Screening, Signalling and Costly Misrepresentation*

Raymond Deneckere[†] Sergei Severinov[‡]

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Abstract

When misrepresenting private information is costly and each type has a different "natural" least-cost signal or message, the availability of multiple signals has a significant effect on signalling equilibria and on the set of implementable outcomes in a screening context. In particular, welfare losses associated with costly signaling disappear as the number of available signals increases, yet at the same time each type is identified with a high degree of precision in every equilibrium satisfying a dominance criterion. In the screening context, we establish conditions under which the principal can implement an arbitrary allocation profile at a small signal cost. This result helps to explain why employers often prefer to screen applicants via multiple interviews rather than via menus of contracts. We also derive an optimal screening mechanism in such a setting. A surprising property of this mechanism is the absence of exclusion.

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†Department of Economics, University of Wisconsin, Madison, 1180 Observatory Drive, Madison, 53706, WI, USA. Email: rjdeneck@wisc.edu.

‡Vancouver School of Economics, University of British Columbia, 6000 Iona Drive, Vancouver, V6T 1L4, BC, Canada; Email: sseverinov@gmail.com.

1 Introduction

In this paper we study mechanism design and signalling in settings where agents incur a cost of conveying and misrepresenting their private information. These costs may exist for several reasons. First, the agents may have to take costly and unproductive actions (e.g. excessive education) in order to signal their types. Second, misrepresenting the truth may require costly effort to manufacture or conceal evidence that reveals the true state of the world. For example, in order to obtain a supplier contract or a loan, win a grant or a promotion, a firm or an individual may need to be perceived as highly productive, successful or creditworthy. This goal may be attained by taking costly actions to produce “evidence” exaggerating prior or current performance and concealing the risk of default or non-performance.[1] Third, an individual may find it costly to misrepresent the truth for psychological or ethical reasons.

Our model and results rely on two main assumptions:

- The first-best or least cost signals vary across agent types;
- Agents can send multiple signals or messages.

As we argue below, these two key parts of our approach are natural, well motivated, and strongly grounded in theory and empirics. In particular, let us highlight the motivation behind them in the context of the model of signaling via education, one of the prime examples in our paper. First, driven by intellectual curiosity, taste for knowledge or entertainment value, most people like to learn, but some people like learning more than others. So, it is natural to envisage that individuals derive a direct utility from learning, and that this utility is correlated with ability. As a consequence, each type has a different least-cost level of education. Alternatively, education may enhance productivity, more so for higher ability individuals. Spence explicitly recognized the latter possibility in his review of the signaling model (Spence, 2002).

Second, in reality the students' GPA i.e., an average of all grades, rather than the time spent in education usually serves as a signal of students' abilities. We view a grade in each course as a different signal. Indeed, in each course a student has to complete course-specific activities, including reading new material, completing assignments, taking tests and exams. Also, many fields and even subfields have their own analytical apparatus and methods learning which requires qualitatively new and independent cognitive effort. This motivates our second main assumption that individuals undergo multiple tests or send multiple messages.

[1] An example of efforts to manipulate signals involves college admission tests. Educators have long been concerned that extensive test preparation by the wealthy skew the measurement of underlying student ability. The College Board recently announced plans to redesign the SAT test to address this (R. Perez-Pena, “Then and Now: A Test that Aims to Neutralize Advantages of the Privileged,” *New York Times*, March 15, 2014).

The major insight of our paper is that the juxtaposition of these two assumptions - the type-dependence of the least cost messages and the availability of multiple messages- has powerful consequences. In the signaling context, when the number of available signals becomes sufficiently large, any perfect Bayesian equilibrium that satisfies a very mild refinement must be very informative and identify elements in very fine partitions of types. Furthermore, in such equilibria the sender's signalling costs become insignificantly small. Plainly, if each signal corresponds to a grade in a different course, then students who need to take multiple courses to earn a degree will overinvest in studying only a little bit compared to their "ideal" investment reflecting their abilities and taste for knowledge. The end result is that a student's GPA will reflect his or her natural ability quite accurately and, importantly, the total overinvestment by each ability type and hence overall inefficiency will be rather small.

These conclusions are important, because ever since Spence's (1973) seminal contribution, economists have been concerned with the potential loss of welfare due to signalling. Specifically, Spence demonstrates that job applicants, engaged in a competitive 'rat race,' will spend too much time and effort on education in order to signal their ability, even though it may not enhance their productivity. Indeed, the concerns about unproductive competition for grades have been expressed both by educators and parents.[2] Thus, our paper identifies important conditions under which such rate races does not occur and welfare losses from signaling remain quite small.

Our results also shed some light on the empirical literature which has tried to document the existence and magnitude of signalling costs in education, but has had trouble doing so. Page (2010) concludes that "the number of studies that use convincing empirical strategies to test the signaling model is short, and the evidence is mixed." Several other studies, including Chevalier et. al (2004) , Clark and Martorell (2014), and Arteaga (2017) indicate that the signaling cost may be quite small. Given the large number of courses taken by modern-day students in college, our model provides a potential explanation for these findings.

In the screening context, we show that with sufficiently many signals/messages the principal can come arbitrarily close to implementing any decision rule and allocation of surplus, keeping misrepresentation costs arbitrarily small. So, the availability of multiple signals allows to avoid significant inefficiencies stemming from the agent's incentive to manipulate private information and the principal's desire to limit the agent's information rents.

This result has practical relevance. It can explain why employers in some industries prefer to screen job applicants thoroughly via multiple interviews, rather than offer self-selecting menus of contracts or strong performance incentives. Indeed, the interviewing process in many

[2] See for example "Crossing the Line: How the Academic Rat Race Is Making Our Kids Sick" by Vicki Abeles, Huffington Post, May 19 2014, http://www.huffpost.com/entry/education-stress_b_5341256.

professional industries is consistent with the idea of requesting multiple messages or signals from the candidate. For example, during a departmental visit on the academic job-market a candidate meets with faculty members working in different fields. Each conversation provides a somewhat different signal of the candidate's ability, because different faculty members assess the candidate from various perspectives and inquire about different aspects of the candidate's knowledge. Such interviewing procedures are also used in professional service industries such as consulting, law or banking. This paper implies that significantly misrepresenting his/her ability would be too costly for a job-candidate who has to go through a sufficient number of such interviews or tests. So, the employer will have an accurate estimate of the candidate's type, and will not have to offer a powerful incentive scheme on the job.

The result that eliciting multiple messages from the agents is a more cost-effective and efficient method of screening also applies in other contexts. For example, in a manager-shareholders relationship a message may correspond to an inspection or an audit. To wit, the board of directors may carry out several accounting and other audits of a firm. Its managers would then have to incur the cost of misrepresenting the true state of the world and fudging the numbers in each audit. Likewise, managers may have to send multiple messages because they have to communicate and convince different audiences, including shareholders, external auditors, financial analysts and credit rating agencies, creditors and employees.

To explain the intuition behind our results, note that increasing the number of signals naturally requires the equilibrium signal cost along a particular signal dimension to fall, at least on average. However, this does not imply that total signal costs will decrease.[3] Thus, the novel contribution of our paper lies in showing that when the least cost signal differs across agent types, total equilibrium message/signaling costs will fall as the number of signals grows.

It may be tempting to reason that with many signals, misrepresentation can be easily prevented by having each agent send only her least cost messages. However, such logic is fallacious. Indeed, if some type θ sends only her least-cost signals then it will be more costly for any other type θ' to mimic θ as the number of signals increases, provided that the least cost signals differ across agent types and the marginal cost of any particular signal is not too sensitive to other signals. This simple intuition explains why global incentive constraints get more relaxed as the number of signals increases. However, crucially it does not apply to local incentive constraints that involve mimicking infinitesimally close types. This is so because the marginal cost of misrepresentation is zero at the least cost message, and adding more signals does not change this fact. So simply asking the agents to send their least cost messages will never satisfy local incentive constraints, no matter how many signals there are.

Importantly, the literature on signaling and screening shows that local incentive constraints

[3] This is illustrated in Example 1 in section 2.2 below.

are the only binding ones in many environments. Connected by the "chains" of local incentive constraints, "higher" types earn informational rents that depend on the allocations given to all "lower" types, even though global incentive constraints are non-binding. So, in order to satisfy local incentive constraints it is necessary to introduce some misrepresentation by almost every type. Yet, we show that the degree of such misrepresentation can be chosen in such a way that the desirable allocation profile is implemented while the expected signaling cost becomes small as the number of signals increases.This happens because a fixed misrepresentation cost, for any type "buys" smaller and smaller changes in beliefs by the principal as the number of signals increases. Consequently, the degree of necessary misrepresentation diminishes at a faster rate than the rate at which the number of messages grows.

The existing literature provides ample motivation for the central assumptions of our paper. First, our assumption that misrepresenting type is costly and type-dependent is motivated by considerable empirical and experimental evidence showing that individuals do not lie as often as warranted by maximizing their material payoffs. In the tax literature, Erard and Feinstein (1994) report that some taxpayers are willing to bear their full tax burden despite financial incentives to underreport their incomes. A large body of experimental evidence indicates that there are intrinsic costs to misrepresentation, which typically depend upon the "size of the lie." Abeler, Nosenzo and Raymond (2016) provide a survey of this literature.

The assumption of type dependent least cost signals is ubiquitous in the signaling literature, spanning a wide range of applications such as limit pricing (Milgrom and Roberts (1982)), advertising (Bagwell (2007)), oligopoly pricing (Mailath (1989)), dividend signaling (Miller and Rock (1985)), electoral competition (Banks (1990)), social norms (Bernheim 1994), bequests (Bernheim and Severinov (2003)) and conspicuous consumption (Ireland (1994)).[4]

There are also several strands of influential literature on screening that make such assumptions. First, there is an extensive literature on certifiable statements or hard evidence whose availability varies with type, originating in the contributions of Milgrom (1981) and Grossman (1981). The extensive literature that flows from these papers (including Seidmann and Winter (1997), Hagenbach, Koessler and Perez-Richet (2014)) makes the strong assumption that every type can costlessly send some set of messages which uniquely identify it and which is not available to any other type. A related branch of literature on mechanism design with hard evidence is more permissive in its assumptions, requiring only partial verifiability, but retains the assumption of binary communication cost so that each message is either costless or infinitely costly for a given type. It explicitly allows for the presentation of multiple pieces of evidence, as we do. This literature includes Green and Laffont (1986), Lipman and Seppi

[4] Also, Frankel and Kartik (2019) employ such assumption in their model of signaling two-dimensional private information via one signal.

(1995), Forges and Koessler (2005), Bull and Watson (2007), Caillaud and Tirole (2007), Deneckere and Severinov (2008), Sher and Vohra (2015), Ben Porath and Lipman (2012), and Hart, Kremer and Perry (2017). A binary communication cost structure also underlies the literature on honesty (Alger and Ma (2003), Alger and Renault (2006, 2007), Severinov and Deneckere (2006)) and Kartik et. al (2007).

In contrast, our paper makes the much milder assumption that all types can send any message available to any other type, but at a cost that is increasing in the magnitude of type misrepresentation. In this assumption we follow the literature on costly state falsification, which originates in the work of Lacker and Weinberg (1989), and includes Maggi and Rodriguez-Clare (1995), Crocker and Morgan (1998), Crocker and Slemrod (2005), Goldman and Slezak (2006), and Picard (2013). The costly state falsification literature explores a wide range of applications, including insurance and tax fraud, financial misreporting, and legal evidence production. In a legal setting, Bull (2008a) studies costly evidence production and disclosure. Bull (2009) compares inquisitorial and adversarial litigation systems when evidence may be suppressed at a cost. Following scandals such as Enron, managerial misreporting has attracted significant attention in the accounting and finance literature (see, e.g., Fisher and Verecchia (2000), Goldman and Slezak (2006), Guttman, Kadan and Kandel (2006) and Caskey, Nagar and Petacchi (2010)).

Our second main assumption, that individuals undergo multiple tests or send multiple messages is also well-motivated and recognized, although it has not been studied extensively. The signaling literature has recognized the availability of multiple signals to convey private information since its early days (although it was explored formally only by a few authors, particularly, Engers (1987)). For example, a firm may use price, advertising, warranties, or brand name to signal the quality of its product. Similarly, the finance literature has considered a variety of signals conveying the future profitability of a firm, such as financial structure (Ross, 1977), dividend policy (Miller and Rock, 1985), stock splits (McNichols and Dravid, 1990) and share buybacks. Biologists have pointed out the multi-component nature of signals animals use to exhibit their quality in courtship, mating or predator deterrence (Johnstone, 1996).

The literature on screening has so far considered only a few environments with multiple messages. First, multiple pieces of evidence serve as messages in the cited literature on mechanism design with hard evidence. Second, legal scholars have studied the presentation of many pieces of evidence and multiple rounds of questioning in court (e.g., Emons and Fluet (2009)),

The rest of the paper is organized as follows. In section 2 we study the signaling version of our model. Section 3 deals with the screening model and optimal mechanisms. Section 4 concludes. The proofs are provided in the Appendix, except for the proofs for section 3.4 which are relegated to the online Appendix at *http://www.severinov.com/Appendix_costlycomm_tech.pdf*.

2 Signaling with Multiple Signals

2.1 Model

We begin by describing a signaling version of our model. It has two actors, a sender and a receiver. The sender first privately observes a realization of a payoff-relevant random variable θ referred to as the sender's type. The set of types is an interval $[\underline{\theta}, \overline{\theta}]$. Reflecting the receiver's uncertainty about the sender's type, we assume that θ is distributed according to a commonly known distribution function $F(.)$ with a continuous and positive density $f(.)$.

After learning θ, the sender sends n signals $\mathbf{m}^n = (m_1, ..., m_n)$ to the uninformed receiver, where m_i belongs to an interval $M_i \subseteq \mathbb{R}$ for all i. Having observed $\mathbf{m}^n$, the receiver forms posterior beliefs $\mu(\mathbf{m}^n)$ about θ and responds by choosing an action $x \in [\underline{x}, \overline{x}]$. The receiver's utility is measured by a function $v(x, \theta)$, the properties of which are described below.

The sender's payoff is given by $u(x, \theta) - C^n(\mathbf{m}^n, \theta)$, where $u(x, \theta)$ is her utility from action x and $C^n(\mathbf{m}^n, \theta)$ is the signaling cost. We assume that for each θ there is a message profile $\gamma^n(\theta) = (\gamma_1(\theta), ..., \gamma_n(\theta))$ that uniquely minimizes $C^n(\mathbf{m}^n, \theta)$ i.e.,

$$C^n(\mathbf{m}^n, \theta) \geq C^n(\gamma^n(\theta), \theta) \text{ for all } \mathbf{m}^n, \tag{1}$$

with strict inequality whenever $\mathbf{m}^n \neq \gamma^n(\theta)$. We also assume that for each $n \geq 1$ and any signal profile $\mathbf{m}^n$, $C^{n+1}(\mathbf{m}^n, \gamma_{n+1}(\theta), \theta) = C^n(\mathbf{m}^n, \theta)$. Without loss of generality, we normalize and assume that $C^n(\gamma^n(\theta), \theta) = 0$ for all θ.[5]

Since sending any signal profile other than $\gamma^n(\theta)$ is costly for type θ, we will henceforth refer to $\gamma^n(\theta)$ as type θ's costless signal profile. The profile $\gamma^n(\theta)$ can be regarded as "truthful" for the sender of type θ, as she would select this signal profile if she did not care about affecting the receiver's beliefs and action.

Many applications fit this simple model. In particular, the sender and receiver pair could be a student and an employer, a firm and its regulator(s), a manager and shareholders, an advisor and a decision-maker, a taxpayer and a tax authority, a bank and a monetary authority, or in the context of status-seeking and social norms problems, an individual and society.

In the student-employer relationship it is intuitive to think of θ as a measure of the student's ability, while signal m_i is the grade in the i-th course or performance in the i-th interview, and the action x is the wage assigned by the employer. In this context, $\gamma^n(\theta)$ is the profile of course grades that a student of ability θ would earn if she studied only as much as she liked and enjoyed, without being concerned about the effect of the grades on the job market prospects. Alternatively, it could be the student's natural performance level in an interview.

[5] A model in which $C^n(\gamma^n(\theta), \theta) = k(\theta) \neq 0$ is equivalent one in which the signal cost and utility function are given by $\hat{C}(\mathbf{m}^n, \theta) = C^n(\mathbf{m}^n, \theta) - k(\theta)$ and $\hat{u}(x, \theta) = u(x, \theta) - k(\theta)$.

In the manager-shareholder or firm-regulator setting, θ is typically a parameter of the production cost or demand facing the firm, while m_i is the i-th piece of evidence presented by the manager, such as the firm's performance characteristic (productivity, capacity utilization), a measure of demand, an accounting measure or data such as labor costs, results of customer surveys or audits, while x is manager's compensations, budget or project assigned to her. Here, $\gamma^n(\theta)$ stands for the true measures of the firm's performance and/or the market conditions. The assumption that message profile $\mathbf{m}^n$ s.t. $\mathbf{m}^n \neq \gamma^n(\theta)$ is more costly reflects that a manager can distort the firm's characteristics and fudge the numbers, but this requires costly effort.

In the bank-monetary authority example θ could represent the quality of the bank's credit portfolio, and m_i could be a reported characteristic of the bank's deposit and credit performance. In this context, x would denote the scale of the regulatory restrictions imposed by the monetary authority on the bank's credit and deposit policies. Similar interpretations apply to $\theta_i, m_i, \gamma_i(\theta_i)$ and x in the taxpayer-tax authority framework.

As another example, Bernheim (1994) studies a signaling model of status-seeking and social norms where the sender is an individual and the receiver is the society at large, and θ corresponds to the intrinsic preference for individual action, such as preferred style of dress, so that $\theta = \gamma(\theta)$. The signal corresponds to chosen individual action, while x is the status conferred on the individual by society. This model is readily extendable to multidimensional signals, as society observes many distinct aspects of individual behavior, including dressing style, conspicuous consumption, participation in charitable activities and giving, etc.

Bernheim and Severinov (2003) explore parents' signaling their preferences to multiple offspring via bequest allocation. Parents' type determines the ideal distribution of their resources between the children, and could potentially be conveyed via several actions and decisions taken by the parents.

A common feature of these examples is that each signal involves an action or a choice which has specific content and, to a certain degree, is independent of the other signals. Therefore, the effort/cost expended on one signal has a limited effect on the cost of the other signals. Particularly, in our reexamination of the job-market signaling, the signals are grades in individual courses in various fields. Taking each course requires performing course-specific activities such as reading the material, completing assignments and exams. Many fields also have different apparatus and methods, mastering which requires qualitatively new and independent cognitive effort. For instance, a student majoring in mathematics has to take courses in Analysis, Geometry, Algebra and other math disciplines, most of which have unique methodologies, tool sets and approaches. So studying hard in one course does not obviate the need to expend effort to earn a good grade in another course. Similarly, in a manager-shareholder relationship achieving (or fudging) large cost savings does not make it much easier for a manager to demonstrate high quality of the product or favorable customer perception of it.

Accordingly, our assumptions on the signalling cost function below require each signal to be independent of the other signals to a certain degree, so that increased misrepresentation along any signal dimension raises the overall signaling costs in a nonnegligible way. This is essential for our results. However, some interdependence between signals is natural. For example, as the sender sends more signals, she may learn to misrepresent her information more effectively. Alternatively, the effort spent generating one signal could fatigue the sender, raising her cost of misrepresentation in other dimensions. Accordingly, we allow the cost of signal m_i to depend on the other signals, as embodied in the non-separability of the communication cost function C^n in $(m_1, ..., m_n)$. Specifically, we make the following assumptions.

