# An Ascending Double Auction

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**Abstract**

We show why the failure of the affiliation assumption prevents the double auction from achieving efficient outcomes when values are interdependent. This motivates the study of an ascending price version of the double auction. It is shown that when there is a sufficiently large, but still finite, number of sellers, this mechanism has an *approximate* perfect Bayesian equilibrium in which traders continue bidding if and only if their true estimates of the 'value' of the object being traded exceed the current price. This equilibrium is ex post efficient and has a *rational expectations property* in the sense that along the equilibrium path traders appear to have made the best possible trades conditional on information revealed by the trading process.

## 1 Introduction

Double auctions have been shown to work very effectively in markets with many buyers and sellers when traders have private values. Rustichini, Satterthwaite, and Williams (1994) have shown that in independent private value environments, an equilibrium of a standard (one-shot) double auction quickly converges to an efficient allocation as the number of traders get large (see also Gresik and Satterthwaite (1989)). More recently, Cripps and Swinkels (2006) have provided similar results for correlated value environments.

Less in known about the properties of double auctions in large interdependent value environments. Interdependence may be important in large markets for a

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number of reasons. For example, auctions of residential condominiums or similar housing units involve interdependence because potential buyers typically care about the types of their future neighbors. Notably, for large condominium developments, several similar units are typically on the market at any given time. Auctions for vacation time shares are common and might have similar interdependencies of valuations. In large procurement auctions interdependencies are often generated endogenously if winning bidders ultimately contract out work to losing bidders. The reason is that strong (or low cost) losing bidders may make it cheaper for the winning bidder to execute the project by contracting out some parts of it. Similarly, in durable goods markets, resale opportunities may be affected not only by information possessed by other traders about the value of future trade, but also because existing traders may be competitors or customers in future resale. In financial markets, traders have diverse information about the underlying value of the securities being traded. Each trader's information potentially affects the value of every other trader.

Perry and Reny (2006), continuing the research agenda of Milgrom (1981) and Pesendorfer and Swinkels (1997) and (2000), have shown that, in some interdependent value environments where trader types are affiliated, the double auction supports equilibrium prices that converge in probability to the full information market-clearing price as the number of traders gets large. In particular, this implies that, when the number of traders is large, the double auction realizes almost all gains from trade with high probability and provides a strategic foundation for *rational expectations equilibrium.*

It is quite surprising that the static double auction (where each trader submits only one bid or ask) supports anything close to the efficient competitive allocation, because this mechanism imposes strong restrictions on the way in which individual trading decisions depend on other traders' types. To see this, observe that a bid or an ask in a double auction is, in fact, a contingent trading plan. For example, a seller's ask is a plan saying that, in an auction with $m$ buyers, the seller wants to sell if the $m$-th lowest value in the ordered list of all traders' bids and asks is above his ask price. This is a very restricted contingent plan since the trading decision can only depend on the realization of the $m$-th lowest order statistic of the bids and asks, but not on the distribution of bids and asks below this order statistic. Furthermore, a seller has to sell *whenever* the value of this order statistic exceeds his ask price. These restrictions on the traders' contingent trading plans are very natural in the private value case, but are very restrictive in the interdependent value case, as we illustrate below.

Notably, the convergence results of Perry and Reny (2006) rely on an affiliation assumption. This assumption is common in auction theory. Nonetheless, it is a restriction.[1] We show by example that, when the affiliation assumption does not

[1]Lauermann and Merzyn (2006) have recently shown that in the case where values are con-

hold, then the restrictiveness of the feasible trading strategies in a double auction, in particular, the traders' inability to condition more finely on the information revealed in the course of bidding, leads to a failure of efficiency of double auctions.[2]

This motivates us to study an ascending-price version of the double auction. We demonstrate that this mechanism supports allocations that become ex post efficient as the number of traders gets large in an interdependent value environment without the affiliation assumption.

Our ascending-price double auction mechanism works as follows. Initially, the price is set sufficiently low that all buyers would be willing to buy at that price and no seller would strictly prefer to sell. At this initial price, all traders simultaneously declare whether they wish to continue bidding. All traders who wish to continue bidding are said to be active at the initial price. Traders who declare that they do not want to continue bidding because they would not want to pay a strictly higher price or because the price is already high enough that they are willing to sell, are considered to be *inactive* bidders who have dropped out of the bidding at the initial price. If the number of active bidders exceeds the number of units for sale, the price is increased according to the procedure that will be described below, and the process is repeated. A trader can become *inactive* at any price and her/his decision to do so is publicly observable. Once the number of active traders is less than or equal to the number of units of output for sale, the auction ends. Each buyer who is active at the final price pays that price and receives a unit of the good.[3] Sellers who are inactive when the auction ends trade and are paid the price at which the auction has ended. Sellers who are active at the final price leave the auction without trading.

We show that this mechanism has an *approximate* perfect Bayesian equilibrium in which traders remain active only so long as their values conditional on the information made public by the bidding process exceed the current price. Since traders' who drop out of the bidding reveal their types in this equilibrium, other traders can condition their bidding decisions much more finely on the distribution of types of the others. In particular, since the bidding process reveals the types of the traders with the lowest values, it ensures that the traders with the highest values end up with the good even though their types are not fully revealed.

The ability to condition the decision whether to remain active on the dropout decisions of other traders is good for efficiency reasons, but bad for strategic ones. This is why the strategies we describe constitute only an approximate perfect

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ditionally independent, affiliation between the state and the highest order statistic of traders' values cannot hold if the number of bidders is random and distributed independently of the state.

[2]Gresik (1991) has shown that it is impossible to achieve ex-post efficiency using static mechanisms in the interdependent value environment with valuation functions linear in traders' types and independent type distributions.

[3]Some buyers who drop out at the final price may also receive units of the good if too many traders drop out of the bidding at the final price. We discuss this in more detail below.

Bayesian equilibrium. In particular, sellers may encounter information sets where they can exploit the possibility that they may be pivotal without taking any significant chance of losing a profitable trade. We describe these information sets, then provide conditions under which the expected payoff associated with pushing up the trading price becomes arbitrarily small when the number of traders is large.

Our approximate equilibrium supports an outcome which is ex post efficient with probability 1. This allows us to address some questions that are left open in the existing literature. For example, Perry and Reny (2006) show that, under affiliation, the equilibrium price in a double auction converges in probability to the full information market clearing price. They do not directly address what happens to the equilibrium allocation, or to the posterior beliefs (both of which are important in the rational expectations story that they are interested in) when the number of traders is large but finite. Furthermore, convergence in probability also leaves open the possibility that unusual arrays of types (i.e. the ones that occur with a small probability) lead to prices that are far from full information prices. However, in the ascending-price double auction these issues do not arise.

As we illustrate below by an example, in the absence of affiliation an indirect mechanism can support an outcome close to a full information or a rational expectations equilibrium only if traders are given an opportunity to respond to the same price differently in different situations. Our mechanism accomplishes this because traders, including sellers, make trading decisions after they have observed some of the decisions of other traders. This feature of our mechanism allows traders to react in a flexible way to information about the types of the other traders.

At the same time, when the number of traders gets large, traders (almost) lose their ability to manipulate the trading price. As a result, our equilibrium has a 'rational expectations' property. Specifically, the trading decision for each trader is the best outcome that is feasible given the equilibrium trading price, and given all the information that is revealed along the equilibrium path associated with the trading process. Traders appear to be making optimal choices in all situations conditional on information revealed by the final equilibrium trading price and their own equilibrium outcome.

It is worth noting that models where traders observe the actions of other traders often provide negative results regarding convergence to equilibrium. The best example might be Wolinsky (1988) where traders are given repeated opportunities to observe others' behavior. Wolinsky shows that this will prevent trade from occurring at the right price even when there are many traders with almost costless opportunities to interact.

A similar result in a different setting is provided by Horner and Jamison (2004) who analyze an infinitely repeated sequence of auctions in which bidders have private (but unchanging) information about the common value of a good being sold. Bidders are repeatedly given the opportunity to observe the bids being made by others, and could potentially use this information to learn the common value. They give examples of equilibria in which no private information is *ever* revealed. Gottardi and Serrano (2005) analyze a series of models somewhat similar in structure to the one analyzed in Wolinsky (1988) and show that aggregation failures are closely related to the traders' market power. Gottardi and Serrano (2005) point out that traders know their actions are being observed, and this provides them with an additional opportunity to manipulate the outcome of the mechanism in their favor. Their behavior becomes less informative as a result.

The ability to directly manipulate others' beliefs is a key reason for the failure of convergence discovered by Wolinsky (1988) and the other authors cited above. One of the advantages of our mechanism is that the impact of such behavior becomes small in a large market.

Our paper also contributes to the design of ascending-price auctions and understanding of their incentive properties. One-sided ascending-price (English) auctions have been studied by Krishna (2003), Izmalkov (2003), Birulin and Izmalkov (2003) and others. Ausubel (2004) constructs an ascending-bid auction for multiple items. In a context considerably different from ours, Ausubel points out that an ascending-bid auction may retain efficiency with interdependent values, while a static one-shot auction does not. This result relates his paper to ours, to the extent that they both point at the advantages of dynamic ascending-price auctions.

The rest of the paper is organized as follows. In section 2 we show why double auctions cannot achieve efficiency without affiliation. In section 3 we present our model. Section 4 contains our main result. In section 5 we discuss the implications. Section 6 concludes. All proofs are relegated to the Appendix.

# 2 Double Auctions

First, let us consider why double auctions are restrictive. Suppose that traders' bids and asks are all monotonically related to their 'types'. Then any ask price announced by a seller is equivalent to a contingent plan according to which this seller agrees to trade if the value of the appropriate order statistic -the $m$-th lowest value among $m$ bids and $n$ asks ordered (in increasing order) in a single array- is above this seller's ask price. Suppose that this order statistic is equal to $p$ for some array of types, and it is ex-post efficient for our seller to trade. By the rules of the double auction, the seller will end up trading under any array of other trader types which gives rise to a higher value of this order statistic. So if the double auction supports an efficient outcome, it must be efficient for this seller to trade when this order statistic has any value higher than $p$. An increase in the value of this order statistic will increase the price at which the seller trades, which certainly makes the seller more willing to trade. However, a higher value of this order statistic also signals that other traders have higher private types. Under standard interdependence assumptions this will mean that the seller will also assign a higher value to the good, so her opportunity cost of trade will be higher. Some condition needs to hold to ensure that the former effect outweighs the latter.

