Who wants to be an auctioneer?

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Abstract

This paper endogenizes the decision whether to post a mechanism or to participate in another trader's mechanism in a competing mechanisms environment. With a population of heterogeneous buyers and sellers facing standard search frictions, each trader in our market has to decide whether to post a mechanism or to visit a mechanism posted by a trader on the other side of the market. We show that the equilibrium in this market is unique and is constrained efficient. Inefficient traders (low-value buyers and high-cost sellers) choose to visit with probability one, while more efficient traders randomize between posting and visiting. The resulting allocation differs substantially from the equilibrium allocation in the market where only one side can post mechanisms, especially when trader heterogeneity is significant. This suggests that decentralized marketplaces should allow participating buyers and sellers to self-select into making or receiving offers. We also provide conditions under which posting decisions are monotone, so that more efficient types post with higher probabilities than less efficient types.

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1 Introduction

This paper studies a large decentralized market populated by continua of heterogeneous buyers and sellers, who differ in their unit costs and valuations, respectively. Our main goal is to understand the traders' decisions whether to offer their own mechanisms or to participate in other traders' mechanisms, and to explore the consequences of these decisions for market outcomes. So, in a departure from the existing literature, in our setting every buyer, who has a unit demand, and every seller, who has a unit of the good for sale, can choose between posting a mechanism and visiting a mechanism posted by a potential trading partner.

This market operates as follows. First, each buyer and seller decides whether to post a mechanism or to visit a mechanism posted by a trader on the other side of the market. Upon observing all posted mechanisms, all non-posting traders simultaneously decide which mechanisms to visit. Then the posted mechanisms operate and the final allocations are determined.

The main results of our paper shows that, in equilibrium, two submarkets emerge and operate: one in which buyers post auction mechanisms visited by sellers, and the other one where the roles of the buyers and sellers are reversed. On a more general level, our contribution lies in establishing the existence and uniqueness of an equilibrium in a complex competing mechanism design setting, and characterizing the equilibrium properties. In particular, we show how the decision to post or visit a mechanism depends on a trader's type. We also evaluate the welfare gains from allowing both buyers and sellers to post mechanisms, compared to the market with fixed roles where the traders on one side must post, and the traders on the other side must visit.

Let us now describe our results in greater detail. We first consider a competing mechanisms problem in which all traders are heterogeneous, but sellers must post mechanisms, and buyers must visit. This is a continuation game that occurs in our general setting after the traders make their posting/visiting decisions. The heterogeneity of the posters distinguishes our setting from the existing literature. We show that this game has a unique equilibrium outcome in which the sellers post efficient mechanisms, such as the second-price auctions with reservations prices equal to their costs. We also provide a detailed characterization of the buyers' unique equilibrium strategies describing their visiting decisions, and their expected payoffs.

We then focus on out central issue exploring the buyers' and sellers' choices whether to post or visit mechanisms. Here, we characterize a constrained efficient outcome solving the planner's problem under market search frictions. These frictions imply symmetric assignment rules requiring that all visitors of the same type be assigned to the mechanisms according to the same probability distribution. Relying on our key technical result - the strict concavity of the constrained welfare function in the visiting/posting decisions-we establish the uniqueness of the constrained efficient allocation.

Next, we show that the decentralized equilibrium and the constrained efficient allocations coincide, and hence the equilibrium outcome is unique. The critical insight here is that a trader's private marginal benefit in equilibrium is equal to her/his marginal contribution to the social welfare, both when posting a mechanism or visiting one.

Thus, our decentralized market with two-sided posting attains maximal possible efficiency subject to the basic search friction. Intuitively, the two-sided posting allows more efficient traders -the higher value buyers and the lower cost sellers- to increase their trading probabilities by posting mechanisms. At the same time, less efficient types can earn positive profits by visiting, even if they cannot themselves attract visitors by posting. Beyond its theoretical importance, this result indicates that a trading platform can increase its volume of trade and profitability by creating multiple submarkets where traders from different sides of the market can post their offers or mechanisms. This advantage of two-sided posting is of general nature and, thus, should apply in other decentralized trading environments.

The welfare gains from allowing both sides to choose between posting and visiting can be substantial. For example, if the costs and values are distributed uniformly on the same interval but only the sellers can post auctions, then 39% of the sellers, who have higher costs, will not be able to attract any visitors and stay out in equilibrium. So, the buyers face a congestion and many of them fail to trade. In contrast, all trader types participate in our market, either by posting or visiting, and earn positive payoffs.