Assumption 1 *(i) There exists $L > 0$ s.t. $|\gamma_i(\theta) - \gamma_i(\theta')| \geq L|\theta - \theta'|$, for all i.*

(ii) There exists $\omega > 0$ such that $C^n(\mathbf{m}^n, \theta) - C^n(\gamma^n(\theta), \theta) \geq \omega||\mathbf{m}^n - \gamma^n(\theta)||^2$.

Assumption 1 (i) requires that the costless signals vary non-trivially with type. In the signaling model of education it means that students of different ability have different "natural grades" which they earn by studying only as much as they enjoy. When the signals are interviews, it says that, without interview preparation, a more able type would come through as such. In the manager-shareholder or firm-regulator context, it means that firms with different productivities or demands generate different true performance measures. This assumption is also natural in Bernheim's (1994) signaling model of social norms, since his notion of type coincides with the type's preferred action and the cost of signaling $g(m-\theta)$ increases in $|m-\theta|$.

Assumption 1 (ii) says that message profiles further away from $\gamma^n(\theta)$ are more costly for the sender and puts a lower bound on these costs. It reflects the relative independence of signals sent along different dimensions, although the cost effect of any particular signal may diminish if there are significant misrepresentations in many other dimensions. Both parts (i) and (ii) are natural in many contexts. For example, in the absence of uncertainty, it is optimal for the firms with higher quality products to set higher prices, offer longer warranties, and advertise more. Similarly, under perfect information, firms with different future investment opportunities will use different debt structures and dividend policies. In these contexts, a deviation from optimal policies in either direction is costly.

The receiver's utility function is denoted by $v(x, \theta)$, which we assume to be strictly concave in x. So for all posterior beliefs μ, the receiver's best response $BR(\mu)$ is unique, where

$$BR(\mu) = \arg\max_{x \in X} \int v(x, \theta) d\mu(\theta). \tag{2}$$

Let $x^*(\theta)$ denote the receiver's best response to point beliefs $\delta(\theta)$ at θ i.e.,

$$x^*(\theta) = \arg\max_{x} v(x, \theta) \tag{3}$$

We focus on pure strategy perfect Bayesian equilibria. Such an equilibrium is a triple $(\widetilde{\mathbf{m}}^n(\theta), \mu(\mathbf{m}^n), \widetilde{x}^n(\mu))$ where $\widetilde{\mathbf{m}}^n(\theta) = (\widetilde{m}_1(\theta), ..., \widetilde{m}_n(\theta))$ denotes the equilibrium signal vector of type θ, $\mu(\mathbf{m}^n)$ denotes the equilibrium beliefs of the receiver after she receives signal $\mathbf{m}^n$, and $\widetilde{x}^n(\mu)$ denotes the unique best response action of the receiver to beliefs μ. Beliefs $\mu(\mathbf{m}^n)$ are required to satisfy the standard consistency condition with the sender's equilibrium signal strategy, and the sender's signal strategy must maximize her expected payoff given the receiver's beliefs.[6] Let $U^n(\theta)$ denote the equilibrium expected utility of type θ. Then we have:

$$U^n(\theta) = u(\widetilde{x}^n(\mu(\widetilde{\mathbf{m}}^n(\theta)), \theta) - C^n(\widetilde{\mathbf{m}}^n(\theta), \theta). \tag{4}$$

With a slight abuse of notation, when it does not cause ambiguity, we will let $\widetilde{x}^n(\theta)$ denote the sender's best response to type θ's equilibrium signal profile $\widetilde{\mathbf{m}}^n(\theta)$.

To deal with the problem of multiplicity of equilibria that usually affects signaling games, we apply the dominance criterion refinement of Cho & Kreps (1987) to eliminate the equilibria that rely on implausible off-equilibrium beliefs. As it is the weakest refinement known to us, we have chosen it to make our results as strong as possible.

Specifically, fix some perfect Bayesian equilibrium of our game, and let $\mu^*(.)$ be the belief function and $\mathbf{m}$ be an out-of-equilibrium signal profile in this equilibrium. Dominance criterion specifies the following condition on the admissible principal's beliefs following $\mathbf{m}$: such beliefs should put no weight on a type θ if there is a message profile $\mathbf{m}'$ which dominates sending the message profile $\mathbf{m}$ for type θ in the following sense:

$$\min_{\mu} u(BR(\mu), \theta) - C^n(\mathbf{m}', \theta) > \max_{\mu} u(BR(\mu), \theta) - C^n(\mathbf{m}, \theta). \tag{5}$$

where $BR(.)$ is defined in (2). The inequality (5) says that it is better for type θ to send the message profile $\mathbf{m}'$ than $\mathbf{m}$, even if the receiver interprets the former in the worst possible way, and the latter - in the best possible way for the sender. When (5) holds, the dominance criterion requires the equilibrium belief $\mu^*(\mathbf{m})$ following $\mathbf{m}$ to put no weight on type θ.[7] If an equilibrium does not satisfy this condition, it is eliminated by the dominance criterion.

2.2 Discussion and Motivating Example

One may wonder whether allowing for multiple signals immediately implies our results. In particular, it is tempting to argue that misrepresentation can be easily prevented when there are many signals, as the signaling costs associated with misrepresentation then become sufficiently large to exceed any potential benefit. However, such logic is fallacious. Indeed, the signalling

[6]The restriction to pure strategy equilibria is imposed mainly for brevity. Allowing for mixed strategy equilibria is straightforward, but requires somewhat more cumbersome notation.

[7]An exception to this rule occurs if $\mathbf{m}$ is dominated for all θ. Then the receiver can hold any belief $\mu(\mathbf{m})$.

costs necessary to prevent misrepresentation may very well be invariant to the dimension of the signal space, as illustrated in the following simple example.

Example 1. Suppose that a sender (such as a job-applicant in education signalling model) has type θ. She sends n signals $(m_1, ..., m_n)$ (perhaps reflecting grades obtained in different courses) at cost $\sum_{i=1}^{n} \frac{m_i^\alpha}{\theta}$, where $\alpha > 0$, and then receives the payoff $v(\mu(m_1, ..., m_n))$, where $\mu(m_1, ..., m_n)$ is the receiver's beliefs about the sender's type given the signal profile $(m_1, ..., m_n)$. It is immediate that for any equilibrium $(\mu^*(m_1, ..., m_n), m_1(\theta), ..., m_n(\theta))$ of the model with n signals, there is an equilibrium of the model with a single signal m in which type θ sends signal $\hat{m}(\theta) = (\sum_{i=1}^{n} m_i(\theta)^\alpha)^{\frac{1}{\alpha}}$ and the receiver forms beliefs $\hat{\mu}(.)$ s.t. $\hat{\mu}(m) = \mu^*(m_1, ..., m_n)$ where $(m_1, ..., m_n)$ is any message profile satisfying $m = (\sum_{i=1}^{n} m_i^\alpha)^{\frac{1}{\alpha}}$. In this equilibrium, the sender incurs the same signaling cost and obtains the same payoff as in the model with n signals.

What is key in this example is not the additivity of the signal cost function, but rather that the lowest cost message is the same (zero) for all sender types. This ultimately implies that the total cost for any one type of a signal profile preventing mimicking of that type by another type is invariant to the number of signals. So, increasing the number of signals induces the sender to send a less costly signal in each dimension, yet results in the same equilibrium allocation (i.e., wage) and signaling cost as in the environment with one signal.

Our basic insight is that increasing the dimension of the signal space has an effect if the least cost signal varies non-trivially with the agent's type. This is illustrated by the following example where any type's signaling cost decreases as the number of signals grows.

Example 2. Consider the following modified version of Spence's education model. A worker can produce q units of output at cost $\frac{q^2}{2}$ which provides benefit θq to the employer, where $\theta \in [0, 1]$ is the worker's type privately known by her and distributed according to $F(\theta)$. The worker's utility function is given by $u(q, t, \theta) = t - \frac{q^2}{2}$, and the employer's net benefit is $v(q, t, \theta) = \theta q - t$, where t is the employer's transfer to the worker.

The worker obtains education before entering the job-market. The observable characteristic of education is a profile of grades $\mathbf{m}^n = (m_1, ..., m_n)$, with $m_i \geq 0$ standing for the worker's grade in course i. Education is not productive, but carries a type-dependent cost given by

$$C^n(\mathbf{m}^n, \theta) = \frac{1}{2} \sum_{i=1}^{n} (m_i - \theta)^2. \tag{6}$$

The employment market is competitive, so that the worker's net utility before education costs is equal to the expected social surplus $\max_q E_\mu(\theta q - \frac{q^2}{2})$, where μ is the employer's beliefs about the worker's type θ.

In a separating equilibrium with equilibrium signalling strategy $\widetilde{m}^n(\theta)$, the beliefs on the equilibrium path are degenerate, i.e. $\mu(\widetilde{m}^n(\theta)) = \delta(\theta)$. The competitive markets assumption implies that the quantity allocation in a separating equilibrium must be the first-best, i.e. $\widetilde{q}(\theta) = \arg\max_q \left(\theta q - \frac{q^2}{2}\right) = \theta$, and that the worker receives all surplus, so that $t(\theta) = \widetilde{q}(\theta)\theta = \theta^2$. Also, because of the symmetry in the signal cost function, all equilibrium grades are the same, i.e. $\widetilde{m}_i(\theta) = \widetilde{m}(\theta)$ for all $i = 1, ..., n$. Using the latter property and $q(\theta) = \theta$ for all θ, the following incentive constraints have to hold for every pair (θ, θ'):

$$U^n(\theta) \equiv \frac{\theta^2}{2} - \frac{n}{2}(\widetilde{m}(\theta) - \theta)^2 \geq \frac{1}{2}(\theta')^2 - \frac{n}{2}(\widetilde{m}(\theta') - \theta)^2 \tag{7}$$

Thus the right-hand side of (7) is maximized at $\theta' = \theta$, producing the first-order condition $\theta - n(\widetilde{m}(\theta) - \theta)\frac{d\widetilde{m}}{d\theta} = 0$, which admits a solution

$$\widetilde{m}(\theta) = (1 + s)\theta, \quad \text{where} \quad s = \frac{1}{2}\left(\sqrt{1 + \frac{4}{n}} - 1\right). \tag{8}$$

Using (8) on the right-hand side of (7) confirms that the payoff that type θ gets by imitating type θ' is concave in θ'. So this signaling strategy, indeed, sustains a separating equilibrium.

To survive the dominance criterion, the beliefs following an off equilibrium message profile $\mathbf{m}$ must satisfy $\mathbf{E}_{\mu(\mathbf{m})} \sum_{i=1}^{n}(m_i - \theta)^2 \leq 2$.[8] So, to support our equilibrium we can utilize the following off-equilibrium point beliefs: $\mu(\mathbf{m}) = \delta(\theta')$, where θ' is the lowest type $\theta \in [0, 1]$ s.t. $\sum_i (m_i - \theta)^2 \leq 2$ and $\mu(\mathbf{m}^n) = \delta(1)$ if the last inequality fails for all $\theta \in [0, 1]$.

The worker's equilibrium signaling cost is given by:

$$\frac{n}{2}(\widetilde{m}(\theta) - \theta)^2 = \frac{n}{2}s^2\theta^2 = \frac{\theta^2}{8}\left[\sqrt{n+4} - \sqrt{n}\right]^2 \tag{9}$$

It is easily verified from (8) and (9), correspondingly, that the misrepresentation in each message and equilibrium signaling cost monotonically decrease to zero with the number of signals. To see how quickly signaling costs decrease, Table 1 expresses the ratio of equilibrium net utility to the first-best surplus $S^{FB}(\theta) = \frac{\theta^2}{2}$ as a function of n:

Table 1.

n	1	2	3	4	5	6	7	8	9	10
$C^n(\mathbf{m}^n, \theta)/S^{FB}(\theta)$	38%	27%	21%	17%	15%	13%	11%	10%	9%	8%

[8] An off equilibrium signal profile $\mathbf{m}$ is not dominated for type θ iff for every message profile $\mathbf{m}'$ we have:.

$$1 - \frac{1}{2}\mathbf{E}_{\mu(\mathbf{m})}\sum_{i=1}^{n}(m_i - \theta)^2 \geq 0 - \frac{1}{2}\sum_{i=1}^{n}(m_i' - \theta)^2$$

Since the right-hand side of this inequality is maximized at $m_i' = \theta$ for all i, the beliefs must be concentrated on θ for which $\sum_i (m_i - \theta)^2 \leq 2$. If this set is empty, no type can ever gain by sending the signal profile $\mathbf{m}$, and we may set $\mu(\mathbf{m}) = \delta(1)$. It is easy to show that these beliefs support the separating outcome.

Recall that, besides the signalling costs, the equilibrium allocation is the same as the first-best one. So, Table 1 indicates that signalling costs fall off and the loss of surplus diminishes quite fast as the number of signals increases.

2.3 The Main Result for the Signalling Model

Our main result for the signalling model is presented in the following Theorem.

Theorem 1 *There exists $K < \infty$ such that in every PBE satisfying the dominance criterion supp.$(\mu(\widetilde{\mathbf{m}}^n(\theta))) \subset [\theta - \frac{K}{\sqrt{n}}, \theta + \frac{K}{\sqrt{n}}]$, $|\widetilde{x}^n(\theta) - x^*(\theta)| \leq \frac{K}{\sqrt{n}}$, $C^n(\widetilde{\mathbf{m}}^n(\theta), \theta) \leq \frac{K}{\sqrt{n}}$ for all θ.*

Thus as n increases, for any type θ: (i) the support of the beliefs given any equilibrium signal profile $\widetilde{\mathbf{m}}^n(\theta)$ collapses to the point θ; (ii) the equilibrium action $\widetilde{x}^n(\theta)$ converges to the principal's first-best action; (iii) the equilibrium signaling cost $C^n(\widetilde{\mathbf{m}}^n(\theta), \theta)$ converges to zero.

According to Theorem 1 To understand the intuition behind this Theorem, let us first think of the model with a single signal and focus on its least-cost (Riley) separating equilibrium, which exists under simple conditions on $u(.)$ and $C(.)$ such as single-crossing. Let $m_1^*(\theta)$ denote type θ's equilibrium signal. Then $C(m_1^*(\theta), \theta)$ is positive and nonnegligible for almost every type θ.

Now let us introduce a second signal, and suppose that the sender selects his least cost message as a second signal, while keeping his first signal unchanged. Then each type's signal cost will remain unchanged. But since $\gamma_2(\theta') \neq \gamma_2(\theta)$ the cost to imitating type θ will increase for every other type θ'. So, this extra signal relaxes global incentive constraints.

More significantly, the message profile $(m_1^*(\theta), \gamma_2(\theta))$ also satisfies local incentive constraints, which are the binding ones in a separating equilibrium. To see this, note that local incentive constraints in a separating equilibrium are characterized by the following first-order condition obtained by differentiating the receiver's equilibrium payoff function $u(x^*(\hat{\theta}), \theta) - C^n(\mathbf{m}^*(\hat{\theta}), \theta)_{|\hat{\theta}=\theta}$ with respect to $\hat{\theta}$:

$$u_x(x^*(\theta), \theta)\frac{dx^*}{d\theta} = \sum_{i=1}^{n} \frac{\partial C^n}{\partial m_i}(\mathbf{m}^*(\theta), \theta)\frac{dm_i^*}{d\theta}. \tag{10}$$

Importantly, $\frac{\partial C^2}{\partial m_2}(m_1^*(\theta), \gamma_2(\theta)) = 0$ since $\gamma_2(\theta)$ is type θ's least cost signal, while the identity $C^2(m, \gamma_2(\theta), \theta) = C^1(m, \theta)$ implies that $\frac{\partial C^2}{\partial m_1}(m, \gamma_2(\theta), \theta) = \frac{\partial C^1}{\partial m_1}(m, \theta)$. Thus $(m_1^*(\theta), \gamma_2(\theta))$ satisfies the first-order condition (10), because $m_1^*(\theta)$ satisfies it with $n = 1$. So $(m_1^*(\theta), \gamma_2(\theta))$ is an equilibrium signal profile with $n = 2$. In fact, no matter how many extra signal dimensions we introduce, the signal profile $(m_1^*(\theta), \gamma_{-1}^n(\theta))$ continues to be an equilibrium profile carrying the same message cost for all n.

However, the dominance criterion eliminates such equilibria as unreasonable. Indeed, suppose that n is large and instead of sending the above signal profile, the sender deviates and sends the profile $\gamma^n(\theta)$ instead. Since this signal profile is prohibitively costly for a type θ' outside a small neighborhood of θ, the dominance criterion requires that the receiver's beliefs following $\gamma^n(\theta)$ be concentrated on such small neighborhood. As a consequence, type θ can earn approximately $u(x^*(\theta), \theta)$ at zero cost, thereby breaking the above equilibrium.

Theorem 1 establishes that a similar logic applies to all equilibria. The argument in the previous paragraph shows that separating equilibria with a non-negligible signaling cost do not survive the dominance criterion. Further, if an equilibrium with a large n contained a pooling interval of a significant size, then the pooling signal profile will be prohibitively costly to a type at one of the endpoints of this interval. So any equilibrium surviving the dominance criterion for large n must be approximately revealing and have small signalling cost.

To see how one can construct equilibria surviving the dominance criterion, recall that at the signal profile $(m_1^*(\theta), \gamma_2(\theta))$ the cost of imitating type θ is larger than the cost of sending a single message $m_1^*(\theta)$ for any type θ'. Further, we can simultaneously lower $|m_1^2(\theta) - \gamma_1(\theta)|$ and raise $|m_2^2(\theta) - \gamma_2(\theta)|$ while preserving the local incentive constraints (10). As a result of this change, the signalling cost of type θ will decrease because the first-order effect of raising $m_2^2(\theta)$ around $\gamma_2(\theta)$ on θ's cost is negligible, while bringing $m_1^2(\theta)$ closer to $\gamma^1(\theta)$ has the first-order effect of lowering θ_1's cost.

Some degree of misrepresentation persists, as the marginal cost of misrepresentation is zero at any type's costless signal profile. However, the signaling costs must vanish as n becomes large because, as shown above, the equilibrium allocation converges to the fully revealing one, $x^*(\theta)$, and the dominance criterion implies that the agent-type θ can ensure almost the same allocation by repeatedly sending her costless message.

In the context of job-market signaling, Theorem 1 implies that a student of every ability gets grades that are very close to her "natural" ones. Significantly, the fact that the overall signalling costs go to zero means that there is no unproductive and exhausting 'rat race' between students trying to achieve better grades in order to obtain higher-paying jobs. This conclusion has significant implication and policy prescriptions for the design of educational programs and degree requirements. Indeed, negative welfare consequences of competition for grades between students have motivated several Business schools, including Wharton, Chicago, Stanford and Berkeley to adopt grade non-disclosure policies which prohibit students from revealing grades to recruiters. Our results provide a different perspective and a prescription for dealing with this problem. Specifically, we argue that effort overinvestment can be avoided if educational programs are constructed of a larger number of more distinct courses. An important element of such design is relative independence and non-overlap of material between courses, making the performance in each course a separate and sufficiently independent signal.