The condition that does this in Perry and Reny (2006) is affiliation. In their formulation, a trader's value for the good depends on her own type and on some common quality $q$. Quality $q$ is unknown and random, and the traders' types are distributed identically and independently conditional on $q$. Let $F_q$ denote the probability distribution from which each trader's type is drawn conditional on $q$. When the number of traders is very large (infinite), the equilibrium price in the double auction coincides with the full information market clearing price, and the latter reveals the actual quality.

Now fix a quality $q$, and let $p_q$ be the corresponding full information price. Consider a buyer whose type $x_q$ is such that he bids $p_q$ in the double auction. Since the double auction price is equal to the full information market clearing price, the outcome should be ex post efficient. So the buyer of type $x_q$ must be just indifferent between trading and not trading at price $p_q$, and the measure of the set of traders whose types are higher than $x_q$ (who get the good in an ex-post efficient outcome) must be exactly equal to the measure of the set of available goods.

Next, consider a lower 'quality' $q'$, i.e. $q' < q$. The full information price $p_{q'}$ corresponding to quality $q'$, and hence the equilibrium price in the double auction, must be lower than $p_q$. But the buyer of type $x_q$ bids $p_q$ irrespective of the actual quality, which she is uncertain of. So he will win a unit of output at the new price $p_{q'}$.

Hence, to maintain ex post efficiency, the reduction in price from $p_q$ to $p_{q'}$ in the double auction has to at least compensate the buyer of type $x_q$ for the reduction in the quality of the good from $q$ to $q'$. This is ensured by the affiliation assumption. To understand why this is so, note that under affiliation the distribution of types conditional on $q$ first-order stochastically dominates the distribution of types conditional on $q'$. Therefore, the measure of the set of traders whose types are above $x_q$ is larger when the quality is $q$ than it is when the quality is $q'$. So when the quality is $q'$, the equilibrium price in the double auction must fall to ensure that some buyers with valuations below $x_q$ bid above this equilibrium price and end up with the good. Otherwise, the market would not clear, as the set of buyers who want to purchase the good at the final price would be smaller than the set of sellers who would like to sell at that price. Precisely, under $q'$ the price would fall to the level $p_{q'}$ at which some type $x_{q'}$, $x_{q'} < x_q$, will be indifferent between trading and not trading the good of quality $q'$. Hence, type $x_q$ would strictly prefer to buy the good of quality $q'$ at $p_{q'}$.

Generally speaking, a double auction works well if an increase in quality causes the $m^{th}$-lowest order statistic of traders' types to rise faster than traders' full information values. What follows is a finite example that shows how things can go wrong. This example will also be used below to illustrate how the ascending auction procedure resolves this problem.

There are 6 traders. Only one of them is a seller, so this is a simple auction environment except for the fact that the seller is strategic and privately informed. Types are commonly known to be integers between 3 and 10. The ex post value function of trader $i$ is given by

$$u(x_i, x_{-i}) = x_i + W\left[\frac{\sum_{j\neq i} x_j}{5}\right] \tag{1}$$

where $W[x]$ denotes the 'whole part' function, i.e. the largest integer that is less than or equal to $x$.

The lowest value that a trader can have in this environment is 6 (when all traders have type 3), and the highest value is 20 which occurs when all traders have type 10. Of particular interest are two states of the world corresponding to the following two type profiles:

$$\text{State 1:} \quad \{9, 8, 8, 8, 8, 8\}$$

and

$$\text{State 2:} \quad \{9, 10, 10, 3, 3, 3\}.$$

In each of these two states, it is the seller who has type 9. The profiles of traders' values in these two states are given by

$$\{17, 16, 16, 16, 16, 16\}$$

and

$$\{14, 15, 15, 8, 8, 8\}$$

respectively.

In state 1 all the buyers have high types, which raises the seller's value because of interdependence. In state 2, only the seller and the first two buyers have high types, while the other buyers have the lowest possible types. The full information market-clearing price can be anything between 16 and 17 in state 1, but must be equal to 15 in state 2. Importantly, for the outcome of the auction mechanism to be efficient, the seller of type 9 would need to sell in state 2, but keep the good in state 1. That is, she needs to sell the good if price does not rise above 15, and to keep it if a *higher* price of 16 is reached. Thus, this example has the plausible but unusual (at least in auction theory) property that the seller's 'supply' of the good is inversely related to price.

Now suppose that the joint distribution of types is such that any trader of type 8 believes that with a very high probability the true state is a permutation of state

1 i.e., 4 of 5 other traders have type 8 while one trader has type 9. Similarly, a trader with type 3 or 10 believes that, with a very high probability, the true state is a permutation of state 2. Finally, any trader of type 9 believes that with very high probability, the true state is either a permutation of state 1 or a permutation of state 2, and that each of the two configurations is equally likely. Types are distinctly not affiliated in this example. A trader whose type rises from 3 to 8 believes that with a very high probability the types of some of the other traders will rise, while some of them will fall.

The double auction cannot support an efficient outcome in both states 1 and 2. A trader of type 8 believes that her/his value is very close to 16, and is almost sure that there is another bidder who has this same value and belief. Therefore, in any equilibrium, a trader of type 8 must bid close to 16 with a high probability. A trader of type 10 is in a similar position. He believes that his value is very close to 15 and is almost sure that there is another trader with the same value and beliefs. As a consequence, such trader's bid must be very close to 15 with high probability. This argument implies that traders' bids will not be monotonically increasing in their types as would be necessary for ex post efficiency.

In particular, the seller will submit the same bid in both states. He could submit a bid above 16 and keep the good in both states. Alternatively, he could bid below 15 and sell in both states. Each of these two strategies would produce ex post inefficiency in one of the two states. Finally, if the seller submits a bid between 15 and 16 he will sell in the wrong state.

The failure of double auction in this example stems from the fact that a seller has to make a bid and a trading decision completely independently of the realized profile of types of the other traders. This problem is mitigated in an ascending double auction.

# 3 The Model

## 3.1 Fundamentals

There are $n$ sellers and $m$ buyers trading in a market. Each seller has one unit of a homogeneous good, while each buyer has an inelastic demand for one unit of this good. Trader $i$'s privately known type, $i \in \{1, ..., m+n\}$, is denoted by $x_i$. We assume that $x_i$ lies in a compact subset $\mathcal{X} \subset \mathbb{R}$. Below we will restrict the set of feasible types $\mathcal{X}$ to be finite. The profile of types of all $m+n$ traders is denoted by $x$ and the profile of traders' other than $i$ is denoted by $x_{-i}$. Thus, $x \in \mathcal{X}^{m+n} \subset \mathbb{R}^{m+n}$. When focussing on trader $i$, we use the notation $(x_i, x_{-i})$ for $x$. We use $x_{(k)}$ to denote the $k^{th}$ lowest element of $x$ from the bottom (the value of the $k^{th}$ lowest order statistic in $x$) for some integer $k \geq 0$, $k \leq m+n$. Similarly $x_{-i_{(k)}}$ denotes the $k^{th}$ lowest element in the vector $x_{-i}$. We also use the notation

$\tilde{x}_i$, $\tilde{x}$, $\tilde{x}_{-i}$ for trader $i$'s type, the type profile of all $m+n$ traders and the profile of types of traders other than $i$, respectively, when viewed as random variables.

Trader $i$'s valuation for the good is given by $u(x_i, x_{-i})$. The value function $u(x_i, x_{-i})$ is assumed to be continuous in $x_i$ and non-decreasing in each of its arguments. Note that *all* traders have the same value function, with the first argument of the function denoting the type of the trader her/himself, and the second argument standing for the profile of the other traders' types.

One implication of this assumption is that the value function $u(.)$ depends on the total number of traders $m+n$, but not on the proportion of sellers or buyers among the traders. So a trader's value $u(x_i, x_{-i})$ is the same irrespectively of whether all the other traders are sellers, or there is only one seller among other traders, as long as the size of the market remains constant.

In our notation we suppress the dependence of the value function $u(.)$ on $m+n$, because this dependence should be clear from the arguments of the function.

The value function in Perry and Reny (2006) is a special case of this. They assume that the full information value of trader $i$ is given by $v(x_i, q)$ where $x_i$ is the trader's own type, and $q$ is the unobservable quality of the good being traded. This can be supported as a special case of our formulation by setting

$$u(x_i, x_{-i}) = \mathbb{E}[v(x_i, q)|(x_i, x_i)]$$

as long as $v$ is increasing in both its arguments, and $x_j$ and $q$ are affiliated for $j \in \{1, ..., m+n\}$.[4] When taking expectations, we use the subscript of the expectation operator to indicate the type whose beliefs are being used to calculate the expectation, and the conditioning operator to indicate additional information that this trader type uses when doing the calculation. The variables over which the expectation is being calculated should be clear from the context.

A buyer's payoff is equal to her value less the price that she pays for the good. A buyer gets zero payoff if she does not buy and pays nothing. A seller's payoff is equal to the price that she receives less her value. A seller gets zero payoff if she does not sell and receives nothing. Recall that the first argument of the utility function always denotes the trader's own type. Also, let $x_{-i-j}$ denote the profile of types of traders other than $i$ and $j$. We make the following assumption throughout:

**Assumption 1** *(Single Crossing Condition) If $x_i > x_j$ (where $x_j$ is the $j^{th}$ component of $x_{-i}$), then $u(x_i, x_j, x_{-i-j}) \geq u(x_j, x_i, x_{-i-j})$.*

This assumption implies that, starting from any profile of types in which two traders have the same types, an increase in the type of one of these traders has more impact on this trader's value than the same increase in the other trader's

[4] Here, the expectation is taken using posterior beliefs about $q$ conditional on the types of all traders. Affiliation between $x_j$, for all $j$, and $q$ is needed to ensure that the expectation is increasing in the type of every trader.

type. In Perry and Reny (2006) this restriction holds because an increase in $x_i$ improves $i$'s perception of the quality of the good $q$ just as much as an increase in $x_j$ does, but an increase in $x_i$ also improves $i$'s valuation of every quality.