Furthermore, we uncover a notable pattern via numerical simulations: welfare gains from bilateral posting tend to increase as trader heterogeneity grows. In particular, Table 2 in the Appendix illustrates that in a two-type case these gains gets bigger, as the spread in valuations/costs increases and the type distribution becomes more uniform. This suggests that markets with two-sided posting are more likely to emerge under greater trader heterogeneity.

As far as the choice between posting and visiting, we demonstrate that less efficient (high cost or low valuation) traders visit with probability one. These traders cannot attract any visitors because they do not generate enough surplus. More generally, efficient types tend to post with a higher probability than less efficient types, because in equilibrium there are more visitors than posters, and so a poster is more likely to trade than a visitor. We provide two results highlighting this. First, we show that posting probabilities are monotone when the costs and valuations are distributed uniformly (and so the same is true when the distributions are not too far from the uniform). Second, we establish such monotonicity in a two-type example for a broad range of parameter values.

However, visiting and posting patterns can be non-monotone in type, In particular, when some types occur with a high likelihood. The presence of such type causes the trading probabilities of nearby types, when visiting, to change substantially depending on whether they are more or less efficient than the high-likelihood type. In contrast, the probability of trading when *posting* a mechanism does not exhibit such a jump in the trader's type. This difference implies that the high-frequency type must post with a high probability compared to all nearby types, in order for the nearby types to get the same payoffs from posting and visiting.

Let us now briefly highlight the markets where buyers and sellers exhibit behavior similar to the one in our bilateral posting market, In particular, where high value buyers and low cost sellers post auctions or mechanisms, and low value buyers and high cost sellers visit these mechanisms. First, consider the interactions between the developers/builders and home-purchasers. Affluent households wishing to acquire high-end homes often act as auctioneers seeking construction bids from developers for their projects. On the other hand, less wealthy households look for houses offered by developers and engage in bidding when they face competition for their preferred house. Similar phenomena are observed in the automobile markets, where car-auction houses typically auction lower-cost cars, while buyers interested in higher-end models can and often to request multiple bids from dealers. Likewise, a classified advertisement can be posted by both buyers and sellers (for more on this type of advertising in labor markets see DeVaro and Gürtler (2018)).

In the existing literature, the article most closely related to ours is Shi and Delacroix (2018) ("SD" below) who also study patterns of posting and visiting in a model of directed search. There are several major differences between their paper and ours, that involve the operation of the markets,[1] feasible mechanisms, the nature of traders' het-

[1] SD work with matching technology functions to match buyers and sellers, allowing rivalry to some-

erogeneity, and the presence of trading costs. SD focus mainly on fixed price mechanism with homogenous traders on each side of the market, while we consider optimal trading mechanism. SD consider an important asymmetry between buyers and sellers in the elasticity of supply or entry, with one side having an elastic size (supply), while the size of the other side is fixed. This asymmetry between buyers and sellers, together with entry and posting costs, determine who organizes trade (posts). In comparison, in our paper the heterogeneity of buyers' valuations and seller's costs plays the central role in the traders' individual decisions whether to post or visit and the resulting posting patterns, as more efficient types tend to post more frequently in our market.[2]

The central focus of SD is the role of entry and posting costs, and they provide several novel and interesting insights. We address such costs in our two-type model in Section 5, and show that both entry and posting costs make visiting relatively more attractive for the inefficient traders. In fact, we numerically identify a set of parameter values under which higher posting costs *increase* the probability of posting by the more efficient types, because of lower competition in posting from the inefficient types.

Our paper is also related to the large directed search literature pioneered by Peters (1984) and applied to various fields such as monetary and labor economics, and industrial organization. McAfee (1993) and Peters and Severinov (1997) are early contributions analyzing competing auctions. These papers maintain the assumption of fixed visiting and active posting side in a single marketplace. McAfee (1993) and more recently Albrecht, Gautier, and Vroman (2014) provide efficiency results for such markets showing that both the sellers' mechanisms and the buyers participation decisions are efficient (subject to the standard buyers' search friction). Peters and Severinov (2006) establish efficiency in a decentralized market where each buyer can bid at many sellers simultaneously. Our equilibrium is also constrained-efficient, but in addition to mechanisms and the visiting decisions, we also show that posting/visiting decisions are made efficiently in our setting. Moreover, the welfare level in our market is higher since both buyers and sellers can post mechanisms.