2.4 Signaling and Endogenous Size of the Signal Space

In some contexts, signals are a natural part of the economic environment, and their presence by itself does not require any additional cost. For example, this is the case when a firm has unobservable product quality: it must select its price, advertising, warranty, financing options, return policy, service time, etc., and all these decisions have a signaling content. However, these characteristics must also be chosen if the firm's quality were known.

In other contexts, there may be fixed costs associated with setting up each signal, such as a test or an interview. For example, offering new courses requires colleges to incur additional expenses associated with their development and administration. Such costs may then endogenously limit the number of available signals. To illustrate, let G denote *per student* fixed cost of a course. The socially optimal number of courses n^* then maximizes per capita expected benefit $EU^n(\theta) - nG = E(u(x^*(\theta), \theta) - C^n(\mathbf{m}^n(\theta), \theta)) - nG$. For the example of Section 2.2, Table shows how n^* and the expected surplus obtained by a student varies with per capita fixed cost expressed as a percentage of expected first-best surplus, $G/ES^{FB}(\theta)$.

Table 2.

$G/ES^{FB}(\theta)$	17%	7.5%	4.5%	3.0%	2.2%	1.6%	1.3%	1.0%	0.85%	.7%
n^*	1	2	3	4	5	6	7	8	9	10
$\frac{EU^n(\theta)-nG}{ES^{FB}(\theta)}$	45%	58%	65.5%	71%	74%	77.4%	79.9%	82%	83.35%	85%

Thus if the fixed cost is 1% of the first-best surplus, eight courses are optimal. If students bore all of the fixed cost, then they would receive 82% of the first-best surplus. Arguably, in most cases per capita fixed cost of each course is significantly less than 1% of the first-best surplus, implying that optimal education programs need to include more than 10 courses.

2.5 One-Sided Signal Costs

So far, we have assumed that each sender type has a unique costless signal, and sending either higher or lower signals is costly. This assumption is natural in many applications, such as signaling quality by price, advertising and warranties. However, in some applications, it is costless for a sender of type θ to send any signal below $\gamma_i(\theta)$. For example, in earning grades or performing on job interviews, it is often easy for a student to pretend to be less capable than her natural level $\gamma_i(\theta)$.[9] In this case, the cost function satisfies $C^n(m_i, \mathbf{m}^n_{-i}, \theta) = C^n(\gamma_i(\theta), \mathbf{m}^n_{-i}, \theta)$ if $m_i \leq \gamma_i(\theta)$. In particular, if $\mathbf{m}^n \leq \gamma^n(\theta)$ then $C^n(\mathbf{m}^n, \theta) = 0$. For all other signal profiles

[9] We thank an anonymous referee for suggesting this line of inquiry.

we maintain Assumption 1, so we have $C^n(\mathbf{m}^n,\theta) - C^n(\gamma^n(\theta),\theta) \geq \alpha \sum_i^n \max\{m_i - \gamma_i(\theta), 0\}^2$. We will refer to such cost functions as "one-sided."

Naturally, informative signaling is only possible in this case if the sender's desired direction of imitation is to exaggerate one's type. To ensure this, we make the following assumptions:

Assumption 2 *(i) $v_{\theta x}(x,\theta) > 0$ for all (x,θ); (ii) $u_x(x^*(\theta'),\theta) > 0$ for all $\theta' \leq \theta$.*

Assumption 2(i) guarantees that higher beliefs induce higher receiver responses. Assumption 2(ii) then implies that every type prefers to be perceived as a higher type. It is then natural to consider monotone perfect Bayesian equilibria in which the sender's equilibrium signal profile is increasing in θ, i.e. $\widetilde{\mathbf{m}}^n(\theta') \geq \widetilde{\mathbf{m}}^n(\theta)$ whenever $\theta' > \theta$. We then have the following analogue of Theorem 1:

Theorem 2 *Consider monotone perfect Bayesian equilibria satisfying the dominance criterion. Then as n increases, the support of the beliefs following equilibrium signal profile $\widetilde{\mathbf{m}}^n(\theta)$ for any θ collapses to the point θ, so the equilibrium action $\widetilde{x}^n(\theta)$ converges to the fully revealing one, $x^*(\theta)$, and signal costs $C^n(\widetilde{\mathbf{m}}^n(\theta),\theta)$ converge to zero for any θ.*

As an illustration, consider the one-sided cost version of the example from Section 2.2.2. The corresponding one-sided signal cost function is $C^n(m_1,...,m_n,\theta) = \frac{1}{2}\sum_i \max\{m_i - \theta, 0\}^2$. Then it is still an equilibrium for the worker of type θ to send a signal $\widetilde{m}_i(\theta) = \frac{1}{2}\left(\sqrt{1+\frac{4}{n}}+1\right)\theta$ in each dimension (which can be interpreted as a grade in each course). Intuitively, this is because with each type sending signals above $\gamma_i(\theta)$, the local incentive constraints under one-sided and two-sided signal costs are identical. ff equilibrium point beliefs that survive the dominance criterion and support the separating outcome are the same as in Section 2.2.2.

The importance of Theorem 2 lies in establishing the robustness of the convergence of equilibria to a perfectly revealing outcome with negligible signalling costs as the number of signals increases under a broader set of cost conditions. In particular, a one-sided cost function reflects a realistic scenario when it is only costly to exaggerate one's type and perform better than at the natural level, and it is important to know that our main results survive in this case.

3 Screening with Multiple Messages

In this section, we study a screening version of our model and show that the main conclusions of the signaling model also apply in the screening context. In addition, we investigate the optimal mechanism with an exogenously fixed and an endogenously chosen number of messages.

To set up the model, suppose that the principal controls the allocation (q,t) which consists of a production/consumption decision $q \in \mathbf{R}_+$ and a monetary transfer t. The agent of type θ

obtains utility $t - h(q,\theta)$ and the principal obtains utility $v(q) - t$, where $v(q) \geq 0$, $h(q,\theta) \geq 0$ and $h(0,\theta) = 0$ for all θ, with $v(.)$ and $h(.)$ twice continuously differentiable. An intuitive interpretation is that the agent produces quantity q at a cost $h(q,\theta)$, and the principal gets benefit $v(q)$ from this quantity. The type θ is randomly drawn according to the distribution function $F(\theta)$ with support $\Theta = [\underline{\theta}, \overline{\theta}]$. The agent has an outside option that yields zero utility.

To this standard screening environment we add a costly signal/message process. [10] Specifically, as in the previous section, when sending a vector of messages $\mathbf{m}^n = (m_1, ..., m_n)$ the agent of type θ incurs the cost $C^n(\mathbf{m}^n, \theta)$, where $C^n(.)$ is a twice continuously differentiable function with type-dependent "least-cost" signal profile denoted by $\gamma^n(\theta)$. The agent's net payoff is thus given by $t - h(q,\theta) - C^n(m_1, ..., m_n, \theta)$.

By the Revelation Principle, a mechanism offered by the principal can without loss of generality be represented by a mapping $(\mathbf{m}^n(.), q(.), t(.))$ from the agent's type space into the space of allocations. One can think of a mechanism as a menu of options $\{\mathbf{m}^n(\theta), q(\theta), t(\theta)\}_{\theta\in\Theta}$ offered to the agent. An allocation includes messages because the latter are payoff-relevant and can be also thought of as the agent's actions. The part of allocation $(q(\theta), t(\theta))$ is implemented after the agent sends message profile $\mathbf{m}^n(\theta)$.[11] Without loss of generality, we can assume that after an off-equilibrium vector of signals $\mathbf{m}^n$ s.t. $\mathbf{m}^n \neq \mathbf{m}^n(\theta)$ for any θ, the principal assigns $(q,t) = (0,0)$.

We will say that an allocation profile $(\mathbf{m}^n(.), q(.), t(.))$ is implementable if the following incentive constraints hold for all θ, θ':

$$U(\theta) = t(\theta) - h(q(\theta),\theta) - C^n(\mathbf{m}^n(\theta),\theta) \geq \max_{\theta'\in\Theta} t(\theta') - h(q(\theta'),\theta) - C(\mathbf{m}^n(\theta'),\theta), \qquad (11)$$

Individual rationality requires that $U(\theta) \geq 0$.

The following example illustrates the main result of this section demonstrating that the principal can implement the first-best allocation profile and obtain almost all surplus, keeping the signaling cost arbitrarily small when the number of messages is large.

Example 3. The agent supplies output $q \in \mathbb{R}_+$ to the principal in exchange for a payment $t \in \mathbb{R}_+$. Output q generates surplus $S(q) = q - \frac{q^2}{2}$ for the principal, so her net payoff is:

$$S(q) - t = q - \frac{q^2}{2} - t.$$

The agent's type $\theta \in [0,1]$ is her constant marginal cost of production. She can submit a

[10] Both terms "signal" and "message" are appropriate here in the screening settings as they refers to agent's actions which possess only informational content.

[11] Another way to think of a mechanism formally is as follows. The agent sends a type report $\hat{\theta}$ and then the principal recommends a profile of costly signal $\mathbf{m}^n(\hat{\theta})$ and assigns the allocation $(q(\hat{\theta}), t(\hat{\theta}))$.

profile of messages $(m_1, ..., m_n)$ about her marginal cost incurring the misrepresentation cost

$$C^n(m_1, ..., m_n, \theta) = \frac{1}{2}\sum_{i=1}^{n}(m_i - \theta)^2$$

Thus the agent's net utility is given by:

$$t - \theta q - C^n(m_1, ..., m_n, \theta) = t - \theta q - \frac{1}{2}\sum_{i=1}^{n}(m_i - \theta)^2 \tag{12}$$

The principal's first-best allocation, which she would implement if she knew the agent's marginal cost, involves output $q^{FB}(\theta) = 1 - \theta$ maximizing the social surplus $q - \frac{q^2}{2} - \theta q$, and a transfer to the agent $t^{FB}(\theta) = \theta(1-\theta)$, which leaves the agent at the reservation utility zero.

The mechanism which the principal offers to the agent can be represented as profile $(m_1(\theta), ..., m_n(\theta), q(\theta), t(\theta))$. Let us show how the principal can choose $(m_1(\theta), ..., m_n(\theta)$ to implement $q(.) = q^{FB}(.)$ so that the misrepresentation cost $C^n(\mathbf{m}^n(\theta), \theta)$ becomes arbitrarily small when n becomes large, and all agent types are kept at their reservation utility level 0.

With $q(.) = q^{FB}(.)$, incentive compatibility condition and the condition that each agent-type's net utility is zero can be written as follows:

$$0 = U^n(\theta) \equiv t^n(\theta) - \theta q^{FB}(\theta) - \frac{1}{2}\sum_{i=1}^{n}(m_i(\theta) - \theta)^2 = \max_{\theta'}\left\{t^n(\theta') - \theta q^{FB}(\theta') - \frac{1}{2}\sum_{i=1}^{n}(m_i(\theta') - \theta)^2\right\} \tag{13}$$

Applying the Envelope Theorem yields the following necessary condition for (13) to hold:

$$0 = U'(\theta) = -q^{FB}(\theta) + \sum_{i=1}^{n}(m_i(\theta) - \theta) \tag{14}$$

Using (14) and choosing the same message in very dimension yields $m_i(\theta) = \theta + \frac{q^{FB}(\theta)}{n} = \theta + \frac{1-\theta}{n}$. So, equilibrium messages/signals become less distorted and converge to the costless signal as this number increases, while the associated communication costs converges to zero:

$$C^n(m_1(\theta), ..., m_n(\theta), \theta) = \frac{1}{2}\sum_{i=1}^{n}(m_i(\theta) - \theta)^2 = \frac{n}{2}(m_i(\theta) - \theta)^2 = \frac{(1-\theta)^2}{2n}.$$

Under our message profile $(m_1(\theta), ..., m_n(\theta)$, the transfer function $t^n(\theta)$ satisfying $U^n(\theta) = 0$ is $t^n(\theta) = \theta q^{FB}(\theta) + \frac{1}{2}\sum_{i=1}^{n}(m_i(\theta) - \theta)^2 = (1-\theta)\left(\theta + \frac{1-\theta}{2n}\right)$, and by (13) the payoff of type θ when imitating θ' is given by:

$$(1-\theta')\left(\theta' + \frac{1-\theta'}{2n}\right) - \theta(1-\theta') - \frac{n}{2}\left(\theta' + \frac{1-\theta'}{n} - \theta\right)^2, \tag{15}$$

It is easy to check that (15) is concave in θ' and its derivative with respect to θ is equal to zero at $\theta' = \theta$. So out contract satisfies the agent's incentive constraints. In our more general case, such concavity is not guaranteed. Nevertheless, we show that the same result holds under simple technical assumptions on the communication cost function.

3.1 Main Result for the Screening Model

To deliver our main result, we need the following Assumption regarding the communication cost function $C^n(\mathbf{m}^n, \theta)$:

Assumption 3 *There exist constants $L, \omega_1, \omega_2 > 0$:*
(i) $|\gamma_i(\theta) - \gamma_i(\theta')| \geq L\ |\theta - \theta'|$, for all i;
(ii) $\omega_1 ||\mathbf{m}^n - \gamma^n(\theta)||^2 \leq C^n(\mathbf{m}^n, \theta) \leq \omega_2 ||\mathbf{m}^n - \gamma^n(\theta)||^2$;
(iii) $C^n_{\theta m_i}(\mathbf{m}^n, \theta) \leq -\omega_1$ for all i and $(\mathbf{m}^n, \theta)$;
(iv) $C^n_{\theta\theta}(\mathbf{m}^n, \theta) \geq \omega_1 n$ whenever $||\mathbf{m}^n - \gamma^n(\theta)|| \leq \omega_1$.

Part (i) and the first inequality in part (ii) of Assumption 3 carry over from the signaling context and explained there. In particular, the latter reflects that signals sent along different dimensions are relatively independent of each other, although the size of this effect may diminish if there are significant misrepresentations in many other dimensions.

The second inequality in Assumption 3 (ii) is rather innocuous, as it only requires $C^n(\cdot, \theta)$ to be majorized by a quadratic function centered around $\gamma^n(\theta)$. Assumption 3(iii) requires $C^n(\mathbf{m}^n, \theta)$ to have increasing differences in $(\mathbf{m}^n, \theta)$, and is used to establish the existence and uniqueness of a message profile that satisfies local incentive constraints. Assumption 3(iv) requires C^n to be convex in θ in a neighborhood of $(\gamma^n(\theta), \theta)$, to a degree that grows with n. In fact, it is just a continuity requirement, since other parts of Assumption 3 imply $C^n_{\theta\theta}(\gamma^n(\theta), \theta) \geq \omega_1 n$. These assumptions are easily satisfied by most common specifications of the cost function and, in particular, by the cost function $C^n(\mathbf{m}^n, \theta) = \frac{1}{2}\sum_{i=1}^n (m_i - \theta)^2$.

The main Theorem for the screening model is presented in the following Theorem:

Theorem 3 *Suppose Assumption 3 holds. Let $(q(\cdot), t(\cdot))$ be any pair of twice continuously differentiable quantity and transfer functions such that $U(\theta) \equiv t(\theta) - h(q(\theta), \theta) \geq 0$ for all θ. Then there exist N such that whenever the number of signals n exceeds N, there are transfer and message functions, $t^n(\cdot)$ and $\mathbf{m}^n(\cdot)$ s.t. the mechanism $(\mathbf{m}^n(\cdot), q(\cdot), t^n(\cdot))$ is incentive compatible and satisfies $U(\theta) = t^n(\theta) - h(q(\theta), \theta) - C^n(\mathbf{m}^n(\theta), \theta)$, with the signal cost satisfying $C^n(\mathbf{m}^n(\theta), \theta) < \frac{1}{n}$ for all θ.*

Theorem 3 establishes that the set of implementable allocation profiles grows as the number of signals increases, because adding extra signals allows to satisfy incentive constraints which do not hold in the profile $(q(\cdot), U(\cdot))$ which we wish to implement. Naturally, the larger is the extent to which some incentive constraints are violated in the absence of costly messages, the higher is the number of costly messages required to ensure that these constraints hold in the presence of misrepresentation costs. Significantly, the magnitude of communication costs decrease quickly with the number of messages n, at the rate $\frac{1}{n}$. Since by construction $t^n(\theta) = t(\theta) + C^n(\mathbf{m}^n(\theta), \theta) \leq t(\theta) + \frac{1}{n}$, the principal's payoff in mechanism $(\mathbf{m}^n(\cdot), q(\cdot), t^n(\cdot))$ differs from her payoff under $(q(\cdot), t(\cdot))$ by at most $\frac{1}{n}$. Thus, the principal can extract (almost) all surplus when the number of messages is large.

To see why any profile $(q(\cdot), U(.))$ becomes implementable for sufficiently high n, observe first that large deviations from the associated signal profile $\mathbf{m}^n(.)$ do not pay for any type θ, because such deviations become prohibitively costly, as assured by Assumptions 3(i)- (ii).

Small deviations from $\mathbf{m}^n(.)$ are not so easily deterred, however, because they necessarily involve small signal costs, even when the dimension of the signal space is large. More specifically, note that implementing an allocation profile $q(\cdot)$ together with agent's surplus function $U(\cdot)$ requires the following (local) envelope condition to hold:

$$U'(\theta) = -h_\theta(q(\theta), \theta) - C^n_\theta(\mathbf{m}^n(\theta), \theta). \tag{16}$$

Condition (16), which is a counterpart of condition (10) for the signaling model, implies that $(q(\cdot), U(.))$ determines the slope of the signal cost function, $C^n_\theta(\mathbf{m}^n(\theta), \theta)$. Assumption 3 (iii) ensures that there exists $\mathbf{m}^n(\theta)$ satisfying (16), and this $\mathbf{m}^n(\theta)$ converges uniformly to $\gamma^n(\theta)$ as the number of signals increases. The second inequality in Assumption 3(ii) ensures that the corresponding cost $C^n(\mathbf{m}^n(\theta), \theta)$ converges to zero. Assumption 3 (iv) implies that for large n the agent's payoff function becomes strictly concave in signals $\mathbf{m}^n$ on a neighborhood of $(\gamma^n(\theta), \theta)$, thereby guaranteeing that sending $\mathbf{m}^n(\theta)$ is a local optimum.

The result of Theorem 3 is of economic significance. Particularly, it suggests that an optimal method of dealing with the problem of asymmetric information regarding employees' abilities may involve the design of testing and interviewing procedures, rather than on-the-job screening via incentive schemes. This can explain why incentive schemes offered in a variety of industries are not as steep and high-powered as incentive literature may suggest.

It also provides an explanation why the job interview process has steadily become lengthier and more complex. Indeed, in most professional industries nowadays this process involves several rounds of interviews.[12] The employers use a variety of techniques, including situational, behavioral, case, stress, technical and panel interviews. All of these techniques and

[12]For anecdotal evidence see e.g. Sue Shellenbarger "How to Deal With a Long Hiring Process: Getting a Job Takes Longer Than Ever ...," Wall Street Journal, Jan. 19, 2016. In "Death by Interview: Revealing the Pain Caused by Excessive Interviews (ERE Recruiting Intelligence, May 2013), John Sullivan reports that "a well-known technology firm ... dictated that every candidate undergo 17 interviews."In 2014, at Google "people sat for 15 to 25 interviews" (T. Popomaronis, "Here's How Many Google Interviews it Takes to Hire a

methods allow to assess a candidate's ability from different perspectives and make the signals obtained through different interviews to a large extent independent of each other. Some of this complexity is undoubtedly due to the principal's desire to reveal multiple dimensions of ability, but much of it is geared towards making it difficult for candidates to exaggerate their capabilities and misrepresent their skills.