Assumption 1, along with the assumption that all traders have the same value function, implies that the trader with the highest type also has the highest value.

The value function is also assumed to possess the following continuity property:

**Assumption 2** *For any $\triangle > 0$, there is $N$ such that if $n + m \geq N$, then*

$$\left| u\left(x_i, x_j, x_{-i-j}\right) - u\left(x_i, x'_j, x_{-i-j}\right) \right| < \triangle$$

*for every pair $x_j, x'_j \in \mathcal{X}$, for every $x_i \in \mathcal{X}$ and $x_{-i-j} \in \mathcal{X}^{n+m-2}$.*

This Assumption ensures that when the number of traders is large, the influence of any single trader's type on any other trader's valuation becomes small. However, it does not imply that the aggregate influence of the profile of the other traders' types on a trader's valuation becomes small, nor does it impose any restriction on the size of the 'fixed' effect which adding another trader has on the existing traders' valuations. The model of Perry and Reny (2006) satisfies Assumption 2, because in their model the effect of any trader's signal on the perception of quality by other traders diminishes as the number of traders gets large.

The traders' types are drawn from the joint probability distribution $F_{mn}$ which is common knowledge. We assume that $F_{mn}$ is symmetric[5] and that the marginal distributions of traders' types are identical and have a *finite* support which is independent of $m$ and $n$, i.e $\mathcal{X}$ is finite. For $x \in \mathcal{X}$, let $x^+$ denote the next higher type in $\mathcal{X}$. We make the following *full support* assumption:

**Assumption 3** *There exists an $\varepsilon > 0$ such that for any $x_i \in \mathcal{X}$ and any $m$ and $n$, $\Pr_{F_{mn}} \{\tilde{x}_i = x_i | \tilde{x}_{-i} = x_{-i}\} \geq \varepsilon$ for all $x_{-i}$.*

This condition is borrowed from Peters and Severinov (2005) and resembles a condition in Cripps and Swinkels (2006). Among other things, it ensures that all the conditional expectations that are used in the sequel are well defined. Conditional independence with a common and full support (as in Perry and Reny (2006)) is consistent with this assumption. Note that we do not impose any of the affiliation assumptions used by Perry and Reny (2006).

[5]This symmetry assumption, like the assumption that all traders have the same valuation function, is used to construct an equilibrium in which all traders use the same bidding rule. The analytical complexities associated with asymmetric bidding rules in the interdependent value environment are daunting, so we do not know whether our results can be extended to asymmetric traders with different prior beliefs. Peters and Severinov (2005) show that in the private value case, there exists an equilibrium in which buyers use a common bidding rule similar to the one constructed in this paper, even without common priors.

### 3.2 Ascending-Price Double Auction Mechanism

Below we provide the description of our ascending-price double auction mechanism. In the course of this auction, all traders (sellers included) bid for the right to own (keep) one of the $n$ objects being auctioned. The auction starts at a price equal to

$$\underline{q} = \mathbb{E}\left[u\left(\tilde{x}_i, \tilde{x}_{-i}\right) | \tilde{x}_i = \tilde{x}_{-i_{(1)}} = \cdots = \tilde{x}_{-i_{(m)}} = \underline{x}\right] \tag{2}$$

where $\underline{x}$ is the lowest type in the set $\mathcal{X}$[6]

In words, the lowest price $\underline{q}$ is defined by the condition that a bidder with the lowest type would just be willing to pay $\underline{q}$ if she wins a unit at the auction. This bidder will win a unit only if at least $m$ of the other bidders have this same type. So, $\underline{q}$ is computed as an expectation of the trader's value conditional on the values of the $m$ lowest elements in the vector of types of the other traders being equal to $\underline{x}$, which means, of course, that all the other types are at least this high. Accordingly, in the computation of $\underline{q}$ we condition on the lowest $m$ elements of $x_{-i}$ being equal to $\underline{x}$ and take an expectation with respect to the other $n-1$ elements of $x$. Similarly, in the sequel when we calculate the expected value of a trader at a certain price, we condition on the values of the $m$ lowest trader types and take an expectation with respect to the other $n-1$ types conditional on them lying above the corresponding critical level. This method is analogous to the one used in the analysis of single-unit ascending auctions.

Thus, at the start of our auction all traders simultaneously declare whether they want to remain *active* at price $\underline{q}$, or whether they want to drop out of the bidding at this price and become *inactive.* If the number of active bidders exceeds the number of units for sale, the auctioneer raises the price according to a formula that will be described below, and this process is repeated. The price increases until the number of active bidders is less than or equal to the number of units of the good for sale.[7] All trades are executed at the price attained at this terminal point. An active buyer is given a unit of the good at this trading price. An inactive seller is paid the trading price in exchange for her unit of the good. Active sellers leave the market without trading. If the number of active traders is below the number of units for sale, the unsold units are randomly awarded to the bidders who dropped out of the bidding at the final trading price.

[6] In (2) and in similar formulas below we use $(\tilde{x}_i, \tilde{x}_{-i})$ to denote the random type profile of all traders. The expectation is taken with respect to this type given its prior distribution $F_{mn}$ and additional information upon which the expectation is conditional.

[7] The mechanism could be modified to allow inactive traders to re-enter the bidding as in Izmalkov (2003). Since the equilibrium that we construct achieves an efficient outcome without reentry and, moreover, reentry would never be optimal for a trader $i$ even if it was allowed - provided that other traders use the equilibrium strategies - we assume re-entry away to simplify the presentation.

The history of the game at any point in the auction consists of a current price $p$, the list of traders who have dropped out of the bidding and the prices at which they have dropped out. Any history of the game generates bidders' beliefs which consist of two components: as assignment of types to the traders who have dropped out and a probability distribution over the type profiles of active traders. We use a recursive procedure to construct these two components of beliefs. This procedure is described in Condition 1 below. Note that the assignment of types to the players who have dropped out, which is described in Condition 1, will also be used in the auction price adjustment rule.

**Condition 1** *Suppose that history $h$ is such that the current auction price is equal to $p$ and $k \geq 0$ bidders have dropped out, and these bidders have been assigned types $\hat{x}_h \equiv \{\hat{x}_1, \ldots, \hat{x}_k\}$ ordered from the lowest to the highest. Any bidder who drops out at history $h$ is assigned type $\hat{x}^p \in \mathcal{X}$ where $\hat{x}^p$ satisfies the following equation:*

$$\mathbb{E}\left[u\left(\tilde{x}_i, \tilde{x}_{-i}\right) | \tilde{x}_{-i_{(1)}} = \hat{x}_1, \ldots, \tilde{x}_{-i_{(k)}} = \hat{x}_k, \tilde{x}_i = \tilde{x}_{-i_{(k+1)}} = \cdots = \tilde{x}_{-i_{(m)}} = \hat{x}^p\right] = p. \tag{3}$$

*If a bidder remains active after history $h$, then her type is at least as large as $\hat{x}^p$.*

*So, an active player $i$'s beliefs about the types of other active players are characterized by probability distribution derived from the prior $F_{mn}$ using Bayes rule and conditioning on the types assigned to the traders who have dropped out, on the cutoff $\hat{x}^p$ and on player $i$'s true type.*

*An inactive player $j$'s beliefs about the types of active players are characterized by probability distribution derived from the prior $F_{mn}$ using Bayesian rule and conditioning on the types assigned to the traders who have dropped out, except for the type assigned to $j$ according to (3), on the cutoff $\hat{x}^p$ and on player $j$'s true type.*

Thus, the key aspect of beliefs formed according to Condition 1 is the assignment of types to inactive traders who have dropped out. Specifically, after any history all traders (except the trader whose type is the subject of beliefs) believe that with probability 1, an inactive trader with order number $i$ (i.e. the one who was the $i$-th to drop out) has type $\hat{x}_i$ -which is computed according to (3) given the history $h'$ at which the trader had dropped out and the beliefs corresponding to that history.

Note that by the monotonicity of the utility function, $\hat{x}^p > \hat{x}_j$ for all $j = 1, ..., k$. Further, since the profile of types $(\tilde{x}_i, \tilde{x}_{-i})$ used to compute (3) is such that $\tilde{x}_{-i_{(m)}} = \hat{x}^p$, the $n-1$ highest elements of $\tilde{x}_{-i}$ weakly exceed $\hat{x}^p$. This fact is implicitly taken into account in the computation of the expectation on the left-hand side of (3).

It is also important to note that in our auction the solution $\hat{x}^p$ to equation (3) will lie in $\mathcal{X}$ after any possible history. That is, inactive players will always be assigned types that occur with a positive probability according to the prior $F_{mn}$. This follows immediately from the price adjustment rule in our auction which we describe below. Specifically, the auction price is adjusted using a recursive dynamic procedure defined by Condition 1 and the formula (3). To see how this procedure works, suppose that $l$ bidders drop out and become inactive at price $p$ after history $h$. Each of these (now inactive) bidders is assigned type $\hat{x}^p$ and an order number between $k+1$ and $k+l$. The ordering is arbitrary. Thus, $\hat{x}_i = \hat{x}^p$ for $i = k+1, ..., k+l$. If the number of remaining active bidders is still strictly higher than $n$ (i.e. there are more active bidders than there are goods for sale), the price is adjusted upwards to $p^+$ which satisfies

$$\mathbb{E}\left[u\left(\tilde{x}_i, \tilde{x}_{-i}\right) | \tilde{x}_{-i_{(1)}} = \hat{x}_1, \ldots, \tilde{x}_{-i_{(k+l)}} = \hat{x}_{k+l}, \tilde{x}_i = \tilde{x}_{-i_{(k+l+1)}} = \cdots = \tilde{x}_{-i_{(m)}} = \hat{x}^{p+}\right] = p^+. \tag{4}$$

Recall that $\hat{x}^{p+}$ stands for the next element of the grid $\mathcal{X}$ which is higher than $\hat{x}^p$. If enough bidders drop out at $p^+$ so that the number of remaining active bidders no longer exceeds the number of units for sale, then the auction ends at this price. Otherwise, the described price adjustment procedure is performed again. If no bidder drops out at $p^+$, then the price is raised to the next level $p^{++}$ which is given by the same expression as in (4) except that $\hat{x}^{p+}$ is replaced by the next element of the grid $(\hat{x}^{p+})^+$, etc.

