\delta))})$ and $\partial R_{k,1}(\mathbf{V}^{f(j(\delta),k(-\delta))})$ are concave since the probability distribution of shocks $\mathbf{s}$ are bounded. Concavity and boundedness from above and below imply that $\frac{\partial R_{j,1}(\mathbf{V}^{f(j(\delta),k(-\delta))})}{\partial\delta}-\frac{\partial R_{k,1}(\mathbf{V}^{f(j(\delta),k(-\delta))})}{\partial\delta}\geq\frac{1}{n}$. So, when $\frac{a}{c}>n((B+\bar{s})T+1)$, the value of (25) is positive. Hence it is not optimal to invest positive amounts in two alternatives. An argument similar to the one in Proposition 1 then establishes that in every period all budget should be invested into a favorite alternative.

Proof of Proposition 5.

First, note that a symmetric Nash equilibrium exists. Indeed, if the decision-maker's strategy is symmetric across the contenders, then all contenders have the same beliefs regarding the decision-maker's investments and random shocks at every period. Therefore, every contender has the same best-response investment function. Further, given that each contender uses the same investment function, it is indeed optimal for the decision-maker to use a strategy which is symmetric across the contenders. The fixed point of these best response functions exists by standard argument and constitutes a symmetric Nash equilibrium.

Next, let $(e_1^*, ..., e_T^*)$ be the equilibrium effort investment profile for each of the contenders. Given this profile and given the distribution of shocks, the decision-maker's problem is exactly the same as in the benchmark model studied in Section 2. The only modification here is that, independently of the decision-maker's investment, the value of contender i now changes by the amount $e_t^* + s_{ti}$ in each period rather than by s_{ti}. So part (i) - the optimality of investing all resources in one contender in each period- follows directly from Proposition 1.

To establish part (ii), again consider the optimal effort profile $(e_1^*, ..., e_T^*)$ of a contender. Suppose that $e_t^* \geq e_{t'}^*$ for some time-periods t and t' s.t. $t > t'$. Then, consider some contender i. Recall that the decision-maker puts all the resources into a favorite alternative in each period. Therefore, by switching effort levels in periods t and t', contender i will keep her overall costs constant. At the same time, this modification raises the probability that i receives the decision-maker's investment in period t' and hence in all later periods. So, this deviation is strictly profitable for contender i. *Q.E.D.*