Similarly, industry observers have documented that the number of reports, surveys, audits, etc. that managers and firms need to provide to shareholders, auditors, boards, institutional investors and regulators has been steadily increasing. For example, a comprehensive Ernst and Young survey reports that "... companies face increasing demands on reporting, driven by both internal and external stakeholders. The number of reports companies issue is growing, and reporting is expanding to include more financial and non-financial aspects."[13] According to the theory developed in this paper, costliness of the misrepresentation in these submissions benefits the stakeholders by making the eliciting of information and the allocation more efficient.

3.2 Screening with One-sided Cost of Misrepresentation

As in the signaling part of the paper, in this subsection we extend our model to a set-up where only one direction of misrepresentation is costly. For example, in a procurement or regulation setting a firm may be able to understate its cost of production at will without any difficulty, simply by omitting to include some inputs in its costs. However, cost padding requires effort and time to generate additional invoices, change databases, etc.

Accordingly, assume that $C^n(m_i, \mathbf{m}_{-i}, \theta) = C^n(\gamma_i(\theta), \mathbf{m}_{-i}, \theta)$ for any $m_i < \gamma_i(\theta)$, implying in particular that $C^n(\mathbf{m}^n, \theta) = C^n(\gamma^n(\theta), \theta) = 0$ whenever $\mathbf{m}^n \leq \gamma^n(\theta)$. We will say that such a signal cost function is "one-sided."[14]

We maintain Assumption 3 on $C^n(\mathbf{m}^n, \theta)$ with the appropriate modification replacing the difference $||\mathbf{m}^n - \gamma^n(\theta)||$ with $||(\mathbf{m}^n \vee \gamma^n(\theta)) - \gamma^n(\theta)||$, where $\mathbf{m}^n \vee \gamma^n(\theta)$ is the coordinate-wise maximum of $\mathbf{m}^n$ and $\gamma^n(\theta)$, in parts (ii) and (iv) of the Assumption.

In this case, the scope of implementability is more restrictive. However, under additional assumption we can extend the result of Theorem 3 as follows.

Theorem 4 *Suppose that the signal cost function $C^n(\mathbf{m}, \theta)$ is one-sided. Also, suppose that $h_q(q, \theta) > 0$, and $h_{q\theta}(q, \theta) > 0$.*

Googler," CNBC "Make It," April 17 2019).

[13] "Connected Reporting: Responding to Complexity and Rising Stakeholder Demands," Ernst and Young FAAS group, 2014, p. 8.

[14] A model in which an agent can send any lower messages/evidence for free but is not able to exaggerate available evidence is studied by Rappoport (2017) . His analysis focuses of a different set of problems, and in particular, studies how the equilibrium rules of inference change with evidence structure.

Then the result of Theorem 3 holds for every pair of twice continuously differentiable functions (q, t) *such that* $q'(\theta) \leq 0$ *and* $t'(\theta) - h_q(q(\theta), \theta)q'(\theta) \geq 0$ *for all* θ.

With one-sided signal costs, it becomes impossible to implement an allocation rule $q(\theta)$ with surplus function $U(\theta) = t(\theta) - h(q(\theta), \theta)$ that generate incentives to deviate downwards. Preventing such deviations would require messages $\mathbf{m}^n$ that are more costly for higher types than for lower types, but such messages are not available under one-sided communication cost. The assumption $t'(\theta) - h_q(q(\theta), \theta)q'(\theta) \geq 0$ rules out local incentives to deviate downwards.

Furthermore, we also need to impose the monotonicity condition $q'(\theta) \leq 0$, which is standard in incentive problems. It becomes necessary because global incentive constraints in the downward direction cannot be ensured by the communication costs alone, since higher types can imitate lower types for free under one-sided communication costs.

However, one-sided communication costs still help to satisfy the upwards incentive constraints, and this direction of imitation is typically the more important one. Indeed, in the set-up of this section it is most natural to interpret θ as a production cost parameter, with higher θ corresponding to a higher marginal cost. The lower-cost types wish to imitate higher cost types, and the one-sided signal costs allow these incentive constraints to be satisfied with a minimal loss of efficiency when the number of messages is large. In fact, one-sided signal costs are sufficient for implementation of the optimal mechanisms considered in the next section.

3.3 Optimal Mechanism

In this section, we characterize the optimal mechanism maximizing the principal's expected payoff with a given number of messages n. Theorem 3 says that ultimately with a large n, the principal can extract full first-best surplus. But when the number of messages is limited, optimal mechanism design is an important issue.

In this analysis, we need some additional assumptions. First, we assume that the payoff functions $v(q)$ and $h(q, \theta)$ are twice continuously differentiable with $v(0) = h(0, \theta) = 0$, $h_{q\theta} > 0$, $v_{qq} - h_{qq} < 0$, $h_\theta \geq 0$ for all q and θ. Second, we assume that $v_q(0) - h_q(0, \overline{\theta}) = 0$ and $\lim_{q\to\infty} v_q(0) - h_q(0, \underline{\theta}) < 0$. These assumptions guarantee that the first-best quantity $q_{FB}(\theta)$ maximizing social surplus $v(q) - h(q, \theta)$ is unique, finite and decreasing, with $q_{FB}(\overline{\theta}) = 0$.

We also make the following monotonicity assumptions:

Assumption 4 *(i)* $v(q) - h(q, \theta) - \frac{F(\theta)}{f(\theta)} h_\theta(q, \theta)$ *has strictly decreasing differences in* (q, θ)*;*

(ii) $C^n(\mathbf{m}^n, \theta) + \frac{F(\theta)}{f(\theta)} C^n_\theta(\mathbf{m}^n, \theta)$ *is submodular in* $\mathbf{m}^n$ *and has strictly decreasing differences in* $(\mathbf{m}^n, \theta)$*;*

(iii) $C^n(\mathbf{m}^n, \theta)$ *is submodular in* $\mathbf{m}^n$ *and has strictly decreasing differences in* $(\mathbf{m}^n, \theta)$.

It is well-known that under Assumption 4(i) in the optimal mechanism with no costly signals the principal selects second best quantity $q_{SB}(\theta)$ maximizing the 'virtual' surplus

$$\Gamma(q,\theta) \equiv v(q) - h(q,\theta) - \frac{F(\theta)}{f(\theta)} h_\theta(q,\theta). \tag{17}$$

Assumption 4(i) implies that $q_{SB}(\theta)$ is strictly decreasing in θ whenever it is nonzero. Let $\theta^* \in (\underline{\theta}, \bar{\theta})$ be the unique solution to $\Gamma_q(0,\theta) = 0$.[15] Then $q^{SB}(\theta) > 0$ if and only if $\theta \in [\underline{\theta}, \theta^*)$. Thus, it is optimal for the principal to exclude all types in $[\theta^*, \overline{\theta}]$, which is socially inefficient. For example, higher-cost firms are excluded from production in a procurement setting.

Next, suppose that $n > 0$. Since the signals are now a part of an allocation, it is tempting to conjecture that an optimal mechanism maximizes augmented virtual surplus:

$$\Lambda^n(q, \mathbf{m}^n, \theta) = v(q) - h(q,\theta) - \frac{F(\theta)}{f(\theta)} h_\theta(q,\theta) - \left[C^n(\mathbf{m}^n,\theta) + \frac{F(\theta)}{f(\theta)} C^n_\theta(\mathbf{m}^n,\theta)\right] \tag{18}$$

To maximize (18) the principal would select the second best quantity $q_{SB}(\theta)$ and a "second best" signal function $\mathbf{m}^n_{SB}(\theta)$ that minimizes the virtual cost $C^n(\mathbf{m}^n,\theta) + \frac{F(\theta)}{f(\theta)} C^n_\theta(\mathbf{m}^n,\theta)$. The agent's information rent in this mechanism is:

$$U^n_{SB}(\theta) = U^n_{SB}(\overline{\theta}) + \int_\theta^{\overline{\theta}} \left[h_\theta(q_{SB}(t),t) + C^n_\theta(\mathbf{m}^n_{SB}(t),t)\right] dt. \tag{19}$$

Assumption 4(ii) implies that $\mathbf{m}^n_{SB}(\theta)$ is strictly increasing in θ, with $\mathbf{m}^n_{SB}(\theta) > \gamma^n(\theta)$, and hence $C_\theta(\mathbf{m}^n_{SB}(\theta),\theta) < 0$ for all $\theta \in (\underline{\theta}, \overline{\theta}]$. So, it is apparent from (19) that costly signals generate an incentive for the agent to understate her type. This incentive offsets the agent's incentive to overstate her type arising because $h_\theta(q,\theta) > 0$, resulting in a decrease of the agent's information rents. For low values of θ, the latter incentive dominates, and the information rents are decreasing in type.[16] But for large θ the incentive to understate the type due to communication costs becomes stronger and makes informational rents increase in type.[17]

The last observation implies that the principal can improve upon the mechanism $(q_{SB}(\theta), \mathbf{m}^n_{SB}(\theta))$ maximizing (18). Specifically, the principal can eliminate the rents of inefficient types by raising the quantities assigned to them above the second best to balance their incentives to understate and overstate their types. As a result, in the optimal mechanism a non-negligible fraction of agent types is held at their reservation utility level. So, the allocation of those types is characterized by the following condition: $U'(\theta) = -h_\theta(q(\theta),\theta) - C^n_\theta(\mathbf{m}^n(\theta),\theta) = 0$.

This argument implies that the expression for virtual surplus needs to be adjusted. To see how, note that when downwards incentive constraints are binding, assigning the allocation

[15] Such a θ^* exists because $\Gamma_q(0,\overline{\theta}) < 0$, $\Gamma_q(0,\underline{\theta}) > 0$ and by Assumption 4(i) $\Gamma_q(0,\theta)$ is decreasing in θ.

[16] Indeed, because q_{SB} and $\mathbf{m}^n_{SB}$ are then close to their first best values, h_θ is large and C^n_θ is close to zero.

[17] q_{SB} and $\mathbf{m}^n_{SB}$ are then far away from their first-best levels, so that h_θ is close to zero and C^n_θ is negative and has a large absolute value.

$(q, \mathbf{m}^n)$ to type θ requires paying information rent $h_\theta(q,\theta) - C^n_\theta(\mathbf{m}^n, \theta)$ to all types below θ, which reduces the principal's expected profits by this amount times $F(\theta)$. But paying this rent also allows to satisfy the participation constraints of those types below θ who would face negative payoffs otherwise. So letting $\delta(\theta)$ denote the shadow marginal value to the principal from satisfying the participation constraints of types below θ and setting $\sigma(\theta) = F(\theta) - \delta(\theta)$, instead of (18), the principal's adjusted virtual surplus from type θ becomes:

$$\Omega^n(q, \mathbf{m}^n, \theta, \sigma(\theta)) = v(q) - h(q,\theta) - \frac{\sigma(\theta)}{f(\theta)} h_\theta(q,\theta) - \left[C^n(\mathbf{m}^n, \theta) + \frac{\sigma(\theta)}{f(\theta)} C^n_\theta(\mathbf{m}^n, \theta) \right] \qquad (20)$$

The optimal mechanism maximizes the adjusted virtual surplus (20) with σ determined endogenously. Our next Theorem provides necessary and sufficient conditions under which the solution has a simple structure, with individual rationality constraint binding for high types and non-binding for low types. To state it, let $\breve{\sigma}(\theta)$ be the solution in σ to $h_\theta(q(\sigma,\theta),\theta) + C^n_\theta(\mathbf{m}^n(\sigma,\theta)),\theta) = 0$, where $q(\sigma,\theta)$ and $\mathbf{m}^n(\sigma,\theta)$ maximize (20) for a fixed value of σ.

Theorem 5 *Suppose that Assumption 4 holds and $\breve{\sigma}'(\theta) \leq f(\theta)$ for all $\theta \in [0,1]$. Then the allocation $(q(.), \mathbf{m}^n(.))$ in the optimal mechanism is continuous, $q(\theta)$ is decreasing and $\mathbf{m}^n(\theta)$ is increasing in θ, with $\mathbf{m}^n(\underline{\theta}) = \gamma^n(\underline{\theta})$, $\mathbf{m}^n(\overline{\theta}) = \gamma^n(\overline{\theta})$ and $\mathbf{m}^n(\theta) > \gamma^n(\theta)$ for all $\theta \in (\underline{\theta}, \overline{\theta})$.*

Furthermore, $q(\theta) > 0$ for all $\theta \in [\underline{\theta}, \overline{\theta})$, $q(\overline{\theta}) = 0$, and there exists $\widehat{\theta}_n \in (0, \theta^*)$ s.t.:

$$\begin{cases} q(\theta) = q^{SB}(\theta),\ U(\theta) > 0 \text{ and } U'(\theta) < 0 & \text{if } \theta \in [\underline{\theta}, \widehat{\theta}_n), \\ q^{SB}(\theta) < q(\theta) < q^{FB}(\theta) \text{ and } U(\theta) = 0 & \text{if } \theta \in (\widehat{\theta}_n, \overline{\theta}). \end{cases}$$

The following Lemma provides conditions on the primitives which guarantee that the assumption $\breve{\sigma}'(\theta) \leq f(\theta)$ holds.

Lemma 1 *Suppose that $\frac{F(\theta)}{f(\theta)}$ is increasing, C^n is convex in $\mathbf{m}^n$, $C^n_{\mathbf{mm}\theta} \leq 0, C^n_{\theta\theta m} \leq 0$, $h_{\theta\theta} \leq 0$, and $h_{\theta\theta q} \geq 0$. Then $\breve{\sigma}'(\theta) \leq f(\theta)$.*

According to Theorem 5, low (efficient) agent types in $[\underline{\theta}, \widehat{\theta}_n)$, are assigned standard second-best quantities and earn positive surpluses. Higher (inefficient) types in $(\widehat{\theta}_n, \overline{\theta}]$, are assigned positive quantities exceeding the second-best ones and get zero surpluses. The boundary type $\widehat{\theta}_n$ is assigned the second-best quantity and gets zero surplus.

Theorem 5 is related to Proposition 1 in Maggi and Rodriguez-Clare (1995) which characterizes the optimal mechanism with a single costly message. Both in their and our models a non-trivial set of types $[\widehat{\theta}_n, \overline{\theta}]$ are held at the reservation utility level. So this property is robust to the number of signals. However, there are important differences between our and their contributions. First, in their model the agent sends one signal, while we explore how the number of signals affects the optimal mechanism (see also Theorem 6 below). Second, these authors impose the restriction $q'(\theta) \geq 0$ and $m'(\theta) \geq 0$ to guarantee that their second-order conditions hold. We dispense with these assumptions.

The third difference between our results and those of Maggi and Rodriguez-Clare (1995) is that we establish that the optimal mechanism exhibits no exclusion i.e., $q(\theta) > 0$ for all $\theta \in [\underline{\theta}, \overline{\theta})$. So no type that can generate a positive surplus is excluded when the principal optimally exploits the misrepresentation costs. This result is important as it underscores that the phenomenon of exclusion, present in the standard case without costly signals, is not robust.

The next result provides the comparative statics properties of the solution.

Theorem 6 *Suppose that there exist $\underline{v}, \bar{v} \in (0, \infty)$ such that $\underline{v} \leq \frac{\partial^2 C^n}{\partial m_i^2} \leq \bar{v}$ and $\underline{v} \leq |\frac{\partial^2 C^n}{\partial \theta \partial m_i}| \leq \bar{v}$ for all i. Then there exists $T < \infty$ such that in the optimal mechanism $\widehat{\theta}_n - \underline{\theta} \leq \frac{T}{n}$, $q^{FB}(\theta) - q(\theta) \leq \frac{T}{n}$, $\mathbf{m}_i^n(\theta) - \gamma_i(\theta) \leq \frac{T}{n}$, $U(\theta) \leq \frac{T}{n}$, and $C^n(\mathbf{m}^n(\theta), \theta) \leq \frac{T}{n}$, for all θ.*

Thus, as n gets large, the quantity allocation in the optimal mechanism converges to the first-best, while the misrepresentation cost becomes negligibly small and almost all agent types get zero surplus (since $\widehat{\theta}_n \to \underline{\theta}$.). So, the principal extracts almost all surplus. The small size of the misrepresentation costs and the near-efficiency of the allocation contrasts with the inefficiency of the second-best solution under adverse selection highlighted in the literature.

An economically significant lesson of this section resonating with our previous results is that the overall efficiency increases and the principal gets more surplus as the number of costly messages increases. The added value of Theorems 5 and 6 lies in showing how exactly the principal achieves this.

3.4 Screening and Endogenous Size of the Signal Space

One factor that we have not considered yet is the possible presence of fixed costs. For example, the principal may incur a cost to develop and administer a test, or process the transmitted information. So in this subsection, we will characterize the optimal number of signals when the principal has to pay a fixed cost G to set-up each signal, and study the relationship between the total fixed costs nG and the agent's communication costs in the optimal mechanism.

In this analysis, we will assume that the agent's signal costs are additively separable across signals, i.e. $C^n(m_1, ..., m_n, \theta) = \sum_{i=1}^n c(m_i, \theta)$. Also, for simplicity, we will henceforth treat n as a continuous variable.

Let $W(n)$ denote the principal's expected surplus, gross of any fixed costs, in the optimal mechanism with n signals. To derive the optimal number of signals n^* that maximize the principal's profits $W(n) - nG$ we establish the following Lemma. Letting $D = \{(m, \theta) : \gamma(\theta) \leq m \leq \gamma(\overline{\theta})$ and $\underline{\theta} \leq \theta \leq \overline{\theta}\}$ define:

$$\underline{K} = \min_{(m,\theta) \in D} \frac{c_\theta c_m}{c_{m\theta} c}(m, \theta) - 1, \qquad \overline{K} = \max_{(m,\theta) \in D} \frac{c_\theta c_m}{c_{m\theta} c}(m, \theta) - 1$$

Then we have:

Lemma 2 *Suppose that* $c_{mm}c_{m\theta} - c_{mm\theta}c_m < 0$ *for all* θ.[18] *Then* $W(n)$ *is a strictly increasing and strictly concave function. Also,* $\underline{K} > 0$*, and in the optimal mechanism agent-type* θ *sends* n *signals* $m(\theta)$ *satisfying:*

$$\frac{G}{\overline{K}} \leq \int_{\underline{\theta}}^{\bar{\theta}} c(m(\theta), \theta) f(\theta) d\theta \leq \frac{G}{\underline{K}}$$

Lemma 2 implies that we can compute the optimal number of signals n^* via the first-order condition $\frac{dW}{dn}(n^*) = G$, unless G is prohibitively large in which case $n^* = 0$. The Lemma also shows that the ratio of the signal fixed cost G to the agent's expected communication cost per signal $\int_{\underline{\theta}}^{\bar{\theta}} c(m(\theta), \theta) f(\theta) d\theta$ is between $\underline{K}$ and $\overline{K}$.

Although the fixed costs limit the optimal number of signals, our previous results show that the optimal allocation profile converges to the first-best fast, at the rate n, as the number of signals increases. This indicates that the associated increase in efficiency is sufficiently large to dominate the set-up costs nG and the agent's communication costs for sufficiently many signals, when G is not too large.