Thus, our price adjustment rule requires that after any possible history $h$, the auctioneer sets new price $p^+$ to satisfy (4) for the assignment of types of inactive players $\{\hat{x}_1, \hat{x}_{k+l}\}$ corresponding to $h$. Comparing (3) and (4), it is a tautology to say that with $p^+$ on the right-hand side of (3) and with the assignment of inactive players' types $\{\hat{x}_1, \hat{x}_{k+l}\}$, the solution to (3) is $\hat{x}^{p+}$. But $\hat{x}^{p+}$ lies in $\mathcal{X}$ by definition. Also, by (2), the solution to (3) at the lowest price $\underline{q}$ and null history is $\underline{x} \in \mathcal{X}$. So, indeed, the solution to equation (3) will always lie in $\mathcal{X}$.

As one can see, the type profile $\{\hat{x}_1, \ldots, \hat{x}_k\}$ assigned to the traders who have dropped out at history $h$ plays a central role in our analysis. The price adjustment rule relies exclusively on it. Moreover, given the price adjustment rule, the symmetry of all traders and the monotonicity of the utility function $u(.)$, there is a one-to-one correspondence between the set of histories in our game and the set $H$ of all tuples $\{\hat{x}_1, \ldots, \hat{x}_k, p\}$, $k \leq m$, representing the profile of types assigned to traders who have dropped out from the auction and the current price. It is straightforward to show this via an iterative application of (3).

So, to simplify the exposition, in the sequel we will with a slight abuse of terminology refer to the tuple $\{\hat{x}_1, \ldots, \hat{x}_k, p\}$ corresponding to $h$ as history $h$. Finally, let $H$ be the set of all such tuples with $k \leq m$. A strategy for each bidder is a map from $H$ into the set of probability distributions over the set $\{a, i\}$ (where $a$ stands for active, and $i$ stands for inactive). Altogether, our price adjustment procedure, the specification of histories and the traders' strategy sets define a dynamic game of incomplete information. Our solution concept for this game is as follows.

**Definition 1** *A $\delta$-perfect Bayesian equilibrium is a set of strategies and beliefs such that no trader can increase his or her payoff by more than $\delta$ by deviating* *from her equilibrium strategy given her beliefs, and such that beliefs satisfy Bayes rule on the equilibrium path.*

# 4 A Strategy Rule for the Ascending-Price Double Auction

We are now ready to describe an equilibrium strategy rule for the traders in this auction mechanism. For every history $h$, a strategy rule specifies whether or not an active trader should continue bidding. To describe our equilibrium strategy rule we start with the following definition:

**Definition 2** *The willingness to pay of an active trader $i$ of type $x_i$ after history $h = \{\hat{x}_1, \ldots, \hat{x}_k, p\}$ is equal to:*

$$\mathbb{E}\left[u\left(\tilde{x}_i, \tilde{x}_{-i}\right)|h; \tilde{x}_i = x_i, \tilde{x}_{-i_{(k+1)}} = \cdots = \tilde{x}_{-i_{(m)}} = \hat{x}^p\right] \tag{5}$$

*where $\hat{x}^p$ is the solution to (3) given $h$ and conditioning on $h$ in the expectation means conditioning on the event $\tilde{x}_{-i_{(1)}} = \hat{x}_1, \ldots, \tilde{x}_{-i_{(k)}} = \hat{x}_k$.*

An active trader's willingness to pay is her expected value conditional on the event that just enough traders drop out at the current price $p$ so that the auction ends. Note that, by the full support assumption, the expectations in this definition are always well-defined.

**Definition 3** *The strategy $\sigma^*$ is defined as follows: a bidder remains active (continues bidding) after history $h = \{\hat{x}_1, \ldots, \hat{x}_k, p\}$ if his willingness to pay is strictly higher than the current price $p$, and becomes inactive otherwise.*

Observe that, according to $\sigma^*$, after history $h = \{\hat{x}_1, \ldots, \hat{x}_k, p\}$ any active bidder whose type is strictly higher than $\hat{x}^p$ will continue bidding at price $p$.

We would like to characterize the outcome of the auction when traders use the strategy $\sigma^*$ as a function of the traders' type profile. To this end, we introduce the notion of *perceived value.*

**Definition 4** *Consider some array of traders' types $x = (x_1, \ldots, x_{n+m})$ and suppose that trader $i$ has the $r^{th}$ lowest type in this array. Let trader $i$'s* perceived value $v_i[x]$ *under the array $x$ be equal to*

$$\mathbb{E}\left\{u(\tilde{x}_i, \tilde{x}_{-i})|\tilde{x}_{-i_{(1)}} = x_{(1)}, \ldots, \tilde{x}_{-i_{(r-1)}} = x_{(r-1)}, \tilde{x}_i = \tilde{x}_{-i_{(r)}} = \cdots = \tilde{x}_{-i_{(m)}} = x_i\right\} \quad \text{if } r \leq m \tag{6}$$

$$\mathbb{E}\left\{u(\tilde{x}_i, \tilde{x}_{-i})|\tilde{x}_{-i_{(1)}} = x_{(1)}, \ldots, \tilde{x}_{-i_{(m)}} = x_{(m)}, \tilde{x}_i = x_i\right\} \quad \text{if } r > m. \tag{7}$$

*Also, let $v_{(m)}[x]$ (or simply $v_{(m)}$ when the array of types is clear from the context) be the perceived value of the trader with the $m^{th}$ lowest type in $x$.*

Trader $i$'s *perceived value* is his expected valuation of the good, conditional either on his knowledge of the $m$ lowest types in the array $x$, or on his knowledge of the types lower than his and the estimate that the $m$-th lowest type in the array $x$ is equal to his type $x_i$. By construction, $v_{(m)}[x]$ is the $m^{th}$-lowest such value. Our main theorem can now be stated.

**Theorem 1** *Let $\delta > 0$ and suppose that Assumptions 1-3 hold. Then there is some $N$ such that there exists a $\delta$-perfect Bayesian equilibrium where all traders use the strategy rule $\sigma^*$ if the number of sellers and hence the number of goods being sold is larger than $N$. For each array of types $x$ that occurs with a positive probability, all trades occur at price $v_{(m)}[x]$. A trader whose type is above $x_{(m)}$ will win a unit of the good for sure, a trader whose type is below $x_{(m)}$ will not win a unit of the good. A trader whose type is $x_{(m)}$ may or may not win a unit - in either case his expected value for the good will be the same as the equilibrium trading price.*

**Proof:** *See the appendix.*

Two observations are in order at this point. First, in the proof we show that the traders with the $n$ highest perceived values always end up with the good. By symmetry and the single-crossing Assumption 1, these are the traders with the $n$ highest types. So the equilibrium outcome of the mechanism is ex post efficient with probability 1.

The second observation is that the traders' bidding decisions on the equilibrium path completely reveal the $m$ lowest types of traders. So once the auction ends, the expected value of the good for a trader $i$ who ends up with a unit of the good, conditional on all information revealed by the bidding process to this trader, is equal to his perceived value $v_i[x]$. So the traders who consume the good have perceived values at least as high as the trading price $v_{(m)}[x]$. A trader who ends up without the good has expected value that is at least as high as her perceived value. However, his expected value is monotonic in his own type and therefore cannot exceed $v_{(m)}[x]$. Consequently, the equilibrium outcome is the best one for every trader conditional on the equilibrium price and the information that traders have at the end of the bidding process. We refer to this as the *rational expectations property.*

However, unlike in a standard rational expectations equilibrium where traders only condition their beliefs on price, the traders' beliefs at the end of the bidding process are conditioned on all the information revealed in the course of bidding. The ascending auction procedure we study here can never reveal the highest trader types, so the *equilibrium* trading price will not generally be fully revealing, and in particular the equilibrium trading price will not coincide with the full information market clearing price.[8]

[8]The equilibrium price in the double auction described in Perry and Reny (2006) does not coincide with the full information price in this sense either. They show that the equilibrium price

Yet since our rational expectations property holds uniformly for all trader types, it holds for every trader under any array of types that generates the same equilibrium outcome for the trader. Thus the ascending double auction has a stronger rational expectations property: the outcome of the mechanism will appear to be the best one for each trader conditional on the information conveyed by her own trading outcome (i.e., the final trading price and whether or not she has won the good). We show by an example below that this property may not hold if a trader conditions her belief only on information conveyed by the final price.

# 5 Discussion

In the example studied in section 2, our bidding rule $\sigma^*$ supports ex post efficient trade in the problematic states that we described above. Recall that in our discussion we have focused on two type profiles/states of the world:

$$\text{State 1:} \quad \{9, 8, 8, 8, 8, 8\}$$
$$\text{State 2:} \quad \{9, 10, 10, 3, 3, 3\}$$

with full information values equal to $\{17, 16, 16, 16, 16, 16\}$ and $\{14, 15, 15, 8, 8, 8\}$ in states 1 and 2 respectively.

Given the utility function (1) and the assumption that the lowest trader type is 3, we can use equation (2) to compute the starting (reserve) price in our mechanism. It is equal to the lowest full information value of 6 which occurs when all traders have type 3.

Let us, first, focus on state 2 with the array of types $\{9, 10, 10, 3, 3, 3\}$. All bidders need to compute the type assigned to any bidder who drops out at price 6. This type is given by the solution to (3), and is equal to 3. At this price, the seller (whose type is 9) and the buyers whose types are 10 have willingness to pay exceeding 6. However, the three buyers with types equal to 3 would drop out immediately according to $\sigma^*$. Successive application of the price adjustment rule causes the price to rise to 14. At price 14, the seller (whose type is 9) would drop out because her willingness to pay no longer exceeds the current auction price. The two high-value buyers with types 10 will then continue to bid until the auction price reaches 15, at which point both of them will drop out. One of them will be chosen at random to trade. Hence, the final trading price will be equal to the full information market clearing price 15.

Now let us consider state 1. Again, bidding will start at price 6. At this point, the seller's willingness to pay is 12, while the buyers' willingness to pay is 11. Each trader's willingness to pay (which changes as the price increases) remains above the price and all traders remain active, until the price reaches 16. At this price, each

in the double auction will be close to the full information price with high probability.

buyer's willingness to pay is also 16. So, all buyers will drop out of the bidding at 16 and will be assigned type 8. The seller, as the only trader who remains active at this price, wins the auction and ends up keeping the good. Again, the final trading price is equal to the full information market clearing price.