Endogenizing posted mechanisms, Eeckhout and Kircher (2010) ask what mechanisms emerge in equilibrium under different matching technologies. Using urn-ball

times prevent visitors from reaching posters. In our set-up, strategic visitors choose which mechanism to participate in, which is observationally equivalent to urnball matching.

[2]SD have a similar monotonicity result in the case where one side has two types, while the other side is homogenous: their Theorem 3 shows that low types visit while high types post in a certain region of parameter space.

matching, Coles and Eeckhout (2003) and Virag (2011) identify several equilibria of the directed search model with different trading mechanisms. Neeman and Vulkan (2010) perform a very different comparison and ask whether centralized or decentralized markets perform better, while we focus on two-sided decentralized markets.

Kultti et al. (2009) consider a dynamic market in which a trader can either search or wait for a partner, and the traders on each side of the market are homogeneous. They compare two trading mechanisms, an auction and bargaining, and provide conditions under which both buyers and sellers search.[3] The main distinguishing feature of our model is the endogenous choice of mechanisms by the posters. We also study how the posting/visiting decisions depend on the traders' types, and examine the difference in welfare between one-sided and two-sided posting markets.

Finally, our paper is related, albeit distantly, to the literature on the organization of trade at platforms. Ambrus and Argenziano (2009) show that mutiple asymmetric platforms can coexist. Virag (2019) studies a model with strategic platforms but with homogeneous buyers and sellers. The literature on the design of trading platforms with *divisible* objects (see e.g. Malamud and Rostek (2017)) is also interested in efficiency of decentralized exchange economies.

The rest of the paper is organized as follows. Section 2 sets up the model, and provides equilibrium existence and efficiency results. Section 3 studies a submarket where the sellers post. Section 4 presents central equilibrium characterization results. Sections 5 deals with a two-type model and its numerical analysis. Section 6 concludes. The proofs are relegated to the Appendices.

2 Model

Consider a market with a mass of size one of sellers and a mass of buyers of the same size.[4] The sellers' costs and the buyers' values are distributed on $[0, 1]$ according to atomless probability distributions F_s and F_b, respectively, with continuous densities $f_s > 0$ and $f_b > 0$, respectively. Each seller can supply one unit of the good, and each buyer wishes to acquire a single unit. All sellers and buyers are risk-neutral. Thus, if a seller with type c sells his unit at price p, she obtains the payoff $p - c$. If a buyer with

[3] Stacey (2019) also studies which market side is more active, allowing only a limited amount of trader heterogeneity.

[4] The assumption that the mass sizes are equal is made for ease of exposition. All our results continue to hold when the masses of buyers and sellers are different.

type v buys a unit at price p, then his payoff is $v - p$. The payoff from not trading is zero for each trader type.

Each buyer and seller can either post his/her own mechanism or visit a mechanism on the other side of the market. The timeline of the events is as follows. First, each buyer and seller chooses between posting a mechanism and not posting and becoming a visitor. This decision and the choice of a mechanism in case of posting are made simultaneously by all traders. All mechanisms are then posted and observed by all visitors, who then simultaneously decide which mechanism to visit. Each visitor can participate in one mechanism only. For ease of exposition, we say that the sellers'/buyers' mechanisms are posted in submarket S/B.

To simplify the analysis and for consistency with the literature on large markets, we restrict the set of available mechanisms to direct anonymous incentive compatible mechanisms which treat all participants identically.[5]

Let $\mathcal{M}_s$ be the set of such seller mechanisms and $\mathcal{M}_b$ be the set of such buyer mechanism. A typical element of $\mathcal{M}_s$ is denoted by $M_s = (Q, T)$, where $Q : [0,1]^{[0,1]} \mapsto \mathcal{P}([0,1])$ and $T : [0,1]^{[0,1]} \mapsto \mathbf{R}_+^{[0,1]}$ are (Lebesgue measurable) allocation function mapping the profile of buyers' type announcements into the set of probability distribution $\mathcal{P}([0,1])$ over the buyers which specifies the allocation of the good, and transfer function mapping the profile of buyers' type announcements into the set of transfer profiles, $\mathbf{R}_+^{[0,1]}$, respectively. The buyers who do not participate in a mechanism are assigned an announcement $v = 0$ by convention. The allocation (winning probability and transfer) of such buyer is restricted to zero, which guarantees non-participating traders their outside payoffs. The anonymity of the mechanism is ensured by imposing the restrictions that: (i) T is permutation invariant; (ii) Q has the same density or same atom at all $i, j \in [0,1]$ when buyers indexed by i and j announce the same type.