To illustrate this, let us consider the following example. Let $v(q) = q - \frac{1}{2}q^2$, $h(q, \theta) = \theta q$, $c(m, \theta) = \frac{1}{2}(m - \theta)^2$, and $F(\theta) = \theta$ for $\theta \in [0, 1]$. Then $\underline{K} = \overline{K} = 1$, so by Lemma 2 the principal's cost nG of setting up the communication system equals the agent's expected communication cost. Table 3 highlights the relation between the fixed cost, the optimal number of signals n^* and the associated change in the principal's surplus. To normalize, we express the fixed cost and the principal's gain of surplus over the second best, $W(n) - W^{SB}$, as a fraction of the maximal potential gain in surplus $W^{FB} - W^{SB}$.

Table 3.

$\frac{G}{W^{FB}-W^{SB}}$	24%	12%	8%	5.5%	4%	3%	2.5%	2%	1.7%	1.4%
n^*	1	2	3	4	5	6	7	8	9	10
$\frac{W(n^*)-W^{SB}}{W^{FB}-W^{SB}}$	33%	50%	60%	66.6%	71.4%	75%	77.8%	80%	81.8%	83.3.%

Thus, in this example the principal will elicit at least 10 signals if the fixed cost of setting-up an extra signal does not exceed 1.4% of the potential increase in the principal's surplus. By Lemma 2, the agent will then incur the expected signal costs of 14% of the potential gain of the principal's surplus, thereby dissipating a part of the benefit. Nevertheless, the principal's surplus will increase by 83.3% percent of the available maximal gain, and hence her profits (net of fixed communication costs) will reach 69.3% of the total potential maximal gain.

[18]Since $c_{mm} > 0$, $c_{m\theta} < 0$, and $c_m > 0$ for $m > \gamma(\theta)$, for this condition to hold it is sufficient that $c_{mm\theta} \leq 0$. More generally, the condition holds if $c_{mm\theta}$ is not too large.

To conclude this section, note that a fixed cost may be associated with a particular signal, rather than an agent. This is especially plausible in employer-employee relationship where a message represents an outcome of a test given to a job-candidate. Then test-specific fixed costs will be amortized over all the job-candidates, and therefore would create less of an obstacle for increasing the number of tests. In this case, our model predicts that larger firms who interview many applicants will put more emphasis on testing and evaluation of candidates, rather than on providing on-the-job incentives. This appears to be broadly consistent with reality.

4 Conclusions

This paper shows that in environments with misrepresentation costs, signalling structure and screening mechanisms in which the agents send multiple signals significantly expand the set of implementable outcomes allowing to attain (near-) efficiency with small signaling/misrepresenentation costs. We also point out that an optimal method of dealing with asymmetric information regarding employees' abilities in such environments involves the design of testing and interviewing procedures, rather than on-the-job incentive schemes. These results suggest that the problem of dissipation of resources in unproductive screening and signalling, the so-called 'rat race,' may not be as significant as previously thought.

Although not shown here, Theorems 1 and 3 continue to hold when the type is multidimensional. This is important since in many contexts an agent's skills and abilities have multiple aspects. For example, a Department hiring new faculty members is concerned about their teaching and research abilities, collegiality, and propensity to contribute to service.

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5 Appendix

Proof of Theorem 1.

Let $\overline{u}$ and $\underline{u}$ be the maximal and minimal possible sender's utilities i.e.,
$\overline{u} = \max\{u(x,\theta) : \theta \in \Theta \text{ and } x = BR(\mu) \text{ for some } \mu \in \Delta(\Theta)\}$,
$\underline{u} = \min\{u(x,\theta) : \theta \in \Theta \text{ and } x = BR(\mu) \text{ for some } \mu \in \Delta(\Theta)\}$.

The continuity of $u(\cdot,\cdot)$ and the compactness of X and $[\underline{\theta},\overline{\theta}]$ imply that $-\infty < \underline{u} < \overline{u} < \infty$.

We now claim that $\mu(\gamma^n(\theta))$, the receiver's posterior following the message profile $\gamma^n(\theta)$, must be supported on a sufficiently small neighborhood of θ. Indeed, let $k = \sqrt{\frac{\overline{u}-\underline{u}}{\omega L^2}}$ and suppose that $||\theta - \theta'|| > \frac{k}{\sqrt{n}}$. Using this inequality and Assumption 1 we obtain:

$$\begin{aligned}&\min_{\mu} u(BR(\mu),\theta') - C^n(\gamma^n(\theta'),\theta') - \max_{\mu} u(BR(\mu),\theta') - C^n(\gamma^n(\theta),\theta')\\ &\geq \underline{u} - \overline{u} + \omega||\gamma^n(\theta') - \gamma^n(\theta)||^2 > \underline{u} - \overline{u} + \omega n L^2||\theta' - \theta||^2 > 0,\end{aligned}$$

So for any type θ' satisfying $||\theta - \theta'|| > \frac{k}{\sqrt{n}}$ sending $\gamma^n(\theta)$ is dominated by sending $\gamma^n(\theta')$. Thus, in any equilibrium satisfying the dominance criterion $supp(\mu(\gamma^n(\theta))) \in \left(\theta - \frac{k}{\sqrt{n}}, \theta + \frac{k}{\sqrt{n}}\right)$

We now claim that $supp(\mu(\widetilde{\mathbf{m}}^n(\theta)))$ converges to $\{\theta\}$ as $n \to \infty$. To this effect, observe that Assumption 1(ii) implies that $C^n(\widetilde{\mathbf{m}}^n(\theta),\theta) \geq \omega||\widetilde{\mathbf{m}}^n(\theta) - \gamma^n(\theta)||^2$. So,

$$\underline{u} \leq u(\widetilde{x}^n(\theta),\theta) - C^n(\widetilde{\mathbf{m}}^n(\theta),\theta) \leq \bar{u} - \omega||\widetilde{\mathbf{m}}^n(\theta) - \gamma^n(\theta)||^2.$$

Hence, $||\widetilde{\mathbf{m}}^n(\theta) - \gamma^n(\theta)|| \leq Lk$. Now let θ_n' be such that $\widetilde{\mathbf{m}}^n(\theta_n') = \widetilde{\mathbf{m}}^n(\theta)$. Then using Assumption 1 (i) we obtain:

$$\begin{aligned}&L\sqrt{n}|\theta_n' - \theta| \leq ||\gamma^n(\theta_n') - \gamma^n(\theta)|| = ||\left(\widetilde{\mathbf{m}}^n(\theta) - \gamma^n(\theta)\right) - \left(\widetilde{\mathbf{m}}^n(\theta) - \gamma^n(\theta_n')\right)|| \leq\\ &||\widetilde{\mathbf{m}}^n(\theta) - \gamma^n(\theta)|| + ||\widetilde{\mathbf{m}}^n(\theta) - \gamma^n(\theta_n')|| \leq 2Lk.\end{aligned} \tag{21}$$

It follows from (21) that $|\theta' - \theta| \leq \frac{2k}{\sqrt{n}}$. So $supp.\mu(\widetilde{\mathbf{m}}^n(\theta)) \in [\theta - \frac{2k}{\sqrt{n}}, \theta + \frac{2k}{\sqrt{n}}]$ establishing that $supp.\mu(\widetilde{\mathbf{m}}^n(\theta))$ converges to $\{\theta\}$ as $n \to \infty$.

Next, since $v(x,\theta)$ is concave in x, $\widetilde{x}^n(\theta) = BR(\mu(\widetilde{\mathbf{m}}^n(\theta))$ is unique and satisfies $\int_{[\underline{\theta},\bar{\theta}]} v_x(\widetilde{x}^n(\theta),\theta)d\mu(\widetilde{\mathbf{m}}^n(\theta))(t) = 0$. By the intermediate value Theorem we then have:

$$v_x(\widetilde{x}^n(\theta),\theta) + \int_{[\underline{\theta},\bar{\theta}]} (t-\theta)v_{x\theta}(\widetilde{x}^n(\theta), s(t,\theta)d\mu(\widetilde{\mathbf{m}}^n(\theta))(t) = 0,$$

where $s(t,\theta)$ lies between t and θ for all t. Combining this equality with $v_x(x^*(\theta),\theta) = 0$, which holds by definition of $x^*(\theta)$, and using the Intermediate value Theorem again yields:

$$(x^*(\theta) - \widetilde{x}^n(\theta))v_{xx}(x',\theta) = \int_{[\underline{\theta},\bar{\theta}]} (t-\theta)v_{x\theta}(\widetilde{x}^n(\theta), s(t,\theta)d\mu(\widetilde{\mathbf{m}}^n(\theta))(t), \tag{22}$$

where $x' \in [\min\{x^*(\theta),\widetilde{x}^n(\theta)\}, \max\{x^*(\theta),\widetilde{x}^n(\theta)\}]$. Since, as established above, $|t'-\theta| \leq \frac{2k}{\sqrt{n}}$ when $t' \in$ *supp*.$\mu(\widetilde{\mathbf{m}}^n(\theta))$, equation (22) then implies:

$$|\widetilde{x}^n(\theta) - x^*(\theta)| \leq \frac{2k}{\sqrt{n}} \frac{\max_{(x',\theta')\in X\times[\underline{\theta},\bar{\theta}]} |v_{x\theta}(x',\theta')|}{\min_{(x',\theta')\in X\times[\underline{\theta},\bar{\theta}]} |v_{xx}(x',\theta')|} \equiv \frac{K_2}{\sqrt{n}}. \tag{23}$$

The argument that establishes (23) relies on *supp*.$\mu(\widetilde{\mathbf{m}}^n(\theta)) \in [\theta - \frac{2k}{\sqrt{n}}, \theta + \frac{2k}{\sqrt{n}}]$. So, since we have also established above that *supp*$(\mu(\gamma^n(\theta))) \in \left(\theta - \frac{k}{\sqrt{n}}, \theta + \frac{k}{\sqrt{n}}\right)$, it follows that $|BR(\mu(\gamma^n(\theta)) - x^*(\theta)| \leq \frac{K_2}{\sqrt{n}}$. Finally, we have:

$$\begin{aligned} &C^n(\widetilde{\mathbf{m}}^n(\theta),\theta) \leq u(\widetilde{x}^n(\theta),\theta) - u(BR(\mu(\gamma^n(\theta)),\theta) \leq \\ &|\widetilde{x}^n(\theta) - BR(\mu(\gamma^n(\theta))| \times |\max_{(x',\theta')\in X\times[\underline{\theta},\bar{\theta}]} u_x(x',\theta')| \leq \frac{2K_2}{\sqrt{n}} \times |\max_{(x',\theta')\in X\times[\underline{\theta},\bar{\theta}]} u_x(x',\theta')|, \end{aligned}$$

where the first inequality holds by incentive compatibility, the second inequality follows from $|\widetilde{x}^n(\theta) - x^*(\theta)| \leq \frac{K_2}{\sqrt{n}}$ and $|BR(\mu(\gamma^n(\theta)) - x^*(\theta)| \leq \frac{K_2}{\sqrt{n}}$.
Setting $K = \max\{k, K_2, 2K_2|\max_{(x',\theta')\in X\times[\underline{\theta},\bar{\theta}]} u_x(x',\theta')|\}$ completes the proof. *Q.E.D.*

Proof of Theorem 2. An argument similar to the one in the proof of Theorem 1 establishes that the lower endpoint of $\mu(\gamma^n(\theta))$ must exceed $\theta - \sqrt{\frac{\bar{u}-\underline{u}}{\omega n L^2}}$. The monotonicity of the receiver's best response correspondence then implies that $BR(\mu(\gamma^n(\theta)) \geq x^*(\theta - \sqrt{\frac{\bar{u}-\underline{u}}{\omega n L^2}})$, so that by sending the message profile $\gamma^n(\theta)$ sender type θ can guarantee itself a payoff of $u(x^*(\theta - \sqrt{\frac{\bar{u}-\underline{u}}{\omega n L^2}}),\theta)$.

We now claim that *supp*$(\mu(\widetilde{\mathbf{m}}^n(\theta)))$ converges to $\{\theta\}$ as $n \to \infty$. Indeed, suppose to the contrary that there exists $\varepsilon > 0$ and for each n a perfect Bayesian equilibrium and a sender type θ_n such that $\theta_n'' - \theta_n' \geq \varepsilon$, where $\theta_n'' = \max$ *supp* $\mu(\widetilde{\mathbf{m}}^n(\theta_n))$ and $\theta_n' = \min \text{supp}\ \mu(\widetilde{\mathbf{m}}^n(\theta_n))$. By monotonicity, it follows that *supp* $\mu(\widetilde{\mathbf{m}}^n(\theta_n)) = [\theta_n', \theta_n'']$.

For any pair θ', θ'' with $\theta'' > \theta'$ let $\mu_{\theta',\theta''}$ denote the conditional distribution of F on $[\theta',\theta'']$. Let $\Delta = \inf\{u(x^*(\theta''),\theta'') - u(BR(\mu_{\theta',\theta''}) : \theta'' - \theta' \geq \varepsilon\}$, and observe that $\Delta > 0$. Then we have $U^n(\theta''_n) = u(\widetilde{x}^n(\theta''_n), \theta''_n) - C^n(\widetilde{\mathbf{m}}^n(\theta''_n), \theta''_n) \leq u(\widetilde{x}^n(\theta''_n), \theta''_n) \leq u(x^*(\theta''_n), \theta''_n) - \Delta$, which for sufficiently large n contradicts that $U^n(\theta''_n) \geq u(x^*(\theta''_n - \sqrt{\frac{\overline{u}-\underline{u}}{\omega n L^2}}), \theta''_n)$. *Q.E.D.*

Proof of Theorem 3. Fix any pair of twice continuously differentiable functions $q : \Theta \to Q$ and $t : \Theta \to \mathbb{R}$. Then $U(\theta) \equiv t(\theta) - h(q(\theta), \theta)$ is also twice continuously differentiable. We will show that there exists $N < \infty$ and a sequence of transfers and messages rules $(t^n, \mathbf{m}^n)$, such that for all $n \geq N$ the mechanism $(q(.), t^n(.), \mathbf{m}^n(.))$ is incentive compatible and provides each type with net payoff $U(\theta)$, i.e.

$$U(\theta) = t^n(\theta) - h(q(\theta), \theta) - C^n(\mathbf{m}^n(\theta), \theta) = \max_{\theta' \in \Theta}\{t^n(\theta') - h(q(\theta'), \theta) - C^n(\mathbf{m}^n(\theta'), \theta)\} \quad (24)$$

The proof goes through a sequence of steps.

Step 1. First, let us construct signal rule $\mathbf{m}^n(\theta)$.

Let $z^n : \Theta \to \mathbb{R}$, $m^n_{i,z}(\theta) = \gamma^n_i(\theta) + z^n(\theta)$ for all i, and $\mathbf{m}^n_z(\theta) = (m^n_{1,z}(\theta), ..., m^n_{n,z}(\theta))$. When incentive constraints (i.e., second equality in (24)) hold, the envelope theorem implies that

$$U'(\theta) = -h_\theta(q(\theta), \theta) - C^n_\theta(\mathbf{m}^n_z(\theta), \theta) \quad (25)$$

Let us show that for each n and θ there exists a unique $z^n(\theta)$ so that (25) holds. To this end, rewrite (25) as:

$$f(z) \equiv C^n_\theta(\mathbf{m}^n_z(\theta), \theta) = -U'(\theta) - h_\theta(q(\theta), \theta) \quad (26)$$

By Assumption 3(iii) we have $f'(z) = \sum_i C^n_{\theta m_i}(\mathbf{m}^n_z(\theta), \theta) < 0$, so there is at most one solution to (26). Let us now show that there exists such a solution.

First, we claim that $C^n_\theta(\gamma^n(\theta), \theta) = 0$. Indeed, differentiating the identity $C^n(\gamma^n(\theta), \theta) \equiv 0$, we obtain $\sum_i C^n_{m_i}(\gamma^n(\theta), \theta)\frac{\partial \gamma^n_i(\theta)}{\partial \theta} + C^n_\theta(\gamma^n(\theta), \theta) = 0$. Since $\mathbf{m}^n = \gamma^n(\theta)$ uniquely minimizes $C^n(\cdot, \theta)$, we have $C^n_{m_i}(\gamma^n(\theta), \theta) = 0$ for all $i \in \{1, ..., n\}$ establishing the claim. Next, letting $m^n_{i,t}(\theta) = \gamma^n_i(\theta) + tz$ for $t \in [0, 1]$, we have:

$$C^n_\theta(\mathbf{m}^n_z(\theta), \theta) - C^n_\theta(\gamma^n(\theta), \theta) = \int_0^1 z \sum_i C^n_{\theta m_i}(\mathbf{m}^n_t(\theta), \theta) dt. \quad (27)$$

Equation (27) and Assumption 3(iii) imply that $f(z) = C^n_\theta(\mathbf{m}^n_z(\theta), \theta) - C^n_\theta(\gamma^n(\theta), \theta) \leq -\omega_1 n z$ if $z > 0$, and so $f(z) \to -\infty$ as $z \to \infty$. Similarly, $f(z) \geq \omega_1 n|z|$ if $z < 0$, so $f(z) \to \infty$ as $z \to -\infty$. Hence, there is a unique solution $z^n(\theta)$ to (25).

Step 2. Let us show that $\mathbf{m}^n_z(\theta) \to \gamma^n(\theta)$ as n grows, uniformly in θ.

Let $g(\theta) = -h_\theta(q(\theta), \theta) - U'(\theta)$. It then follows from (25), (27) and Assumption 3(iii) that:

$$|g(\theta)| = |C^n_\theta(\mathbf{m}^n_z(\theta), \theta) - C^n_\theta(\gamma^n(\theta), \theta)| \geq \omega_1 n |z^n(\theta)| \quad (28)$$

Furthermore, since g is continuous and Θ is compact, it follows from the Weierstrass Theorem that there exists a constant $\lambda > 0$ s.t. $|g(\theta)| \leq \lambda$ for all θ. Hence (28) implies that $|z^n(\theta)| \leq \frac{\lambda}{\omega_1} n^{-1} \to 0$ as $n \to \infty$. We conclude that $\mathbf{m}_z^n(\theta) \to \gamma^n(\theta)$ as $n \to \infty$, uniformly in θ.

Step 3. In this step we establish the following auxiliary Lemma:

Lemma 3 *A mechanism $\{\mathbf{m}^n(\cdot), q(\cdot), t(\cdot),\}$ is incentive compatible if and only if*

$$t(\theta) = \min_{\theta' \in \Theta} \{U(\theta') + h(q(\theta), \theta') + C^n(\mathbf{m}^n(\theta), \theta')\}, \tag{29}$$

where $U(\theta) = t(\theta) - h(q(\theta), \theta) - C^n(\mathbf{m}^n(\theta), \theta)$.

Proof of Lemma 3: Suppose $\{\mathbf{m}^n(\cdot), q(\cdot), t(\cdot)\}$ is incentive compatible. Then:

$$U(\theta') \geq t(\theta) - h(q(\theta), \theta') - C^n(\mathbf{m}^n(\theta), \theta'),$$

for all θ, θ', with equality at $\theta' = \theta$. Hence (29) holds.

Conversely, suppose that (29) holds. Then $\{\mathbf{m}^n(\cdot), q(\cdot), t(\cdot)\}$ is incentive compatible because

$$U(\theta) = t(\theta) - h(q(\theta), \theta) - C^n(\mathbf{m}^n(\theta), \theta) \geq t(\theta') - h(q(\theta'), \theta) - C^n(\mathbf{m}^n(\theta'), \theta).$$

Q.E.D.

Step 4. Set $t^n(\theta) = U(\theta) + h(q(\theta), \theta) + C^n(\mathbf{m}_z^n(\theta), \theta)$.