The seller's "odd" desire to keep the good when the price is high and trade it when the price is low is accommodated by the fact that the seller can reserve his decision whether or not to remain active when the price exceeds 15 until after he observes the bidding decisions of the other traders. When he observes three buyers drop out at price 6, he concludes that there is no point holding out for a high price once bidding reaches 15. When all buyers continue to bid until the price reaches 15, the seller concludes that the value of the good is even higher, and bids more aggressively to hold on to it.

The same example, but with a different state of the world, can be used to illustrate why $\sigma^*$ is only an approximate equilibrium and why we need a large number of sellers. For example, in the state

$$\{7,7,3,3,3,3\}$$

the trading price is 10 if traders use $\sigma^*$. When the auction price reaches 10, both the seller and the buyer of type 7 have to decide whether to continue bidding. According to $\sigma^*$, both of them should drop out at price 10. If they do so, the seller may or may not sell.

The problem is that the seller knows that he is pivotal when the price reaches 10, since at this point only he and another buyer are active. If the seller drops out at this point, the process ends and he earns zero profit. If he continues bidding, there are two possibilities. One is that the buyer will drop out, which will happen if the buyer's type is 7, as in the current type profile. The seller will then win the auction and earn zero profit. The other possibility is that the buyer will remain in the bidding -which would happen if the buyer's type is 8 or greater. In that case, the price will rise to at least 12 (by formula (4)). The seller might win the auction at price 12 if the buyer drops out, but this does not create a problem because her surplus is zero in that case, just as it is if he drops out at price 10. Alternatively, if the seller drops out at price 12 before the buyer does, the seller will get a higher price from a buyer - whose type in this case must be at least 8. So dropping out of bidding at price 10 is suboptimal for the seller.

In this example, the expected gain to the seller from continuing to bid is significant. However, when the number of sellers and hence the number of goods being auctioned is large, an active seller becomes pivotal when there are exactly $n$ other active traders each of whom has a willingness to pay exceeding the current price. When $n$ is large, the full support Assumption 3 guarantees that with a high probability *at least* one of the other active bidders has willingness to pay equal to the current price. This makes it very unlikely that the seller can prolong the auction by continuing to bid. This also explains why we only need the number of sellers $n$ to be large (but not the number of buyers $m$) for Theorem 1 to hold.

We can also use this example to illustrate the rational expectations property of our mechanism. This property was described in the previous section. Consider the outcome in state 1 where the profile of types is $\{9, 8, 8, 8, 8, 8\}$. When all the buyers drop out at price 16, the seller believes that each of the buyers has type 8. Bidding ends and the seller fails to trade. Ex post, the fact that all the buyers dropped out simultaneously at price 16 reveals the true state to the seller. Conditional on this belief about the state, the seller's demand correspondence at price 16 consists of only one outcome, not trading, which is what happens in the auction. So the outcome is in the seller's 'demand' correspondence conditional on all the information the outcome reveals. In state 2 where the profile of types is $\{9, 10, 10, 3, 3, 3\}$, the seller also learns the state once bidding ends. Conditional on this information he would prefer to trade at price 15. Once again, the outcome is in his demand correspondence given the price and posterior beliefs.

However, it would be wrong to think that the seller can choose the best outcome if she conditions her decision only on information revealed by price. For, consider the states $\{8, 7, 7, 7, 7, 7\}$ and $\{8, 9, 9, 3, 3, 3\}$ which may also arise in the example which we have considered. Recall that a trader's utility function is equal to $u(x_i, x_{-i}) = x_i + \left[\frac{\sum_{j \neq i} x_j}{5}\right]$, and so the full information values in these two states are $\{15, 14, 14, 14, 14\}$ and $\{13, 14, 14, 9, 9, 9\}$, respectively. The equilibrium trading price predicted by Theorem 1 is 14 in both states but the seller keeps the good in the first state and sells in the second. Our ascending-price auction mechanism can deliver this outcome, because the seller learns a lot more about the types of the others in the course of bidding. In contrast, the described outcome would be infeasible in a static rational expectations equilibrium where the seller only gets to see the price.

# 6 Conclusions

We have studied an ascending-price double auction mechanism and provided a strategy rule which constitutes a $\delta$-perfect Bayesian equilibrium for this mechanism when the number of sellers is large enough. To the best of our knowledge, dynamic double auctions have not yet been studied in the literature. The allocation supported by this equilibrium in our mechanism is ex post efficient. This property holds even when the types of the traders are not affiliated, so our mechanism delivers an efficient outcome in cases where a standard double auction would not. This is so because in our mechanism traders acquire information about the types of the other traders in the course of the bidding.

The equilibrium we describe is only an approximate equilibrium. It is difficult to say whether there is a way around this problem. The same property of the ascending-price mechanism that supports the revelation of information in the course of bidding also allows sellers to realize whether they are pivotal and attempt to manipulate the price. When the market is large, there is little for sellers to gain from this information, so their incentive to deviate becomes arbitrarily small.

# 7 Appendix - Proof of Theorem 1

The proof is provided via a series of Lemmas. Lemma 1 shows that the traders' equilibrium beliefs are consistent with strategy profile $\sigma^*$, i.e. are derived by applying Bayes rule on the path of the game where all traders use strategy $\sigma^*$.

Lemma 2 shows that the traders who follow the strategy $\sigma^*$ behave as if their valuations are equal to their *perceived values* defined by (6) or (7).

Lemma 2 allows us to characterize the outcome of the auction when all but one trader follow the strategy $\sigma^*$, and the remaining trader follows an arbitrary strategy. This is done in Lemmas 3-5. An immediate implication of this is the characterization of the auction outcome when all traders follow $\sigma^*$, as stated in Theorem 1.

Finally, we use this characterization to complete the proof in Lemma 6. This Lemma shows that for any $\delta$, no trader can gain more than $\delta$ by deviating from the strategy $\sigma^*$, if all other traders also follow this strategy and the number of sellers is large enough. The argument consists of two parts. First, we show that buyers never gain by deviating from $\sigma^*$, as long as they hold beliefs described in Condition 1. Sellers, on the other hand, can increase their payoffs by deviating from $\sigma^*$ in some information sets. So the second part of the Lemma shows that this gain becomes arbitrarily small as the number of traders becomes large.

## 7.1 Beliefs

Recall that the traders' beliefs in the ascending-price double auction are determined by the prior type distribution and observations of the dropout decisions of the other traders. Precisely, the beliefs are constructed recursively using the procedure described in Condition 1 of section 2. Our first result establishes the necessary consistency of these beliefs with strategy $\sigma^*$.

**Lemma 1** *The traders' beliefs constructed according to Condition 1 are consistent with all traders using the strategy $\sigma^*$.*

*Proof:* We need to show that the traders' beliefs constructed according to Condition 1 satisfy Bayes rule after any history that occurs with a positive probability on the path of the auction, if all traders use the strategy $\sigma^*$. Take any such history $\{\hat{x}_1, ..., \hat{x}_k, p\}$. According to $\sigma^*$, an active trader $i$ drops out after this history if and only if her willingness to pay (5) after this history is less than or equal to $p$. Comparing (3) and (5), observe that after history $\{\hat{x}_1, ..., \hat{x}_k, p\}$ trader $i$'s willingness to pay is less than or equal to $p$ if and only if $x_i \leq \hat{x}^p$, where $\hat{x}^p$ solves (3) for this history. In fact, if $x_i < \hat{x}^p$, then the price adjustment rule in the auction implies that trader $i$ using strategy $\sigma^*$ would have dropped out at a price below $p$.

So, Bayes rule implies that the type of a trader, who uses $\sigma^*$ and drops out (remains active) after history $\{\hat{x}_1, ..., \hat{x}_k, p\}$, is exactly equal to (greater than) $\hat{x}^p$. The beliefs formed according to Condition 1 prescribes exactly this. So, the beliefs are consistent. *Q.E.D.*

In the rest of the proof, we will employ the following notion:

**Definition 5** *Say that $i$ believes that the array of types $x_{-i}$ is possible after history $h$ if the beliefs given by Condition 1 put a positive probability on $x_{-i}$.*

Significantly, if $h = \{\hat{x}_1, ..., \hat{x}_k, p\}$ and $i$ thinks that the array of types $x_{-i}$ is possible, then the following is true. If $i$ is active (inactive, i.e. has dropped out) at history $h$, then $x_{-i(k')} = \hat{x}_{k'}$ for all $k' \in \{1, ..., k\}$ (for all $k' \in \{1, ..., k\}$ except for $k'$ s.t. $\hat{x}_{k'}$ is assigned to $i$). This fact will be used below several times.

## 7.2 Traders Behavior

The following lemma provides a result that is central to the logic of the proof of the theorem. Fix an array of types $(x_1, \ldots, x_{m+n}) \equiv (x_i, x_{-i})$ and calculate the corresponding array of perceived values $(v_1, \ldots, v_{m+n})$ defined by (6) or (7). Then trader $i$ with type $x_i$, who uses $\sigma^*$ and believes that: (i) other traders follow $\sigma^*$; (ii) the profile of types $x_{-i}$ is possible, will act just as if he had private value equal to $v_i(x_1, \ldots, x_{m+n})$. If trader $i$ deviates from $\sigma^*$, then she acts in the same way as a trader with some type $x'_i$ following $\sigma^*$. So, trader $i$ can figure out the impact of his deviation by calculating the new array of perceived values associated with the array of types $(x_1, \ldots, x'_i, \ldots, x_{m+n})$ then applying $\sigma^*$ to obtain the new outcome.