Similarly, a typical element of $\mathcal{M}_b$ is denoted by $M_b = (P, X)$, where $P : [0,1]^{[0,1]} \mapsto \mathcal{P}([0,1])$ and $X : [0,1]^{[0,1]} \mapsto \mathbf{R}_+^{[0,1]}$ are allocation and transfer functions, respectively, that map the profile of the sellers' type announcements (costs or values) into the probability distribution determining the allocation of the good among the sellers and into the profile of the transfers to the sellers where similar restrictions apply as to M_s.

Next, consider the visitors' participation (mechanism selection) strategies, which we assume to be identity-invariant and to depend on their types only. To describe

[5] The restriction to direct mechanisms rules out equilibria in which a mechanism depends directly or indirectly on other mechanisms. For analysis of such equilibria see e.g. Peters and Szentes (2012).

them formally, let A_s denote the profile of sellers' mechanisms and let $\mathcal{A}_s$ be the space of mechanism profiles. Then a participation strategy of a buyer type v, $D_v(.)$, is a probability distribution over A_s i.e., $D_v(A_s) \in \mathcal{P}(A_s)$ for all $A_s \in \mathcal{A}_s$.[6]

For the market B, we use similar notation A_b to denote a profile of buyers' mechanisms, with $\mathcal{A}_b$ denoting the space of mechanism profiles, and $R_c(.)$ denoting the participation strategy of a seller with cost c, so that $R_c(A_b) \in \mathcal{P}(A_b)$ for all $A_b \in \mathcal{A}_b$.

To conclude this section, let us recap our anonymity and symmetry assumptions. Recall that the traders are restricted to offering anonymous mechanisms which treat all visitors identically. This assumption is natural in a large market where the designers are typically unable to identify the visitors personally. Similarly, we restrict consideration to identity-invariant participation strategies that depend only on the traders' types. In particular, any two buyers with the same value and any two sellers with the same cost post with equal probabilities and, when visiting, follow the same probability distribution over the set of posted mechanisms. This assumption reflects market frictions arising from the lack of coordination in large markets, and is standard in directed search literature. It is also natural given the anonymity of the mechanisms. We refer to equilibria that satisfy these assumptions and in which the posters use pure strategies, interchangeably, as *identity-independent or symmetric.* Given those assumptions, equilibrium visiting decisions boil down to choosing a distribution of posting *types* visited, while visiting probabilities for mechanisms not posted in equilibrium are pinned down by the market utility assumption as it appears in the equilibrium definition at the end of Section 3.1 and in the analysis in Section 3.2 and in Section 4.[7]

[6]Formally, we can view A_s as a measurable function from $[0,1]$, the set of sellers, into $\mathcal{M}_s^{[0,1]}$. This representation associates a mechanism with every seller. This is without loss of generality, since by convention a visiting seller is assigned a *null mechanism* $M^0 = (Q^0, T^0)$, where $Q^0 = T^0 = \mathbf{0}$. Then $\mathcal{A}_s$ is the space of measurable functions from $[0,1]$ to $\mathcal{M}_s^{[0,1]}$. To define the space of probability distributions, $\mathcal{P}(A_s)$ over $A_s \in \mathcal{A}_s$, we endow A_s with the weak convergence topology and take the Borel σ-algebra generated by it.

[7]In Sections 3 and 4, we show that such an equilibrium is essentially unique. Namely, all such equilibria have the same outcome in which all posters offer efficient mechanisms and each visitor's participation decision boils down to randomizing uniformly over the set of mechanisms with sufficiently low reservation prices.

3 Equilibrium in a Submarket

3.1 Preliminaries and equilibrium definition

In this subsection, we assume that there are positive measures of both sellers and buyers in the submarket S. The cumulative distributions of the sellers' costs and the buyers' values in this submarket are denoted by G_s and G_b, respectively, with corresponding positive densities g_s and g_b on $[0, 1]$. This assumption is endogenized in the sequel when we consider the traders' equilibrium choices between positing and visiting. and show how G_s and G_b are generated by the prior type distribution and the traders' equilibrium strategies. Bearing this in mind and taking into account that the mass of buyers and sellers could be less than their respective total masses of 1s, for now we allow that $G_s(1) = \kappa_s$ and $G_b(1) = \kappa_b$ where $\kappa_i \in (0, 1]$, $i \in \{s, b\}$.