By Lemma 3 the allocation $\{q(\cdot), t(\cdot), \mathbf{m}^n(\cdot)\}$ is implementable if:

$$t^n(\theta) = \min_{\theta' \in \Theta} \{U(\theta') + h(q(\theta), \theta') + C^n(\mathbf{m}_z^n(\theta), \theta')\} \tag{30}$$

Let us show that (30) holds for sufficiently large n.

First, consider θ' such that $|\theta' - \theta| > \omega_1$. It follows from Assumption 3(ii) that

$$C^n(\mathbf{m}_z^n(\theta), \theta') - C^n(\mathbf{m}_z^n(\theta), \theta) \geq \omega_1 ||\mathbf{m}_z^n(\theta) - \gamma^n(\theta')||^2 - \omega_2 ||\mathbf{m}_z^n(\theta) - \gamma^n(\theta)||^2. \tag{31}$$

Now $m_{i,z}^n(\theta) = \gamma_i(\theta) + z^n(\theta)$ for all i, and by Step 2 we have $|z^n(\theta)| \leq \frac{\lambda}{\omega_1} n^{-1}$. It follows that the second term in (31) converges to zero uniformly in θ. For the same reason, the expression $\mathbf{m}_z^n(\theta) - \gamma^n(\theta')$ in the first term of (31) converges to $\gamma^n(\theta) - \gamma^n(\theta')$, uniformly in θ. Now by Assumption 3(i) we have $||\gamma^n(\theta) - \gamma^n(\theta')||^2 \geq nL^2 |\theta' - \theta|^2$ implying that the first term on the right side of (31) goes to infinity, whenever $|\theta' - \theta| > \omega_1$. Now let $\kappa = |\max_{\theta, \theta'} \{h(q(\theta), \theta) - h(q(\theta), \theta') + U(\theta) - U(\theta')\}|$. It then follows that there exists $N_1 < \infty$ s.t. for all $n \geq N_1$ the right side of (31) exceeds κ, so that (30) holds for all $|\theta' - \theta| > \omega_1$.

It remains to show that (30) holds for all $|\theta' - \theta| \leq \omega_1$. To this end let $V(\theta', \theta)$ denote the minimand in (30), and note that by (25), we have $\frac{\partial V(\theta',\theta)}{\partial \theta'}_{\theta'=\theta} = 0$ for all $\theta \in \Theta$. Furthermore,

$$\frac{\partial^2 V(\theta', \theta)}{\partial (\theta')^2} = U''(\theta') + h_{\theta\theta}(q(\theta), \theta') + C^n_{\theta\theta}(\mathbf{m}^n_z(\theta), \theta').$$

Let $\beta = \max_{\theta',\theta \in \Theta} |U''(\theta') + h_{\theta\theta}(q(\theta), \theta')|$. By Assumption 3(iv) we have $\frac{\partial^2 V(\theta',\theta)}{\partial(\theta')^2} \geq -\beta + \omega_1 n$ whenever $||\mathbf{m}^n - \gamma^n(\theta)|| = \sqrt{n}|z_n(\theta)| \leq \omega_1$. From step 2, we have $|z^n(\theta)| \leq \frac{\lambda}{\omega_1} n^{-1}$. Select N_2 such that $\omega_1 n - \beta > 0$ and $\lambda n^{-\frac{1}{2}} \leq \omega_1^2$ for all $n \geq N_2$. Then for all θ and all $n \geq N_2$, the function $V(\theta', \theta)$ is strictly convex in θ' over the domain $|\theta' - \theta| \leq \omega_1$, and attains its unique minimum at $\theta' = \theta$, so that (30) holds for all $|\theta' - \theta| \leq \omega_1$.

Step 5. Assumption 3(ii) and Step 2 above imply that

$$C^n(\mathbf{m}^n_z(\theta), \theta) \leq \omega_2 n |z^n(\theta)|^2 \leq \left(\frac{\lambda}{\omega_1}\right)^2 \omega_2 n^{-1}$$

Consequently, $C^n(\mathbf{m}^n_z(\theta), \theta) \to 0$ as n grows, uniformly in θ. From (24), this also implies that $t^n(\theta) \to t(\theta)$, uniformly in θ, completing the proof. *Q.E.D.*

Proof of Theorem 4: First, we select the message rule $\mathbf{m}^n_z(\theta)$ as in the Step 1 of the proof of Theorem 3 to satisfy condition (25). Note that $U'(\theta) + h_\theta(q(\theta), \theta) = t'(\theta) - h_q(q(\theta), \theta)\frac{dq(\theta)}{d\theta}$ which is nonnegative by Assumption. Hence, (25) implies that $C^n_\theta(\mathbf{m}^n_z(\theta), \theta) \leq 0$. Therefore, $\mathbf{m}^n_z(\theta)$ solving (25) satisfies $\mathbf{m}^n_z(\theta) \geq \gamma^n(\theta)$ and so the one-sided nature of the signal cost function in this case does affect the cost of equilibrium signals.

Then Steps 2 and 5 in the proof of Theorem 3 apply verbatim to establish that $\mathbf{m}^n_z(\theta) \to \gamma^n(\theta)$ and $C^n(\mathbf{m}^n_z(\theta), \theta) \to 0$ as n grows, uniformly in θ.

It remains to show that our mechanism is incentive compatible which, with transfer $t(\theta)$ defined as in the proof of Theorem 3, is equivalent to showing that for all $\theta, \theta' \in \Theta$ we have:

$$U(\theta') \geq U(\theta) + h(q(\theta), \theta) - h(q(\theta), \theta') + C^n(\mathbf{m}^n_z(\theta), \theta) - C^n(\mathbf{m}^n_z(\theta), \theta'). \quad (32)$$

First, for θ' s.t. $\gamma^n(\theta') \leq \mathbf{m}^n_z(\theta)$ the proof that (32) holds is the same as in Theorem 3. Now suppose that θ' is such that $\gamma^n(\theta') > \mathbf{m}^n_z(\theta)$ and so $C^n(\mathbf{m}^n_z(\theta), \theta') = 0$. Then there exists $\theta'' \in (\theta, \theta')$ s.t. $\gamma^n(\theta'') = \mathbf{m}^n_z(\theta)$ and for which by the above argument we have:

$$U(\theta'') \geq U(\theta) + h(q(\theta), \theta) - h(q(\theta), \theta'') + C^n(\mathbf{m}^n_z(\theta), \theta)$$

Therefore, (32) holds if $U(\theta') - U(\theta'') \geq h(q(\theta), \theta'')$. Finally, note that the latter inequality holds because $U(\theta') - U(\theta'') \geq -\int_{\theta''}^{\theta'} h_\theta(q(s), s)ds \geq -\int_{\theta''}^{\theta'} h_\theta(q(\theta), s)ds = h(q(\theta), \theta'')$. The first inequality in this sequence holds because $U'(\theta) + h_\theta(q(\theta), \theta) = t'(\theta) - h_q(q(\theta), \theta)\frac{dq(\theta)}{d\theta} \geq 0$ and the second inequality holds because $q(.)$ is decreasing, $h_{q\theta} \geq 0$ and $\theta \leq \theta'' \leq s$. *Q.E.D.*

6 Online Appendix (Not For Publication). Proofs of the Results in Section 3.4.

Proof of Theorem 5: In the optimal mechanism the principal selects "quantity" and transfer functions, $q(\cdot)$ and $t(\cdot)$, and a vector of signals $\mathbf{m}^n(\cdot)$ to solve:

$$\max_{q(\theta),t(\theta),\mathbf{m}^n(\theta)} \int_{\underline{\theta}}^{\overline{\theta}} (v(q(\theta)) - t(\theta))f(\theta)d\theta$$

subject to the following incentive and the individual rationality constraints, respectively:

$$t(\theta) - h(q(\theta),\theta) - C^n(\mathbf{m}^n(\theta),\theta) \geq t(\theta') - h(q(\theta'),\theta) - C^n(\mathbf{m}^n(\theta'),\theta) \text{ for all } \theta \text{ and } \theta' \quad (33)$$

$$U(\theta) \equiv t(\theta) - h(q(\theta),\theta) - C^n(\mathbf{m}^n(\theta),\theta) \geq 0 \text{ for all } \theta. \quad (34)$$

Using (34) to substitute $t(\theta)$ from the objective, and replacing the incentive constraints (33) by the first-order condition associated with the agent's utility maximization, yields the following "relaxed" problem:

$$\max_{q(\theta),\mathbf{m}^n(\theta),U(\theta)} \int_{\underline{\theta}}^{\overline{\theta}} \{v(q(\theta)) - h(q(\theta),\theta) - C^n(\mathbf{m}^n(\theta),\theta) - U(\theta)\} f(\theta)d\theta \quad (35)$$

subject to individual rationality constraint (34) and the first-order condition:

$$U'(\theta) = -h_\theta(q(\theta),\theta) - C^n_\theta(\mathbf{m}^n(\theta),\theta). \quad (36)$$

The proof of the Theorem proceeds as follows. First, we obtain the solution to the relaxed program. Then we will verify that the solution to the relaxed program satisfies the incentive constraints (33) and hence also solves the unrelaxed problem.

To solve the relaxed problem, define the Hamiltonian:

$$H = (v(q) - h(q,\theta) - C^n(\mathbf{m}^n,\theta) - U) f(\theta) - \sigma (h_\theta(q,\theta) + C^n_\theta(\mathbf{m}^n,\theta)) + \rho U \quad (37)$$

Maximizing (37) w.r.t. q and $\mathbf{m}^n$ yields the first order conditions:

$$\begin{aligned} \{v_q(q) - h_q(q,\theta)\}f(\theta) - \sigma h_{q\theta}(q,\theta) &\leq 0 \; (= 0, \text{ if } q > 0) \quad (38)\\ \frac{\partial C^n}{\partial m_i}(\mathbf{m}^n,\theta)f(\theta) + \sigma\frac{\partial^2 C^n}{\partial m_i \partial\theta}(\mathbf{m}^n,\theta) &= 0 \quad (39) \end{aligned}$$

The costate equation is

$$\sigma'(\theta) = f(\theta) - \rho(\theta), \quad (40)$$

Furthermore, the solution has to satisfy complementary slackness conditions

$$\rho(\theta)U(\theta) = 0, \; \rho(\theta) \geq 0, \text{ and } U(\theta) \geq 0, \quad (41)$$

We also have the following transversality conditions: $\sigma(\underline{\theta})U(\underline{\theta}) = 0$, $\sigma(\bar{\theta})U(\bar{\theta}) = 0$, $\sigma(\underline{\theta}) \leq 0$ and $\sigma(\bar{\theta}) \geq 0$.

The rest of the proof proceeds through a number of Claims.

Claim 1. *For $\theta \in [\underline{\theta}, \overline{\theta}]$ and $\sigma \geq 0$ let $q(\sigma, \theta)$ and $\mathbf{m}^n(\sigma, \theta)$ maximize the Hamiltonian H in (37) w.r.t. q and $\mathbf{m}^n$, respectively. Then $q(\sigma, \theta)$ is decreasing in σ, strictly so whenever $q(\sigma, \theta) > 0$, and $\mathbf{m}^n(\sigma, \theta)$ is strictly increasing in σ*, while $\mathbf{m}^n(\sigma, \theta) \geq \gamma^n(\theta)$ with strict inequality when $\sigma > 0$.

By definition, $q(\sigma, \theta)$ is the solution in q to

$$\max_{q \geq 0} \left\{ v(q) - h(q, \theta) - \frac{\sigma}{f(\theta)} h_\theta(q, \theta) \right\}, \tag{42}$$

and $\mathbf{m}^n(\sigma, \theta)$ is the solution in $\mathbf{m}^n$ to

$$\min_{\mathbf{m}^n \in \mathbb{R}^n} \left\{ C^n(\mathbf{m}^n, \theta) + \frac{\sigma}{f(\theta)} C^n_\theta(\mathbf{m}^n, \theta) \right\}. \tag{43}$$

The existence of $q(\sigma, \theta)$ and $\mathbf{m}^n(\sigma, \theta)$ is guaranteed by the Weierstrass theorem because, respectively: (i) $q(\sigma, \theta)$ belongs to $[0, q^{FB}(\theta)]$; (ii) the value of (43) goes to ∞ as $||\mathbf{m}^n|| \to \infty$.[19]

Next, observe that the cross partial of the objective (42) in (q, σ) is equal to $-\frac{1}{f(\theta)} h_{q\theta}(q, \theta) < 0$. Hence it has strictly decreasing differences in (q, σ), and so $q(\sigma, \theta)$ must be decreasing in σ. Similarly, $\mathbf{m}^n(\sigma, \theta)$ is increasing in σ. Indeed, by Assumption 4(iii) the cross-partial of the objective in (43) w.r.t. $\mathbf{m}^n$ and θ equals $\frac{1}{f(\theta)} \frac{\partial^2 C^n}{\partial m_i \partial \theta}(\mathbf{m}^n, \theta) < 0$, so this objective has decreasing differences in $(\mathbf{m}^n, \sigma)$. Furthermore the objective in (43) is submodular in $\mathbf{m}^n$ because $\frac{\partial^2 C^n}{\partial m_i \partial m_j} + \frac{\sigma}{f(\theta)} \frac{\partial^3 C^n}{\partial m_i \partial m_j \partial \theta} < 0$ for all $j \neq i$. This inequality follows from Assumption 4 (iii) when $\frac{\partial^3 C^n}{\partial m_i \partial m_j \partial \theta} \leq 0$; and from Assumption 4 (ii) and the fact that $0 \leq \sigma \leq F(\theta)$ when $\frac{\partial^3 C^n}{\partial m_i \partial m_j \partial \theta} > 0$.

Finally, recall that $C_{m_i}(\mathbf{m}^n(\sigma, \theta), \theta) < 0$ if $m_i^n(\theta) < \gamma_i^n(\theta)$, and $\frac{\partial C^n(\mathbf{m}^n, \theta)}{\partial m_i \partial \theta} < 0$. So, if $\mathbf{m}^n(\sigma, \theta)$ is such that $m_i^n(\theta) \leq \gamma_i^n(\theta)$ then $\frac{\partial C^n(\mathbf{m}^n, \theta) + \frac{\sigma}{f(\theta)} C^n_\theta(\mathbf{m}^n, \theta)}{\partial m_i} = C^n_{m_i}(\mathbf{m}^n, \theta) + \frac{\sigma}{f(\theta)} C^n_{m_i \theta}(\mathbf{m}^n, \theta) \leq 0$ with strict inequality either if $m_i^n(\theta) < \gamma_i^n(\theta)$ or if $m_i^n(\theta) = \gamma_i^n(\theta)$ and $\sigma > 0$. Therefore, we must have $m_i^n(\theta) \geq \gamma_i^n(\theta)$ for all i, with strict inequality when $\sigma > 0$.

Claim 2. *Let $U'(\sigma, \theta) = -h_\theta(q(\sigma, \theta), \theta) - C^n_\theta(m^n(\sigma, \theta), \theta)$. Then for all $\theta \in [\underline{\theta}, \overline{\theta}]$ there exists a unique $\breve{\sigma}(\theta)$ such that $U'(\sigma, \theta) < 0$ if $\sigma < \breve{\sigma}(\theta)$ and $U'(\sigma, \theta) > 0$ if $\sigma > \breve{\sigma}(\theta)$. Furthermore, $\breve{\sigma}(\overline{\theta}) = 0$ and $\breve{\sigma}(\theta) > 0$ for all $\theta < \overline{\theta}$.*

Since $h_{q\theta} > 0$ and $\frac{\partial^2 C^n}{\partial m_i \partial \theta} < 0$ for all i, it follows from Claim 1 that $U'(\sigma, \theta)$ is strictly increasing in σ. Hence there is at most one value σ such that $U'(\sigma, \theta) = 0$. To establish

[19] Henceforth, we will also assume that $q(\sigma, \theta)$ and $\mathbf{m}^n(\sigma, \theta)$ are unique. This can always be guaranteed by assuming that the objective function in (42) is quasiconcave in q, and that the objective function in (43) is quasiconvex in $\mathbf{m}^n$.

that such value exists, let us show that $U'(\sigma,\theta) \leq 0$ when $\sigma = 0$, and $U'(\sigma,\theta) > 0$ when σ is sufficiently large.

First, consider $\sigma = 0$. It follows from (42) and (43) that $q(0,\theta) = q^{FB}(\theta) \geq 0$ and $\mathbf{m}^n(0,\theta) = \gamma^n(\theta)$ for all θ. Since $C^n_\theta(\gamma(\theta),\theta) = 0$, we have $U'(0,\theta) = -h_\theta(q_{FB}(\theta),\theta) \leq 0$. This inequality is strict for all $\theta < \overline{\theta}$ because $q_{FB}(\theta) > 0$ for all such θ. It follows that $\breve{\sigma}(\theta) > 0$ for all $\theta < \overline{\theta}$. Furthermore, since $q_{FB}(\overline{\theta}) = 0$ we have $U'(0,\overline{\theta}) = -h_\theta(0,\overline{\theta}) = 0$, so $\breve{\sigma}(\overline{\theta}) = 0$.

Next, let us show that $U'(\sigma,\theta) > 0$ for all $\sigma > \overline{\sigma}(\theta)$, where $\overline{\sigma}(\theta) = \frac{v'(0)-h_q(0,\theta)}{\min_{q\in[0,q^{FB}(\theta)]} h_{q\theta}(q,\theta)} f(\theta)$. To this end, we first claim that $q(\sigma,\theta) = 0$ for all $\sigma \geq \overline{\sigma}(\theta)$. Indeed, suppose to the contrary that $q(\sigma,\theta) > 0$ for some $\sigma \geq \overline{\sigma}(\theta)$. Since $q(\sigma,\theta)$ satisfies (38), we have:

$$v_q(q(\sigma,\theta)) - h_q(q(\sigma,\theta),\theta) = \frac{\sigma}{f(\theta)} h_{q\theta}(q(\sigma,\theta),\theta) \geq \frac{\overline{\sigma}}{f(\theta)} h_{q\theta}(q(\sigma,\theta),\theta) \geq v'(0) - h_q(0,\theta) \quad (44)$$

But (44) contradicts the assumption that $v_q(q) - h_q(q,\theta)$ is strictly decreasing in q.

Since $q(\sigma,\theta) = 0$ for all $\sigma \geq \overline{\sigma}(\theta)$, it follows that $U'(\sigma,\theta) = -C^n_\theta(\mathbf{m}^n(\sigma,\theta),\theta)$ for all $\sigma \geq \overline{\sigma}(\theta)$. By Claim 1, $\mathbf{m}^n(\sigma,\theta) > \gamma^n(\theta)$ whenever $\sigma > \overline{\sigma}(\theta)$, because $\overline{\sigma}(\theta) \geq 0$. Since $C^n_\theta(\gamma^n(\theta),\theta) = 0$ and $\frac{\partial^2 C^n}{\partial\theta\partial m_i} < 0$, it then follows that $C^n_\theta(\mathbf{m}^n(\sigma,\theta),\theta) < 0$, thereby establishing that $U'(\sigma,\theta) > 0$ for all $\sigma > \overline{\sigma}(\theta)$. Combining this with $U'(0,\theta) \leq 0$ and the uniqueness of $\breve{\sigma}(\theta)$ s.t. $U'(\breve{\sigma}(\theta),\theta) = 0$ implies that $U'(\sigma,\theta) < 0$ if $\sigma < \breve{\sigma}(\theta)$, and $U'(\sigma,\theta) > 0$ if $\sigma > \breve{\sigma}(\theta)$.