**Lemma 2** *Suppose that the beliefs of all traders satisfy Condition 1. Take some history $h_1 = \{\hat{x}_1, \ldots, \hat{x}_r, p\}$, $r \leq m$, and let $x_{-i}$ be an array of types of traders other than $i$ that $i$ believes is possible conditional on the history $h_1$. Then there is $x'_i$ such that each active trader $j$'s ($j \neq i$) willingness to pay in $h_1$, given the array of types $x_{-i}$, is larger than the price $p$ associated with $h_1$ if and only if trader $j$'s perceived value under the array of types $(x'_i, x_{-i})$ is larger than $p$. If $i$ is active in $h_1$, then $x'_i = x_{-i_{(m)}}$. Otherwise, $x'_i \leq x_{-i_{(m)}}$.*

*Proof:* As pointed out after Definition 5, since trader $i$ believes that the type profile $x_{-i}$ is possible after history $h_1 = \{\hat{x}_1, ..., \hat{x}_r, p\}$, $x_{-i}$ must be consistent with this history. This implies the following. If $i$ is active (inactive) at history $h_1$, then $x_{-i(k)} = \hat{x}_k$ for all $k \in \{1, ..., r\}$ (for all $k \in \{1, ..., r\}$ except for $k^i$ s.t. $\hat{x}_{k^i}$ is assigned to $i$) and $x_{-i(k)} \geq \hat{x}^p$ for all $k \in \{r+1, ..., m+n-1\}$ ($k \in \{r, ..., m+n-1\}$), where $\hat{x}^p$ solves equation (3) for history $h_1$.

We will use this to replace $\hat{x}_k$ with $x_{-i(k)}$ in the computation of a trader's willingness to pay. In particular, any active trader $j$'s beliefs about the type profile of traders other than $i$ after history $h_1$ are given by $\{\hat{x}_1, ..., \hat{x}_r\}$ and, hence, by $\{x_{-i(1)}, ..., x_{-i(k)}\}$ if $i$ is active at $h$; or by $\{\hat{x}_1, ..., \hat{x}_k\} \setminus \hat{x}_{k^i}$ and, hence, by $\{x_{-i(1)}, ..., x_{-i(k)}\} \setminus x_{-i(k^i)}$ if $i$ is inactive at $h$.

Let us now set $x'_i$. If $i$ is active at $h_1$, let $x'_i = x_{-i_{(m)}}$. If $i$ is inactive, we set $x'_i = \hat{x}_{k^i}$, where $\hat{x}_{k^i}$ is the type which $i$ was assigned according to equation (3) in Condition 1 when he became inactive. Note that in the latter case, $\hat{x}_{k^i} \leq \hat{x}^p \leq x_{-i(m)}$, as claimed in the statement of the Lemma.

**"Only if" Part: If active trader $j$'s willingness to pay after history $h_1$ is larger than $p$, then $j$'s perceived value under the array of types $(x'_i, x_{-i})$ is larger than $p$.**

Consider some trader $j$ with type $x_j$ who is active after history $h_1$. Then $j$'s willingness to pay at this point is given by:

$$\mathbb{E}\left[u\left(\tilde{x}_j, \tilde{x}_{-j}\right)|h_1; \tilde{x}_j = x_j, \tilde{x}_{-j_{(r+1)}} = \cdots = \tilde{x}_{-j_{(m)}} = \hat{x}^p\right] \tag{8}$$

where $\hat{x}^p$ is defined by the solution to (3), i.e. it satisfies

$$\mathbb{E}\left[u\left(\tilde{x}_j, \tilde{x}_{-j}\right)|h_1; \tilde{x}_j = \tilde{x}_{-j_{(r+1)}} = \cdots = \tilde{x}_{-j_{(m)}} = \hat{x}^p\right] = p. \tag{9}$$

By assumption, (8) is larger than $p$. Since the utility function is increasing in trader's type, it follows that $x_j > \hat{x}^p \geq \hat{x}_r$. We will make use of this inequality below.

Now, let us compute trader $j$'s perceived value under the profile of types $(x'_i, x_{-i})$. Let $(x'_i, x_{-i})_{(k)}$ denote the $k$-th lowest value in the vector $(x'_i, x_{-i})$ and take $t$ such that $x_j = (x'_i, x_{-i})_{(t)}$. Next, let $t' = \min\{t, m\}$ and $x'_j = (x'_i, x_{-i})_{(t')}$. Then by Definition 4, $j$'s perceived value is equal to

$$\mathbb{E}\left[u\left(\tilde{x}_j, \tilde{x}_{-j}\right)|\tilde{x}_j = x_j, \tilde{x}_{-j_{(1)}} = (x'_i, x_{-i})_{(1)}, \ldots, \tilde{x}_{-j_{(t'-1)}} = (x'_i, x_{-i})_{(t'-1)}, \tilde{x}_{-j_{(t')}} = \cdots \tilde{x}_{-j_{(m)}} = x'_j\right]. \tag{10}$$

Note we have defined $x'_i$ in such a way that for all $k \in \{1, ..., r\}$, $(x'_i, x_{-i})_{(k)} = \hat{x}_k$. Since $x_j = (x'_i, x_{-i})_{(t)}$ and, as observed above, $x_j \geq \hat{x}^p \geq \hat{x}_r$, it follows that $t \geq r$. Since $m \geq r$ also, we have $t' \geq r$. Also, as pointed above, since trader $i$ believes that the type profile $x_{-i}$ is possible at history $h_1$, $x_{-i(k)} \geq \hat{x}^p$ for all $k \in \{r+1, ..., m+n-1\}$ if $i$ is active ($k \in \{r, ..., m+n-1\}$ if $i$ is inactive) at $h_1$. So, given the way we have defined $x'_i$, $(x'_i, x_{-i})_k \geq \hat{x}^p$ for all $k \geq r+1$ and, in particular, $x'_j \geq \hat{x}^p$. Given the monotonicity of the utility function, we conclude that $j$'s perceived value in (10) is exceeds the following:

$$\left[u\left(\tilde{x}_j, \tilde{x}_{-j}\right)|\tilde{x}_j = x_j, \tilde{x}_{-j_{(1)}} = \hat{x}_1, \ldots, \tilde{x}_{-j_{(r)}} = \hat{x}_r, \tilde{x}_{-j(r+1)} = \cdots = \tilde{x}_{-j(m)} = \hat{x}^p\right]$$

$$= \mathbb{E}\left[u\left(\tilde{x}_j, \tilde{x}_{-j}\right)|h_1; \tilde{x}_j = x_j, \tilde{x}_{-j_{(r+1)}} = \cdots = \tilde{x}_{-j_{(m)}} = \hat{x}^p\right].$$

The last expression is trader $j$'s willingness to pay.

**"If" Part: If active trader $j$'s willingness to pay in $h_1$ is less than or equal to $p$, then her perceived value under the array of types $(x_i', x_{-i})$ is also less than or equal to $p$.**

Recall that trader $j$'s willingness to pay in $h_1 = \{\hat{x}_1, \ldots, \hat{x}_r, p\}$ is given by (8) where $\hat{x}^p$ satisfies (9). So, by monotonicity of the value function, $x_j \leq \hat{x}^p$. Recall that $x_i'$ was defined in such a way that $\hat{x}_k = (x_i', x_{-i})_{(k)}$ for all $k \in \{1, ..., r\}$. Using this and the fact that $x_j \leq \hat{x}^p$, we obtain:

$$\mathbb{E}\left[u\left(\tilde{x}_j, \tilde{x}_{-j}\right)|h_1; \tilde{x}_j = x_j, \tilde{x}_{-j_{(r+1)}} = \ldots = \tilde{x}_{-j_{(m)}} = \hat{x}^p\right] \geq$$
$$\mathbb{E}\left[u\left(\tilde{x}_j, \tilde{x}_{-j}\right)|\tilde{x}_j = x_j, \tilde{x}_{-j_{(1)}} = \min\{x_j, (x_i', x_{-i})_{(1)}\}, \ldots, \tilde{x}_{-j_{(m)}} = \min\{x_j, (x_i', x_{-i})_{(m)}\}\right].$$

The expression on the left-hand side (right-hand side) is equal to $j$'s willingness to pay (perceived value). So, $j$'s perceived value is also no larger than $p$. *Q.E.D.*

### 7.3 The outcome associated with $\sigma^*$

Let $h = \{\hat{x}_1, \ldots, \hat{x}_r, p\}$ be a history and $x_{-i}$ be a set of types that $i$ thinks is possible in $h$. Let $h' = \{\hat{x}_1, \ldots, \hat{x}_{r'}, p'\}$ be a successor to $h$ that will be realized when the traders other than $i$ have a profile of types $x_{-i}$ and follow strategy $\sigma^*$ while $i$ follows some strategy. By Lemma 2, when the profile of types of traders other than $i$ is $x_{-i}$, there is a type $x_i'$ such that bidder $i$ can predict whether trader $j$ will choose to continue bidding after history $h'$ by calculating $j$'s perceived value under $(x_j, x_i', x_{-i-j})$ and comparing it to $p'$, for all $j \neq i$. This provides a simple necessary and sufficient condition for bidding to end at price $p'$ when traders other than $i$ are using $\sigma^*$: the number of traders whose perceived values exceed $p$ should not be higher than $n$.

Recall that $v_i[x]$ is trader $i$'s perceived value under the array of types $x$ of $m+n$ traders and $v_{(m)}[x]$ is the perceived value of the trader with the $m^{th}$ lowest type in $x$.