Claim 3. *$U(F(\theta),\theta)$ has a unique minimizer $\widehat{\theta}_n \in (0,\theta^*)$. Moreover, $U'(F(\theta),\theta) < 0$ for $\theta < \widehat{\theta}_n$ and $U'(F(\theta),\theta) > 0$ for $\theta > \widehat{\theta}_n$.*

It is easy to see that $U^n_{SB}(\theta) \equiv U(F(\theta),\theta)$ is strictly quasiconvex. So it has a unique global and local minimizer which we take to be $\widehat{\theta}_n$. So $U'(F(\theta),\theta) < 0$ for all $\theta \in [\underline{\theta},\widehat{\theta}_n)$ and $U'(F(\theta),\theta) > 0$ for all $\theta \in [\widehat{\theta}_n,\overline{\theta}]$.

We will establish that $U'(F(\underline{\theta}),\underline{\theta}) < 0$ and $U'(F(\theta^*),\theta^*) > 0$, implying that $\widehat{\theta}_n \in (0,\theta^*)$. By Claim 2, $\breve{\sigma}(\underline{\theta}) > 0$ and hence $U'(F(\underline{\theta}),\underline{\theta}) = U'(0,\underline{\theta}) < 0$. Now, $q(F(\theta^*),\theta^*) = q^{SB}(\theta^*) = 0$ so that $h_\theta(q(F(\theta^*),\theta^*),\theta^*) = 0$. Also, since $F(\theta^*) > 0$, by Claim 1 $\mathbf{m}^n(F(\theta^*),\theta^*) > \gamma^n(\theta^*)$ and so $C^n_\theta(\mathbf{m}^n(F(\theta^*),\theta^*),\theta^*) < 0$. Thus $U'(F(\theta^*),\theta^*) > 0$.

Claim 4. *On the interval $[\underline{\theta},\widehat{\theta}_n)$ the solution to the relaxed program is such that $q(\theta) = q^{SB}(\theta)$, $m^n(\theta) = m^n(F(\theta),\theta)$* and $U'(\theta) < 0$.

By Claim 3 it suffices to prove that $\sigma(\theta) = F(\theta)$ for all $\theta \leq \widehat{\theta}_n$. Note that $\sigma(\theta) \leq F(\theta)$ by the transversality condition $\sigma(\underline{\theta}) \leq 0$, and the fact that $\sigma'(\theta) \leq f(\theta)$, which is implied by the costate equation (40) and the condition $\rho(\theta) \geq 0$.

Now recall from the proof of Claim 2 that $U(\sigma,\theta)$ is strictly increasing in σ. It follows from $\sigma(\theta) \leq F(\theta)$ that $U'(\theta) = U'(\sigma(\theta),\theta) \leq U'(F(\theta),\theta) < 0$ for all $\theta \leq \widehat{\theta}_n$, where the final inequality holds by Claim 3. Since $U(\widehat{\theta}_n) \geq 0$, we therefore have $U(\theta) > 0$ for all $\theta < \widehat{\theta}_n$. The transversality condition then implies that $\sigma(\underline{\theta}) = 0$, and the complementary slackness condition (41) then implies that $\rho(\theta) = 0$ so that $\sigma'(\theta) = f(\theta)$ for all $\theta < \widehat{\theta}_n$. This establishes that $\sigma(\theta) = F(\theta)$ for all $\theta < \widehat{\theta}_n$, and hence by continuity also at $\widehat{\theta}_n$.

Claim 5. $U(\widehat{\theta}_n) = 0$, *and* $\sigma(\theta) < F(\theta)$ *for all* $\theta > \widehat{\theta}_n$.

Since $\sigma(\widehat{\theta}_n) = F(\widehat{\theta}_n)$ by Claim 4 and $\sigma'(\theta) \leq f(\theta)$ for all θ by (40), it follows that $\sigma(\theta) \leq F(\theta)$ for all $\theta > \widehat{\theta}_n$.

Let $\theta' = \max\{\theta \geq \widehat{\theta}_n : \sigma(\theta) = F(\theta)\}$, and suppose that contrary to this Claim $\theta' > \widehat{\theta}_n$. Let us show that in this $\theta' = \overline{\theta}$ then. Indeed, since $\sigma(\theta') = F(\theta')$, we must have $\sigma(\theta) = F(\theta)$ for all $\theta \in (\widehat{\theta}_n, \theta']$. It then follows from Claim 3 that $U'(\theta) = U'(\sigma(\theta), \theta) = U'(F(\theta), \theta) > 0$ on $(\widehat{\theta}_n, \theta']$, implying that $U(\theta') > 0$. Thus if $\theta' < \overline{\theta}$, there exists a right neighborhood V of θ' on which the individual rationality constraint $U(\theta) \geq 0$ is not binding. It then follows from the complementary slackness condition (41) that on this neighborhood we have $\rho(\theta) = 0$, and hence by the costate equation (40) that $\sigma'(\theta) = f(\theta)$. Thus $\sigma(\theta) = F(\theta)$ for all $\theta \in V$, contradicting the definition of θ', thereby establishing that $\theta' = \overline{\theta}$.

Next, let us show that $\breve{\sigma}(\theta) < F(\theta)$ for all $\theta > \widehat{\theta}_n$. Indeed, since $\sigma(\theta) = F(\theta)$ for all $\theta > \widehat{\theta}_n$, Claim 3 yields $U'(\theta) = U'(F(\theta), \theta) > 0$ for all $\theta > \widehat{\theta}_n$. Because $U'(\sigma, \theta)$ is strictly increasing in σ this implies that $\breve{\sigma}(\theta) < F(\theta)$ for all $\theta > \widehat{\theta}_n$.

Since $q(\theta) = q(\sigma(\theta), \theta) = q(F(\theta), \theta)$ and $\mathbf{m}^n(\theta) = \mathbf{m}^n(\sigma(\theta), \theta) = \mathbf{m}^n(F(\theta), \theta)$, and since $0 < \breve{\sigma}(\theta) < F(\theta)$ for all $\theta > \widehat{\theta}_n$, it follows from Claim 1 that $q(\theta) < q(\breve{\sigma}(\theta), \theta) < q^{FB}(\theta)$ and $\gamma^n(\theta) < \mathbf{m}^n(\breve{\sigma}(\theta), \theta) < \mathbf{m}^n(F(\theta), \theta) = \mathbf{m}^n(\theta)$ for all $\theta \in (\widehat{\theta}_n, \overline{\theta})$. But then the value of the relaxed program can be strictly increased by setting $U(\theta) = 0$ and assigning $(q(\breve{\sigma}(\theta), \theta), \mathbf{m}^n(\breve{\sigma}(\theta), \theta))$ on the interval $[\widehat{\theta}_n, \overline{\theta}]$, as can be seen from

$$v(q(\breve{\sigma}(\theta), \theta)) - h(q(\breve{\sigma}(\theta), \theta), \theta) - C^n(\mathbf{m}^n(\breve{\sigma}(\theta), \theta), \theta) > v(q(\theta)) - h(q(\theta), \theta) - C^n(\mathbf{m}^n(\theta), \theta)$$

This contradiction establishes that $\sigma(\theta) < F(\theta)$ for all $\theta > \widehat{\theta}_n$.

It follows that there exists a decreasing sequence $\{\theta_\ell\} \subset (\widehat{\theta}_n, \overline{\theta})$ converging to $\widehat{\theta}_n$, such that $\rho(\theta_\ell) > 0$ and hence $U(\theta_\ell) = 0$. So, by continuity of $U(\cdot)$ we have $U(\widehat{\theta}_n) = 0$.

Claim 6. $U(\theta) = 0$ *for all* $\theta > \widehat{\theta}_n$ *if and only if* $\breve{\sigma}'(\theta) \leq f(\theta)$ *for all* $\theta > \widehat{\theta}_n$.

If $U(\theta) = 0$ for all $\theta > \widehat{\theta}_n$, then $\sigma(\theta) = \breve{\sigma}(\theta)$ and hence $\sigma'(\theta) = \breve{\sigma}'(\theta)$ for all $\theta > \widehat{\theta}_n$. Since $\rho(\theta) \geq 0$, by the costate equation we have $\breve{\sigma}'(\theta) = \sigma'(\theta) = f(\theta) - \rho(\theta) \leq f(\theta)$ for all $\theta > \widehat{\theta}_n$.

Conversely, suppose that $\breve{\sigma}'(\theta) \leq f(\theta)$ for all $\theta > \widehat{\theta}_n$, and that contrary to the statement of the claim there exists some $\theta \geq \widehat{\theta}_n$ such that $U(\theta) > 0$. Let $\theta_1 = \inf\{\theta \geq \widehat{\theta}_n : U(\theta) > 0\}$.

We now claim that $\sigma(\theta_1) = \breve{\sigma}(\theta_1)$. If $\theta_1 > \widehat{\theta}_n$ this is immediate, since we then have $U(\theta) = 0$ and hence $\sigma(\theta) = \breve{\sigma}(\theta)$ for all $\theta \in [\widehat{\theta}_n, \theta_1]$. Now if $\theta_1 = \widehat{\theta}_n$, it follows from Claim 5 that for every $\varepsilon > 0$ there exists $(\theta', \theta'') \subset (\widehat{\theta}_n, \widehat{\theta}_n + \varepsilon)$ on which $U(\theta) = 0$ and hence $\sigma(\theta) = \breve{\sigma}(\theta)$. The continuity of the functions $\sigma(\cdot)$ and $\breve{\sigma}(\cdot)$ then implies that $\breve{\sigma}(\widehat{\theta}_n) = \sigma(\widehat{\theta}_n)$.

Next, we establish that $\sigma(\theta) > \breve{\sigma}(\theta)$ for all $\theta > \theta_1$. Indeed, $\sigma(\theta) \geq \breve{\sigma}(\theta)$ for all $\theta > \theta_1$, since on any interval on which $U(t) > 0$ we have $\sigma'(t) = f(t) \geq \breve{\sigma}'(t)$, and on any interval on which $U(t) = 0$ we have $\sigma(t) = \breve{\sigma}(t)$. But we cannot have $\sigma(\theta) = \breve{\sigma}(\theta)$ for any $\theta > \theta_1$, as this would imply that $\sigma'(t) = f(t) = \breve{\sigma}'(t)$ for all $t \in (\theta_1, \theta)$, hence that $\sigma(t) = \breve{\sigma}(t)$ for all $t \in [\theta_1, \theta]$. But then we have $U(t) = U(\theta_1)$ for all $t \in [\theta_1, \theta]$, contradicting the definition of θ_1.

It follows from the fact that $\sigma(\theta) > \breve{\sigma}(\theta)$ for all $\theta > \theta_1$ that $U'(\theta) = U'(\sigma(\theta), \theta) > U'(\breve{\sigma}(\theta), \theta) = 0$, and hence that $U(\theta)$ is strictly increasing for all $\theta > \theta_1$. As in the proof of Claim 5, we can then show that the value of the relaxed program can be improved by setting $U(\theta) = 0$ and assigning $(q(\breve{\sigma}(\theta), \theta), \mathbf{m}^n(\breve{\sigma}(\theta), \theta))$ for all $\theta > \theta_1$. This contradiction establishes that $U(\theta) = 0$ for all $\theta > \widehat{\theta}_n$.

Claim 7. $\mathbf{m}^n(\underline{\theta}) = \gamma^n(\underline{\theta})$, $\mathbf{m}^n(\overline{\theta}) = \gamma^n(\overline{\theta})$ and $\mathbf{m}^n(\theta) > \gamma^n(\theta)$ for all $\theta \in (\underline{\theta}, \overline{\theta})$. *Furthermore,* $q(\theta) > 0$ *for all* $\theta \in [\underline{\theta}, \overline{\theta})$.

By Claim 4 $\sigma(\theta) = F(\theta)$ for all $\theta \leq \widehat{\theta}_n$, and by Claim 6 $\sigma(\theta) = \breve{\sigma}(\theta)$ for all $\theta \geq \widehat{\theta}_n$. Also, by Claim 2, $\breve{\sigma}(\overline{\theta}) = 0$ and $\breve{\sigma}(\theta) > 0$ for all $\theta < \overline{\theta}$. So, $\sigma(\underline{\theta}) = \sigma(\overline{\theta}) = 0$, and $\sigma(\theta) > 0$ for all $\theta \in (\underline{\theta}, \overline{\theta})$. Then from Claim 1 it follows that $\mathbf{m}^n(\theta) > \gamma^n(\theta)$ for all $\theta \in (\underline{\theta}, \overline{\theta})$. Further, from (43) it follows that $\mathbf{m}^n(0, \theta) = \gamma^n(\theta)$. Hence, $\mathbf{m}^n(\underline{\theta}) = \gamma^n(\underline{\theta})$, $\mathbf{m}^n(\overline{\theta}) = \gamma^n(\overline{\theta})$.

To establish that $q(\theta) > 0$ for all $\theta < \overline{\theta}$, note that $q(\theta) = q^{SB}(\theta) > 0$ for all $\theta \in [\underline{\theta}, \widehat{\theta}_n)$. Now, if $q(\theta) = 0$ for some $\theta \in [\widehat{\theta}_n, \overline{\theta})$, then since $h_\theta(0, \theta) = 0$ equation (36) implies that $0 = U'(\theta) = C^n_\theta(\mathbf{m}^n(\theta), \theta)$. But from $\mathbf{m}^n(\theta) > \gamma^n(\theta)$, $C^n_\theta(\gamma^n(\theta), \theta) = 0$ and $C^n_{m_i} < 0$ for all i it follows that $C^n_\theta(\mathbf{m}^n(\theta), \theta) < 0$, a contradiction. Thus $q(\theta) > 0$ for all $\theta \in [\widehat{\theta}_n, \overline{\theta})$.

Claim 8. *Global incentive compatibility of the solution to the relaxed program.*

It remains to show that this solution satisfies incentive constraints (33) i.e., for any pair of types (θ, θ') we have:

$$U(\theta) - U(\theta') + h(q(\theta'), \theta) + C^n(\mathbf{m}^n(\theta'), \theta) - h(q(\theta'), \theta') - C^n(\mathbf{m}^n(\theta'), \theta') \geq 0 \tag{45}$$

First, suppose that $\theta' \in [\widehat{\theta}, \overline{\theta}]$ i.e., $U(\theta') = 0$. We will consider the case $\theta' > \theta$. The proof for the case $\theta' < \theta$ is similar. Then we have:

$$\begin{aligned} &U(\theta) - U(\theta') + h(q(\theta'), \theta) + C^n(\mathbf{m}^n(\theta'), \theta) - h(q(\theta'), \theta') - C^n(\mathbf{m}^n(\theta'), \theta') = \\ &U(\theta) - U(\theta') - \int_\theta^{\theta'} h_\theta(q(\theta'), s) + C^n_\theta(\mathbf{m}^n(\theta'), s) ds > -\int_\theta^{\theta'} h_\theta(q(\theta'), \theta') + C^n_\theta(\mathbf{m}^n(\theta'), \theta') ds = 0 \end{aligned} \tag{46}$$

The first inequality holds because $U(\theta) \geq 0 = U(\theta')$ The last equality holds because $\theta' \in [\widehat{\theta}_n, \overline{\theta}]$ and so $U'(\theta') = -h_\theta(q(\theta'), \theta') - C^n_\theta(\mathbf{m}^n(\theta'), \theta') = 0$, establishing the incentive compatibility of our mechanism for this case.

Next, suppose that $\theta, \theta' \in [\underline{\theta}, \widehat{\theta}_n]$. Over this region, the solution is $\{q^{SB}(\theta), \mathbf{m}^n(F(\theta),\theta), F(\theta)\}$, and incentive compatibility holds if $q^{SB}(\theta)$ is decreasing in θ, and $\mathbf{m}^n(F(\theta),\theta)$ is increasing in θ (Guesnerie and Laffont, 1984, Theorem 2). That $q^{SB}(\theta)$ is decreasing in θ follows from Assumption 4(i). Next, as a maximizer of the Hamiltonian H, $\mathbf{m}^n(F(\theta),\theta)$ minimizes $C^n(\mathbf{m}^n,\theta) + \frac{F(\theta)}{f(\theta)} C^n_\theta(\mathbf{m}^n,\theta)$. By Assumption 4(ii), this objective has strictly increasing differences in $(\mathbf{m}^n,\theta)$, and is supermodular in $\mathbf{m}^n$. Therefore, $\mathbf{m}^n(F(\theta),\theta)$ is increasing in θ, establishing incentive compatibility in this case.

Finally, let us show that incentive constraints hold for any pair (θ,θ') such that $\theta \in (\hat{\theta}, 1]$ and $\theta' \in [\underline{\theta}, \hat{\theta}]$. Let us rewrite the left-hand side of (45) as follows:

$$\begin{aligned}
&U(\theta) - U(\theta') + h(q(\theta'),\theta) + C^n(\mathbf{m}^n(\theta'),\theta) - h(q(\theta'),\theta') - C^n(\mathbf{m}^n(\theta'),\theta') = \\
&U(\theta) - U(\widehat{\theta}) + \Big(h(q(\theta'),\theta) + C^n(\mathbf{m}^n(\theta'),\theta) - h(q(\theta'),\widehat{\theta}) - C^n(\mathbf{m}^n(\theta'),\widehat{\theta})\Big) + \\
&-\Big(U(\theta') - U(\widehat{\theta})\Big) + \Big(h(q(\theta'),\widehat{\theta}) + C^n(\mathbf{m}^n(\theta'),\widehat{\theta}) - h(q(\theta'),\theta') - C^n(\mathbf{m}^n(\theta'),\theta')\Big)
\end{aligned} \tag{47}$$

To confirm that the incentive constraint between θ and θ' holds, we need to show that the expression in (47) is nonnegative. To this end, we will establish separately that both the second line and the third line in (47) are nonnegative. Start with the second line. We have $U(\theta) = U(\widehat{\theta}) = 0$. Further, consider the expression in brackets in the second line. We have:

$$\begin{aligned}
&h(q(\theta'),\theta) + C^n(\mathbf{m}^n(\theta'),\theta) - h(q(\theta'),\widehat{\theta}) - C^n(\mathbf{m}^n(\theta'),\widehat{\theta}) = \int_{\widehat{\theta}}^{\theta} h_\theta(q(\theta'),s) + C^n_\theta(\mathbf{m}^n(\theta'),s)ds \geq \\
&\int_{\widehat{\theta}}^{\theta} h_\theta(q(\theta'),\widehat{\theta}_n) + C^n_\theta(\mathbf{m}^n(\theta'),\widehat{\theta}_n)ds > \int_{\widehat{\theta}}^{\theta} h_\theta(q(\widehat{\theta}),\widehat{\theta}) + C^n_\theta(\mathbf{m}^n(\widehat{\theta}),\widehat{\theta}_n)ds = 0.
\end{aligned} \tag{48}$$

The first inequality in (48) holds because $h_{\theta\theta} \geq 0$ and $C^n_{\theta\theta} > 0$. The second inequality holds because $q(\theta') > q(\widehat{\theta})$ and $\mathbf{m}^n(\theta') < \mathbf{m}^n(\widehat{\theta})$ (as established above in this Claim), while $h_{\theta q} > 0$ and $C^n(\mathbf{m}^n,\theta)$ has strictly decreasing differences in $(\mathbf{m}^n,\theta)$. The last equality holds because by Claim 8 $h_\theta(q(\widehat{\theta}),\widehat{\theta}) + C^n_\theta(\mathbf{m}^n(\widehat{\theta}),\widehat{\theta}) = 0$. So, the second line in (47) is nonnegative.

Now consider the third line in (47). We have:

$$\begin{aligned}
&-\Big(U(\theta') - U(\widehat{\theta})\Big) + \Big(h(q(\theta'),\widehat{\theta}) + C^n(\mathbf{m}^n(\theta'),\widehat{\theta}) - h(q(\theta'),\theta') - C^n(\mathbf{m}^n(\theta'),\theta')\Big) = \\
&-\int_{\theta'}^{\widehat{\theta}} h_\theta(q(s),s) + C^n_\theta(\mathbf{m}^n(s),s)ds + \int_{\theta'}^{\widehat{\theta}_n} h_\theta(q(\theta'),s) + C^n_\theta(\mathbf{m}^n(\theta'),s)ds = \\
&\int_{\theta'}^{\widehat{\theta}} (h_\theta(q(\theta'),s) - h_\theta(q(s),s)) + \big(C^n_\theta(\mathbf{m}^n(\theta'),s) - C^n_\theta(\mathbf{m}^n(s),s)\big)\, ds > 0
\end{aligned} \tag{49}$$

The first equality holds because $U'(\theta) = -h_\theta(q(\theta),\theta) - C^n_\theta(\mathbf{m}^n(\theta),\theta)$. The inequality holds because, as shown in this Claim, $q(\theta') > q(s)$ and $\mathbf{m}^n(\theta') < \mathbf{m}^n(s)$ for all $s \in (\theta', \widehat{\theta}_n]$, while $h_{\theta q} > 0$ and $C^n(\mathbf{m}^n,\theta)$ has decreasing differences in $(\mathbf{m}^n,\theta)$. So, (47) is nonnegative. *Q.E.D.*

Proof of Lemma 1: Let us show that $\breve{\sigma}'(\theta) \leq f(\theta)$. By definition, $\breve{\sigma}(\theta)$ satisfies $U'(\breve{\sigma}(\theta), \theta) = 0$, and so $U'_\sigma \breve{\sigma}'(\theta) + U'_\theta = 0$. Because $U'_\sigma > 0$, $\breve{\sigma}'(\theta) \leq f(\theta)$ if and only if

$$U'_\sigma(\breve{\sigma}(\theta), \theta) f(\theta) + U'_\theta(\breve{\sigma}(\theta), \theta) \geq 0. \tag{50}$$

Recall that $U'(\sigma, \theta) = -h_\theta(q(\sigma, \theta), \theta) - C^n_\theta(\mathbf{m}^n(\sigma, \theta), \theta)$. Hence, $U'_\sigma = -h_{q\theta} q_\sigma - C^n_{\theta\mathbf{m}} \mathbf{m}^n_\sigma$ and $U'_\theta = h_{q\theta} q_\theta - h_{\theta\theta} - C^n_{\theta\mathbf{m}} \mathbf{m}^n_\theta - C^n_{\theta\theta}$. Thus (50) is equivalent to

$$-h_{q\theta}[q_\sigma f + q_\theta] - h_{\theta\theta} - C^n_{\theta\mathbf{m}}[\mathbf{m}^n_\sigma f + \mathbf{m}^n_\theta] - C^n_{\theta\theta} \geq 0 \tag{51}$$

So it is sufficient to verify that the assumptions of the Lemma guarantee that the inequality (51) holds. In fact, we will show that $h_{q\theta}[q_\sigma f + q_\theta] + h_{\theta\theta} \leq 0$ and $C^n_{\theta\mathbf{m}}[\mathbf{m}^n_\sigma f + \mathbf{m}^n_\theta] + C^n_{\theta\theta} \leq 0$.

To this effect, let us first calculate q_σ and q_θ. As a maximizer of (42), $q(\sigma, \theta)$ satisfies the first-order condition $v'(q) - h_q(q, \theta) - \frac{\sigma}{f(\theta)} h_{q\theta}(q, \theta) = 0$, from which it follows that:

$$\left(v''(q) - h_{qq} - \frac{\sigma}{f} h_{qq\theta}\right) q_\sigma f - h_{q\theta} = 0$$
$$\left(v''(q) - h_{qq} - \frac{\sigma}{f} h_{qq\theta}\right) q_\theta - h_{q\theta}\left(1 - \frac{\sigma f'}{f^2}\right) - \frac{\sigma}{f} h_{q\theta\theta} = 0$$

Because $q(\sigma, \theta)$ is a maximizer, the second order condition $v''(q) - h_{qq} - \frac{\sigma}{f} h_{qq\theta} \leq 0$ holds. Consequently, the inequality $h_{q\theta}[q_\sigma f + q_\theta] + h_{\theta\theta} \leq 0$ is equivalent to

$$h_{q\theta}\left[h_{q\theta}\left(2 - \frac{\sigma f'}{f^2}\right) + \frac{\sigma}{f} h_{q\theta\theta}\right] + h_{\theta\theta}[v''(q) - h_{qq} - \frac{\sigma}{f} h_{qq\theta}] \geq 0 \tag{52}$$

The second term in (52) is positive because $h_{\theta\theta} \leq 0$ by assumption of the Lemma.

Next, $1 - \frac{\sigma f'}{f^2} > 0$. This is immediate if $f' \leq 0$. If $f' > 0$ then, since $\frac{F(\theta)}{f(\theta)}$ is increasing by assumption, it follows that $1 - \frac{\sigma f'}{f^2} \geq 1 - \frac{F f'}{f^2} = 1 - \left(\frac{F}{f}\right)' > 0$. So the assumption that $h_{q\theta} \geq 0$ and $h_{q\theta\theta} \geq 0$ also guarantees that the first term in (52) is positive.

Similar algebraic steps establish that $C^n_{\theta\mathbf{m}}[\mathbf{m}^n_\sigma f + \mathbf{m}^n_\theta] + C^n_{\theta\theta} \leq 0$ under the conditions of the Lemma.

Proof of Theorem 6: First, we claim that there exists K such that $\sigma(\theta, n) \leq \frac{K}{n}$. Since $\frac{\partial C^n}{\partial m_i}(\gamma_i(\theta), \mathbf{m}^n_{-i}, \theta) = 0$, it follows from the mean value Theorem that $\frac{\partial C^n}{\partial m_i}(m_i, \mathbf{m}^n_{-i}, \theta) = \frac{\partial^2 C^n}{\partial m_i^2}(\bar{m}_i, \mathbf{m}^n_{-i}, \theta)(m_i - \gamma_i(\theta))$ for some $\bar{m}_i \in (m_i, \gamma_i)$. From (39) we then have:

$$m_i(\theta) - \gamma_i(\theta) = -\sigma(\theta, n) \frac{\frac{\partial^2 C^n}{\partial\theta\partial m_i}(\mathbf{m}^n(\theta), \theta)}{\frac{\partial^2 C^n}{\partial m_i^2}(\bar{m}_i, \mathbf{m}^n_{-i}(\theta), \theta)} \tag{53}$$

Using the fact that $C^n_\theta(\gamma^n(\theta), \theta) = 0$, and applying the mean value Theorem once more yields:

$$C^n_\theta(\mathbf{m}^n(\theta), \theta) = \sum_{i=1}^n \frac{\partial^2 C^n}{\partial\theta\partial m_i}(\bar{\mathbf{m}}^n, \theta)(m_i(\theta) - \gamma_i(\theta)), \tag{54}$$

where $\bar{\mathbf{m}}^n = \gamma^n(\theta) + \varepsilon(\theta)(\mathbf{m}^n(\theta) - \gamma^n(\theta))$, for some $\varepsilon(\theta) \in (0,1)$. Recall that by Claim 2 of Theorem 5 $\sigma(\theta, n) \geq 0$. Using the assumption that $0 \leq \frac{\partial^2 C^n}{\partial m_i^2} \leq \bar{v}$, $\left|\frac{\partial^2 C^n}{\partial\theta\partial m_i}\right| \geq \underline{v} > 0$, (53) and (54) then yield:

$$C_\theta^n(\mathbf{m}^n(\theta), \theta) \leq -n\sigma(\theta, n)\frac{\underline{v}^2}{\bar{v}} \tag{55}$$

Next, $h_\theta(q(\theta), \theta) \leq h_\theta(q^{FB}(\theta), \theta) \leq q^{FB}(\underline{\theta}) \max h_{q\theta}$, where the first inequality follows from $h_{q\theta} > 0$, and the second inequality holds because $h_\theta(0, \theta) = 0$ and $q^{FB}(\theta)$ is decreasing in θ. Since $U'(\theta) \leq 0$ on the interval $[\underline{\theta}, \overline{\theta}]$, equation (36) implies

$$0 \geq U'(\theta) = -h_\theta(q(\theta), \theta) - C_\theta^n(\mathbf{m}^n(\theta), \theta) \geq -q^{FB}(\underline{\theta}) \max h_{q\theta} + n\sigma(\theta, n)\frac{\underline{v}^2}{\bar{v}}.$$

Setting $K = \frac{\bar{v}q^{FB}(\underline{\theta})}{\underline{v}^2} \max h_{q\theta}$ then establishes the claim.

By (53) we have $m_i(\theta) - \gamma_i(\theta) \leq \sigma(\theta, n)\frac{\bar{v}}{\underline{v}} \leq \frac{K}{n}\frac{\bar{v}}{\underline{v}}$. Furthermore, by (38) we have

$$\sigma(\theta, n)h_{q\theta}(q(\theta), \theta) = \{v_q(q(\theta)) - h_q(q(\theta), \theta)\}f(\theta) = f(\theta)(v_{qq}(q_1(\theta)) - h_{qq}(q_1(\theta), \theta)(q(\theta) - q^{FB}(\theta)),$$

for some $q_1(\theta) \in (q(\theta), q^{FB}(\theta))$. Hence $q^{FB}(\theta) - q(\theta) \leq \sigma(\theta, n)M \leq \frac{KM}{n}$, where $M = \frac{\max h_{q\theta}}{\min f(\theta) \times \min |v_{qq} - h_{qq}|}$. Hence $q(\theta) \to q^{FB}(\theta)$ and $m_i(\theta) \to \gamma_i(\theta)$ uniformly in θ.

Next, recall that $q^{SB}(\hat{\theta}(n)) = q(\hat{\theta}(n))$ for all n. Hence we have:

$$q^{FB}(\hat{\theta}(n)) - q^{SB}(\hat{\theta}(n)) = \frac{F(\hat{\theta}(n))}{f(\hat{\theta}(n))}\frac{h_{q\theta}}{|v_{qq} - h_{qq}|}(q_2(\hat{\theta}(n)), \hat{\theta}(n) \leq \frac{KM}{n},$$

for some $q_2(\hat{\theta}(n)) \in (q^{SB}(\hat{\theta}(n)), q^{FB}(\hat{\theta}(n))$. Since $F(\hat{\theta}(n)) \geq \left(\hat{\theta}(n) - \underline{\theta}\right) \min_{\theta \in [\underline{\theta}, \hat{\theta}(n)]} f(\theta)$, we have $\hat{\theta}(n) - \underline{\theta} \leq \frac{KMP}{n}$, where $P = \frac{\max f(\theta)}{\min f(\theta)} \frac{\min |v_{qq} - hqq|}{\max h_{q\theta}}$. Thus $\hat{\theta}(n) \to \underline{\theta}$.

Since $U(\theta)$ is decreasing and $U(\theta) = 0$ for all $\theta \in [\widehat{\theta}_n(n), \overline{\theta}]$, we have:

$$U(\theta) \leq U(\underline{\theta}) \leq (\widehat{\theta}_n(n) - \underline{\theta}) \max_{\theta \in [\underline{\theta}, \widehat{\theta}_n(n)]} |U'(\theta)| \leq (\widehat{\theta}_n(n) - \underline{\theta}) \max_{\theta \in [\underline{\theta}, \widehat{\theta}_n(n)]} h_\theta(q(\theta), \theta) \leq \frac{KMPQ}{n},$$

where $Q = \max_{\theta \in [\underline{\theta}, \widehat{\theta}_n(n)]} h_{q\theta}(q(\theta), \theta)$. Hence $U(\theta) \to 0$ uniformly in θ.

Finally, since $C^n(\gamma^n(\theta), \theta) = 0$ and $C_\theta^n(\gamma^n(\theta), \theta) = 0$, a Taylor series expansion yields:

$$C^n(\mathbf{m}^n(\theta), \theta) = \sum_{i=1}^n \frac{(m_i - \gamma_i(\theta))^2}{2}\frac{\partial^2 C^n}{\partial m_i^2}(\underline{\mathbf{m}}^n, \theta) \leq \frac{\bar{v}}{2}\sum_{i=1}^n (m_i - \gamma_i(\theta))^2 \leq \frac{1}{n}\frac{K^2\bar{v}^3}{\underline{v}^2} \tag{56}$$

where $\underline{\mathbf{m}}^n \in (\mathbf{m}^n(\theta), \gamma^n(\theta))$. Thus $C^n(\mathbf{m}^n(\theta), \theta) \to 0$, uniformly in θ. *Q.E.D.*

Proof of Lemma 2: Let us first show that $W(n)$ is strictly concave. Since the optimal mechanism is unique, $W(n)$ is continuously differentiable and (1999, p. 217), $\frac{dW(n)}{dn} = \int_{\underline{\theta}}^{\bar{\theta}} \frac{\partial H}{\partial n}(q, m, U, \sigma, n, \theta)d\theta$.

By the first-order condition (39), we have $\frac{c_{m\theta}(m_i^n(\theta),\theta)}{c_m(m_i^n(\theta),\theta)} = \frac{c_{m\theta}(m_j^n(\theta),\theta)}{c_m(m_j^n(\theta),\theta)}$ for all $i,j \in \{1,...,n\}$. So, the assumption that $c_{mm}c_{m\theta} - c_{mm\theta}c_m < 0$ implies that the optimal $m_i^n(\theta)$ is unique and is the same for all i, so we can drop the subscript i and denote the optimal message by $m^n(\theta)$.

Using equation (37) yields $\frac{\partial H}{\partial n} = -c(m^n(\theta),\theta)f(\theta) - \sigma(\theta)c_\theta(m^n(\theta),\theta)$. Substituting for $\sigma(\theta)$ from (39) we then obtain $\frac{\partial H}{\partial n} = \left(\frac{c_\theta c_m}{c_{m\theta}} - c\right) f$, so we have:

$$\frac{dW(n)}{dn} = \int_{\underline{\theta}}^{\bar{\theta}} \left(\frac{c_\theta(m^n(\theta),\theta)c_m(m^n(\theta),\theta)}{c_{m\theta}(m^n(\theta),\theta)} - c(m^n(\theta),\theta)\right) f(\theta)d\theta \tag{57}$$

Thus, $W(n)$ is concave in n if $\left(\frac{c_\theta c_m}{c_{m\theta}} - c\right)$ decreases in n. Note that

$$\frac{d}{dn}\left(\frac{c_\theta c_m}{c_{m\theta}} - c\right) = \frac{c_\theta}{c_{m\theta}^2}\left(c_{mm}c_{m\theta} - c_{mm\theta}c_m\right)\frac{\partial m^n}{\partial n}. \tag{58}$$

Now, from the first-order condition (39) and Claim 7 in Theorem 5, $m^n(\theta)$ satisfies $c_m(m^n(\theta),\theta)f(\theta) + c_{m\theta}(m^n(\theta),\theta)F(\theta) = 0$ for $\theta \in [\underline{\theta}, \widehat{\theta}_n(n))$. So, $\frac{\partial m^n(\theta)}{\partial n} = 0$ for $\theta \in [\underline{\theta}, \widehat{\theta}_n)$.

Also, $h_\theta(q(\theta),\theta) + nc_\theta(m^n(\theta),\theta) = 0$ for all $\theta \in [\widehat{\theta}_n, \overline{\theta}]$. Since $h_{q\theta} > 0$ and $c_{m\theta} < 0$, $h_\theta(q(\theta),\theta) + nc_\theta(m^n(\theta),\theta) = 0$ can hold only if $\frac{\partial m^n(\theta)}{\partial n} < 0$ for $\theta \in [\widehat{\theta}_n, \overline{\theta}]$. Also, $\widehat{\theta}_n$ is decreasing in n. Finally, by Theorem 5 $m^n(\theta) > \gamma(\theta)$, so $c_\theta < 0$. It then follows from the assumption $c_{mm}c_{m\theta} - c_{mm\theta}c_m < 0$ that (58) is strictly negative on $[\underline{\theta}, \widehat{\theta}_n)$ and equal to zero on $[\widehat{\theta}_n, \overline{\theta}]$. So $W(n)$ is strictly concave.

Now let $\eta(m,\theta) = \frac{c_\theta c_m}{c_{m\theta}} - c$, and observe that $c(\gamma(\theta),\theta) = 0$ and $c_\theta(\gamma(\theta),\theta) = 0$ for every θ, so that $\eta(\gamma(\theta),\theta) = 0$. Furthermore, since $c_\theta(m,\theta) < 0$ for $m > \gamma(\theta)$ and $c_{mm}c_{m\theta} - c_{mm\theta}c_m < 0$, we have $\eta_m(m,\theta) > 0$ and hence $\eta(m,\theta) > 0$ for $m > \gamma(\theta)$. Since $m^n(\theta) > \gamma(\theta)$, it follows that the integrand in (57) is strictly positive for all θ, so $W(n)$ is strictly increasing in n.

To see that $\underline{K} > 0$, note that $\phi(m,\theta) = \frac{c_\theta c_m}{c_{m\theta}c} - 1$ is continuous in m, and satisfies $\phi(m,\theta) = \frac{\eta(m,\theta)}{c(m,\theta)} > 0$ for all $m > \gamma(\theta)$. Furthermore, applying l'Hospital's rule, we have

$$\lim_{m\to\gamma(\theta)} \phi(m,\theta) = \lim_{m\to\gamma(\theta)} \frac{c_{m\theta}c_m + c_\theta c_{mm}}{c_{m\theta}c_m + c_{mm\theta}c} = \lim_{m\to\gamma(\theta)} \frac{c_{m\theta} + \frac{c_\theta}{c_m}c_{mm}}{c_{m\theta} + \frac{c}{c_m}c_{mm\theta}} = 2,$$

where the final inequality holds because by l'Hospital's rule we have $\lim_{m\to\gamma(\theta)} \frac{c_\theta}{c_m} = \frac{c_{m\theta}}{c_{mm}}$ and $\lim_{m\to\gamma(\theta)} \frac{c}{c_m} = \lim_{m\to\gamma(\theta)} \frac{c_m}{c_{mm}} = 0$.

Thus $\phi(m,\theta)$ is continuous at $m = \gamma(\theta)$, and satisfies $\phi(\gamma(\theta),\theta) = 2$. It follows that $\underline{K}$, the minimum of $\phi(m,\theta)$ over the set D, is strictly positive.

Finally, it follows from Theorem 5 that $m^n(\theta) \in [\gamma(\theta), \gamma(\overline{\theta})]$. The definition of $\underline{K}$ and $\overline{K}$ then imply that $\underline{K}c \le \frac{c_m c_\theta}{c_{m\theta}} - c \le \overline{K}c$ for all θ, and so

$$0 < \underline{K}\int_{\underline{\theta}}^{\bar{\theta}} c(m(\theta),\theta)f(\theta)d\theta \le \frac{dW(n)}{dn} = G \le \overline{K}\int_{\underline{\theta}}^{\bar{\theta}} c(m(\theta),\theta)f(\theta)d\theta.$$

Q.E.D.