**Lemma 3** *Let $h$ be a history and $x_{-i}$ be a profile of types that trader $i$ believes is possible after history $h$. Suppose that in the continuation following $h$ all traders other than $i$ use strategy $\sigma^*$ and form beliefs according to Condition 1. Suppose also that bidding ends after some history $h' = \{\hat{x}_1, \ldots, \hat{x}_m, q\}$ and that trader $i$ wins a unit of output. Then $q$ cannot exceed $v_{(m)}[x_{-i_{(m)}}, x_{-i}]$.*

*Proof:* The proof is by contradiction, so suppose that $q > v_{(m)}[x_{-i_{(m)}}, x_{-i}]$. Since all traders other than $i$ use strategy $\sigma^*$ and form beliefs according to Condition

1, there must be some predecessor history $h'' = \{\hat{x}_1, \ldots, \hat{x}_r, q''\}$ of $h'$ with $r < m$ from which the price is increased to $q$ and $\hat{x}^q$ solves

$$\mathbb{E}\left[u\left(\tilde{x}_i, \tilde{x}_{-i}\right)|h''; \tilde{x}_i = \tilde{x}_{-i_{(r+1)}} = \cdots = \tilde{x}_{-i_{(m)}} = \hat{x}^q\right] = q.$$

Since $r < m$ and since $i$ is active at $h'$, at least $n+1$ trader types in the array of types $(x_{-i(m)}, x_{-i})$ are at least $\hat{x}^q$. Each of these traders has a willingness to pay of at least $q$ which strictly exceeds $v_{(m)}[x_{-i_{(m)}}, x_{-i}]$. By Lemma 2 each of these traders has a perceived value that strictly exceeds $v_{(m)}[x_{-i_{(m)}}, x_{-i}]$. But by definition $v_{(m)}[x_{-i_{(m)}}, x_{-i}]$ is the $m^{th}$ lowest perceived value, so at most $n$ perceived values can strictly exceed $v_{(m)}[x_{-i_{(m)}}, x_{-i}]$, a contradiction. *Q.E.D.*

**Lemma 4** *Suppose that in the continuation after history $h$ all traders other than trader $i$ use strategy $\sigma^*$ and form beliefs according to Condition 1, the type profile of traders other than $i$ is given by $x_{-i}$ and $i$ thinks that profile $x_{-i}$ is possible after history $h$. Also, suppose that trader $i$ is awarded a unit of the good at price $q$ when the auction ends. Let $x'_i = x_{-i_{(m)}}$. Then $q \geq v_{(m)}[x'_i, x_{-i}]$.*

*Proof:* The proof is by contradiction, so suppose that the trading price at the end of the auction is equal to $p$ s.t. $p < v_{(m)}[x'_i, x_{-i}]$. Since all traders other than $i$ are using strategy $\sigma^*$, by Lemma 2, those of them whose perceived values exceed $v_{(m)}[x'_i, x_{-i}]$ will all remain active at price $p$. There are at least $n$ such traders. So the auction can end at $p$ only if buyer $i$ drops out, but in this case $i$ will not win the good. *Q.E.D.*

Lemmas 2-4 imply the following result:

**Lemma 5** *Suppose that the type profile of traders other than $i$ is given by $x_{-i}$, the history of the game $h = \{\hat{x}_1, ..., \hat{x}_r, p\}$ is such that trader $i$ believes that the type profile $x_{-i}$ is possible, and in the continuation after $h$ all traders other than $i$ follow the strategy $\sigma^*$ and form beliefs according to Condition 1. Then*

1. *If trader $i$ drops out of the bidding so that all traders believe that $i$'s type is $x'_i$, then all trades will occur at price $v_{(m)}[x'_i, x_{-i}]$.*

2. *Suppose that trader $i$ is active at $h$, and also follows $\sigma^*$ and forms beliefs according to Condition 1 in the continuation. Then $i$ will win a unit of the good at price $v_{(m)}[x_i, x_{-i}]$, if $x_i > x_{-i(m)}$, and will not win the unit of the good if $x_i < x_{-i(m)}$.*

*Proof:* Since at history $h = \{\hat{x}_1, ..., \hat{x}_r, p\}$ trader $i$ thinks that the array of types $x_{-i}$ is possible, we have $\hat{x}_k = x_{-i(k)}$ for all $k \in \{1, ..., r\}$. Further, since all players other than $i$ follow $\sigma^*$ in the continuation, at any successor history $h' = \{\hat{x}_1, ..., \hat{x}_{r'}, p'\}$ we have: (i) $\hat{x}_k = x_{-i(k)}$ for all $k \in \{1, ..., r'\}$ s.t. $\hat{x}_k$ is not assigned to $i$; (ii) $i$ thinks that $x_{-i}$ is possible. This is so because the players' beliefs constructed according to Condition 1 allow them to infer correctly the types of players who drop-out while following $\sigma^*$.

To prove Part 1, consider any successor history $h'' = \{\hat{x}_1, ..., \hat{x}_{r''}, p''\}$ of $h$ in which agent $i$ has dropped out and has been assigned type $x'_i$. By Lemma 2, it follows that an active trader $j$'s willingness to pay in $h''$ exceeds the current price $p''$ if and only if her perceived value $v[x_j, x'_i, x_{-i-j}]$ exceeds $p''$. Therefore, trader $j$ drops out at price $p$ such that her perceived value $v[x_j, x'_i, x_{-i-j}]$ is equal to $p$.

By Assumption 1, for any two traders $j$ and $k$ who remain active after trader $i$ has dropped out, $v[x_j, x'_i, x_{-i-j}] > v[x_k, x'_i, x_{-i-k}]$ if and only if $x_j > x_k$. So, in the continuation of $h''$ trader $k$ drops out when $j$ is still active. Since the auction ends as soon as $m$ traders drop out, it follows that the auction will end at price $v_{(m)}[x'_i, x_{-i}]$.

Next, we prove part 2. So suppose that, as all other traders, trader $i$ also follows $\sigma^*$ in the continuation of $h$. If $x_i < x_{-i(m)}$, then $i$'s perceived value is less than the perceived value of at least $n$ other active traders. So, by Lemma 2, $i$ will drop when at least $n$ other traders remain active and will not trade.

If $x_i > x_{-i(m)}$, then $i$'s perceived value is greater than the perceived value of at least $m$ other traders. So, by Lemma 2 those traders will drop out while $i$ will remain active, and $i$ will get a unit of the good. By Lemmas 3 and 4, the final price at which the auction will stop will be equal to $v_{(m)}(x_{-i(m)}, x_{-i})$. But notice that by definition of perceived value (see Definition 4) $v_{(m)}(x_{-i(m)}, x_{-i}) = v_{(m)}(x_i, x_{-i})$ when $x_i > x_{-i(m)}$. Compare expressions (6) and (7) to see this. *Q.E.D.*

As an immediate implication of Lemma 5, we can characterize the outcome of the auction when all traders use the strategy $\sigma^*$ from the beginning. Suppose that this is so and that the type profile is given by $x$. Then, by Lemma 5, all trades will occur at price $v_{(m)}[x]$. A trader whose type is above $x_{(m)}$ will win a unit of the good for sure, a trader whose type is below $x_{(m)}$ will not win a unit of the good. A trader whose type is $x_{(m)}$ may or may not win a unit - in either case his expected (i.e. perceived) value for the good will be the same as the equilibrium trading price. This establishes the first part of Theorem 1.

So, to complete the proof of the Theorem we only need to show that there is no profitable deviation from $\sigma^*$. This is done in the following Lemma:

**Lemma 6** *If Assumptions 1-3 hold, and the number of sellers $n$ is large enough, then the strategy rule $\sigma^*$ for all players, and belief system constructed according to Condition 1 constitute a $\delta$-perfect Bayesian equilibrium.*

*Proof:* **Buyer's Part.** Let us establish that a buyer cannot gain by deviating from $\sigma^*$. Let $x_{-i}$ be an array of types that buyer $i$ thinks is possible after some history. If all traders follow $\sigma^*$ in the continuation and $x_i > x_{-i_{(m)}}$ then, by Lemma 5, $i$ will trade at price $v_{(m)}[x_i, x_{-i}]$ under this array of types. If $i$ deviates from

$\sigma^*$ and follows some alternative strategy which causes the trading prices to stay below $v_{(m)}[x_i, x_{-i}]$ then, by Lemma 4, $i$ fails to trade.

So it only remains to show that buyer $i$ cannot get a higher payoff by deviating from $\sigma^*$ in his decision to drop out of the bidding. Suppose that $x_i < x_{-i_{(m)}}$. Then by Lemma 5, if buyer $i$ follows $\sigma^*$, then he will not trade and hence obtain zero payoff. If he deviates from $\sigma^*$ and ends up trading, then by Lemmas 3 and 4, he will pay the price of at least $v_{(m)}[x_{-i_{(m)}}, x_{-i}]$. But $v_{(m)}[x_{-i_{(m)}}, x_{-i}] \geq v_{(m)}[x_i, x_{-i}] > v_i[x_i, x_{-i}]$. By Lemma 2, $i$'s willingness to pay is also below the trading price, so such deviation will not be profitable.

Similarly, if $x_i > x_{-i_{(m)}}$ (i.e. $v_i[x_i, x_{-i}] > v_{(m)}[x_i, x_{-i}]$), then $i$ will trade at price $v_{(m)}[x_i, x_{-i}]$ and receive a strictly positive surplus, which is better than what he could get by dropping out before the auction ends. So following $\sigma^*$ is a best reply for buyers if all other traders are using $\sigma^*$.

It remains to consider the case $x_i = x_{-i_{(m)}}$. By Lemma 5, in this case $i$ may or may not win an auction if he follows $\sigma^*$. If he does win, he will pay the price $v_{(m)}[x_i, x_{-i}]$ which is equal to both his perceived value and his willingness to pay. By Lemma 4, there is no strategy for $i$ which allows him to trade at a price below $v_{(m)}[x_i, x_{-i}]$, so any $i$'s deviation from $\sigma^*$ is unprofitable. This completes the proof for the buyers.

**Seller's Part.** A different approach is needed for sellers, because, unlike a buyer, a seller gets a positive payoff only if he is inactive at the end of the auction and trades. Let us fix a seller $i$ of type $x_i$. We will separately consider two sets of type profiles $x_{-i}$ of agents other than $i$. *Case 1* will deal with type profiles $x_{-i}$ such that $x_i > x_{-i_{(m)}}$. *Case 2* will deal with type profiles $x_{-i}$ such that $x_i \leq x_{-i_{(m)}}$.

*Case 1.* Let us fix some history $h$ and an array of types $x_{-i}$ such that $x_i > x_{-i_{(m)}}$ and seller $i$ thinks $x_{-i}$ is possible conditional on history $h$.

Since $x_i > x_{-i_{(m)}}$, we have $v_i[x_i, x_{-i}] > v_{(m)}[x_i, x_{-i}]$. If seller $i$, along with all other traders, follows strategy $\sigma^*$ after history $h$, then the auction will end at price equal to $v_{(m)}[x_i, x_{-i}]$ and seller $i$ will not trade and receive zero payoff. If seller $i$ deviates to some alternative strategy and does not sell as a result of this deviation, she still receives zero payoff.

Now suppose that seller $i$ sells after deviating from $\sigma^*$. Then by part 1 of Lemma 5 there is $x'_i$, satisfying $x'_i < x_{-i_{(m)}}$, such that after $i$'s deviation the final price in the auction will be $v_{(m)}[x'_i, x_{-i}]$. Seller $i$'s payoff in this case is equal to $v_{(m)}[x'_i, x_{-i}] - u(x_i, x_{-i})$. Since $x'_i < x_i$, we have $v_{(m)}[x'_i, x_{-i}] \leq v_{(m)}[x_i, x_{-i}]$.

Consider another array of types $\tilde{x}_{-i}$ which has the same $m$ lowest components as $x_{-i}$. Then seller $i$ must also think that $\tilde{x}_{-i}$ is possible after history $h$. Moreover, by definition of perceived value, we have $v_{(m)}[x'_i, x_{-i}] = v_{(m)}[x'_i, \tilde{x}_{-i}]$ and $v_{(m)}[x_i, x_{-i}] = v_{(m)}[x_i, \tilde{x}_{-i}]$. So, $v_{(m)}[x'_i, \tilde{x}_{-i}] \leq v_{(m)}[x_i, \tilde{x}_{-i}]$. Therefore, taking an expectation over all such $\tilde{x}_{-i}$ we get:

$$
\begin{aligned}
& v_{(m)}[x_i', \tilde{x}_{-i}] - \mathbb{E}\left[u\left(\tilde{x}_i, \tilde{x}_{-i}\right)|\tilde{x}_i = x_i, \tilde{x}_{-i_{(1)}} = x_{-i_{(1)}}, ..., \tilde{x}_{-i_{(m)}} = x_{-i_{(m)}}\right] \leq \\
& v_{(m)}[x_i, \tilde{x}_{-i}] - \mathbb{E}\left[u\left(\tilde{x}_i, \tilde{x}_{-i}\right)|\tilde{x}_i = x_i, \tilde{x}_{-i_{(1)}} = x_{-i_{(1)}}, ..., \tilde{x}_{-i_{(m)}} = x_{-i_{(m)}}\right].
\end{aligned}
$$

The last expression is less than zero because $x_i > x_{-i_{(m)}} = \tilde{x}_{-i(m)}$. Furthermore, this expression is equal to the difference between the expected trading price that seller $i$ receives after deviating and bidding as type $x_i'$ and seller $i$'s expected utility conditional on the knowledge of the $m$ lowest values in the array of types of other traders. Since these lowest $m$ values $(x_{-i(1)}, ..., x_{-i(m)})$ were chosen arbitrarily (provided that $x_i > x_{-i_{(m)}}$), we conclude that no deviation from $\sigma^*$ is profitable in Case 1.

*Case 2.* To prove the result in this case, we will show that seller $i$ could not gain more than $\delta$ by remaining in the bidding after a history in which the auction price exceeds $i$'s willingness to pay. By staying in the bidding after such history, seller $i$ could raise the price at which she ultimately trades. The downside is that he may lose a profitable trade when he does so.

As in case 1, fix some history $h = \{\hat{x}_1, \ldots, \hat{x}_r, p\}$ and consider an array of types $x_{-i}$ that seller $i$ of type $x_i$ thinks is possible conditional on $h$ and that satisfies $x_i \leq x_{-i_{(m)}}$. Then, in the continuation of $h$ where all traders follows equilibrium strategy $\sigma^*$, seller $i$ should drop out of bidding and trade at price $v_{(m)}(x_i, x_{-i})$.

Now consider $i$'s expected gain if she deviates from $\sigma^*$ and bids as a trader of some type $x_i'$ s.t. $x_i' > x_i$. For the equations that follow, understand the notation $Pr_{x_i}$ to mean the probability computed using the distribution $F_m n$ conditional on the type $x_i$. As always, this is the appropriate probability computation using beliefs of a trader of type $x_i$. It is implicitly understood that the distribution function $F_m n$ depends on the number of buyers and sellers involved. Taking an expectation over all type profiles $\tilde{x}_{-i}$ consistent with $h$ and satisfying $x_i \leq \tilde{x}_{-i_{(m)}}$, this gain is equal to

$$
\begin{aligned}
& - Pr\left\{\tilde{x}_{-i_{(m)}} < x_i' | h, \tilde{x}_i = x_i \leq \tilde{x}_{-i_{(m)}}\right\} \times \qquad (11) \\
& \times \mathbb{E}\left[v_{(m)}[\tilde{x}_i, \tilde{x}_{-i}] - u_i\left(\tilde{x}_i, \tilde{x}_{-i}\right)|h, \tilde{x}_i = x_i \leq \tilde{x}_{-i_{(m)}} < x_i'\right] \\
& + Pr\left\{\tilde{x}_{-i_{(m)}} \geq x_i' > \tilde{x}_{-i_{(m-1)}} | h, \tilde{x}_i = x_i \leq \tilde{x}_{-i_{(m)}}\right\} \times \\
& \times \mathbb{E}\left[v_i[x_i', \tilde{x}_{-i}] - v_i[\tilde{x}_i, \tilde{x}_{-i}] | h, \tilde{x}_{-i_{(m)}} \geq x_i' > \tilde{x}_{-i_{(m-1)}}, \tilde{x}_i = x_i \leq \tilde{x}_{-i_{(m)}}\right] + \\
& + Pr\left\{\tilde{x}_{-i_{(m-1)}} \geq x_i' | h, \tilde{x}_i = x_i \leq \tilde{x}_{-i_{(m)}}\right\} \\
& \times \mathbb{E}\left[v_{(m)}[x_i', \tilde{x}_{-i}] - v_{(m)}[\tilde{x}_i, \tilde{x}_{-i}] | h, \tilde{x}_{-i_{(m-1)}} \geq x_i', \tilde{x}_i = x_i \leq \tilde{x}_{-i_{(m)}}\right].
\end{aligned}
\qquad (12)
$$

The first summand in (12) is non-positive and reflects the cost to seller $i$ of losing a trade that he would have made had he not deviated. The second summand reflects the seller's gain when he is pivotal and raises the trading price from $v_i[x_i, \tilde{x}_{-i}]$ to $v_i[x_i', \tilde{x}_{-i}]$ by deviating. The third summand represents the impact that seller $i$ has on the price by making others believe that his type is higher and thereby inducing them to bid more aggressively, even though seller $i$ is non-pivotal. This effect causes the price at which $i$ trades to increase from $v_{(m)}[x_i, \tilde{x}_{-i}]$ to $v_{(m)}[x_i', \tilde{x}_{-i}]$. We will show that the last two summands become arbitrarily small (and in particular smaller than $\delta$) when $n$ gets large.

Consider the expectation in the third summand of (12). We have:

$$\begin{aligned}
&\mathbb{E}\left[v_{(m)}[x_i', \tilde{x}_{-i}] - v_{(m)}[\tilde{x}_i, \tilde{x}_{-i}] | h, \tilde{x}_{-i_{(m-1)}} \geq x_i', \tilde{x}_i = x_i \leq \tilde{x}_{-i_{(m)}}\right] = \\
&\mathbb{E}\left[u(\tilde{x}_{-i_{(m-1)}}, x_i', \tilde{x}_{-i} \setminus \tilde{x}_{-i_{(m-1)}}) | \tilde{x}_{-i_{(1)}} = \hat{x}_1, ..., \tilde{x}_{-i_{(r)}} = \hat{x}_r, \tilde{x}_{-i_{(m-1)}} \geq x_i', \tilde{x}_i = x_i \leq \tilde{x}_{-i_{(m)}}\right] - \\
&\mathbb{E}\left[u(\tilde{x}_{-i_{(m-1)}}, \tilde{x}_i, \tilde{x}_{-i} \setminus \tilde{x}_{-i_{(m-1)}}) | \tilde{x}_{-i_{(1)}} = \hat{x}_1, ..., \tilde{x}_{-i_{(r)}} = \hat{x}_r, \tilde{x}_{-i_{(m-1)}} \geq x_i', \tilde{x}_i = x_i \leq \tilde{x}_{-i_{(m)}}\right].
\end{aligned} \tag{13}$$

By Assumption 2, the difference of the utility values under expectation sign in (13) is arbitrarily small uniformly in $\tilde{x}_{-i}$ if $n$ is large enough. So, the expectation of the difference is also arbitrarily small when $n$ is large.

The utility difference in the second summand remains bounded from above and below as $n$ grows. However, consider the probability $Pr\left\{\tilde{x}_{-i_{(m)}} \geq x_i' > \tilde{x}_{-i_{(m-1)}} | h, \tilde{x}_i = x_i\right\}$ in the second summand. Note that trader $i$ is pivotal, i.e. $\tilde{x}_{-i_{(m)}} \geq x_i' > \tilde{x}_{-i_{(m-1)}}$, only if there is a successor history of $h'$ with associated price $p'$, at which all remaining active traders continue bidding, so that by dropping out trader $i$ ends the auction. Let $S(h)$ be the set of pairs $(h', p')$ such that $h'$ is a successor to $h$ and $p'$ is the price that prevails in $h'$. Then,

$$\begin{aligned}
&Pr\left\{\tilde{x}_{-i_{(m)}} \geq x_i' > \tilde{x}_{-i_{(m-1)}} | h, \tilde{x}_i = x_i \leq \tilde{x}_{-i_{(m)}}\right\} \leq \\
&\max_{(h',p') \in S(h)} Pr\{\tilde{x}_j > \hat{x}^{p'} \text{ for every active bidder } j \neq i \text{ at } p' | h', \tilde{x}_i = x_i \leq \tilde{x}_{-i_{(m)}}\}.
\end{aligned}$$

By Assumption 3, $\Pr_{F_{mn}}\{\tilde{x}_j = \hat{x} | \tilde{x}_{-j} = x_{-j}\} \geq \varepsilon$ for every $\hat{x}$ and every $x_{-j}$. Since the number of active traders in each successor $h'$ of $h$ is at least $n$ (otherwise the auction would have terminated),
$\Pr\{\tilde{x}_j > \hat{x}^{p'} \text{ for every active bidder } j \neq i \text{ at } p' | h', \tilde{x}_i = x_i \leq \tilde{x}_{-i_{(m)}}\} \leq (1-\varepsilon)^n$ for every pair $(h', p') \in S(h)$. Hence, the probability that bidder $i$ is pivotal is arbitrarily small at every price level, provided that the number of sellers $n$ is large enough.

Since the first summand in (12) is negative, we conclude that a seller's gain from deviating at any price level becomes smaller than $\delta$ when $n$ is sufficiently large. *Q.E.D.*

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