The visiting buyers' equilibrium participation strategies naturally depend on the payoffs that they expect to get in the posted mechanisms. Significantly, given that there is a continuum of small traders in our market, it is natural to assume that no single seller has an effect on the payoffs that the other traders get in other sellers' mechanisms. Reflecting this, we adopt the market utility approach in assuming that, irrespective of the mechanism offered by a particular seller, a buyer of type v expects to get a payoff $u(v) \geq 0$ when she participates elsewhere, optimally choosing between other posted mechanisms. We will refer to the schedule u as the buyers' expected market utility, and will characterize equilibrium u below.

For now, note that, since each buyer type can mimic any other type and visit any posted mechanism, incentive compatibility implies that u must be increasing and continuous and, when differentiable, $u'(v)$ must be equal to the expected equilibrium probability of trading for type v.[8] The latter must be increasing in v also by incentive compatibility, so $u(v)$ must be convex.

So, in order to attract a buyer type v an individual seller must offer a payoff of at least $u(v)$ to her. At the same time, offering a payoff strictly larger than $u(v)$ to such buyer is suboptimal, because in this case all buyers with valuations close to v will choose to visit this seller. With a large number of visitors each getting a payoff close to $u(v)$ and a single unit of the good for sale, this seller will get a negative net payoff.

[8]The monotonicity of u implies that $u'(v)$ exists almost everywhere and its left-hand derivative, $u'_{-}(v)$, and the right-hand derivative, $u'_{+}(v)$, exist at any $v \in [0, 1]$ and one of them is equal to the probability that type v trades.

Thus, in contrast to a monopolistic mechanism design situation, in our competing mechanisms market S an individual seller cannot affect the buyers' payoffs, and can attract buyers in competition with other sellers only by offering them sufficient payoffs. The competition for the buyers naturally conflicts with the objective of extracting surplus from them and, as we will show below, ultimately causes the sellers to offer efficient mechanisms in equilibrium.

To describe the equilibrium conditions on the traders' strategies in market S, we need to introduce some additional notation. In particular, for a given buyers' participation strategy profile $D_v(.)$, $v \in [0,1]$ and a profile of posted mechanisms A_s, by Bayes' rule the traders' posterior about the type of a buyer participating in a mechanism $M_i \in A_s$ are characterized by the density $g_b(v|M_i, A_s, D) = \frac{g_b(v)D_v(A_s)(M_i)}{\int_0^1 g_b(x)D_x(A_s)(M_i)dx}$. Then, assuming the existence of an equilibrium in mechanism M_i for any number of participating buyers n and any buyers' beliefs, let $U(v|M_i, n, D)$ be the expected equilibrium payoff of buyer type v in mechanism M_i, when it is visited by n buyers who use participation strategies $D_v(A_s)$. Also, let $V(c|M_i, n_i, D)$ be the expected payoff of the seller with cost c who offers mechanism M_i and is visited by n buyers.

The expected number/queue of buyers visiting mechanism M_i is equal to $\bar{\lambda} = \int_0^1 g_b(x)D_x(A_s)(M_i)dx$, and so the number of buyers in this mechanisms follows a Poisson distribution with parameter $\bar{\lambda}$.[9] Accordingly, let $\hat{U}(v|M_i, D)$ and $\hat{V}(c|M_i, D)$ be the ex ante expected payoffs of a buyer type v participating in mechanism M_i and of a seller type c offering mechanism M_i, respectively.

Recall that we restrict consideration to identity-independent *equilibria*. We will also assume that the posters use pure strategies in mechanism choice.[10] Then mechanism strategy profile $\{M_s^*(c)\}$, $c \in [0,1]$, where $M_s^*(c)$ denotes the mechanism posted by seller of type c, and the visiting strategy profile $\{D_v^*(.)\}$, $v \in [0,1]$ of the buyers,

[9]With a continuum of buyers and sellers, the distribution of visitors in a mechanism is the limit of the binomial distribution as the number of visitors in the market grows to infinity. Hence, if the expected number of visitors with valuations in some set $V \subseteq [0,1]$ is equal to λ, the probability distribution of the number of visitors with valuations in V is Poisson with parameter λ. This result is due to Kolchin et al. (1978). To provide some connection to their analysis, since the visiting decisions in our market are made independently by the traders and everyone who decides to visit a particular mechanism reaches it with probability 1, this behavior corresponds to what is called urnball matching technology.

[10]While the assumption that visitors use identity-independent strategies is crucial, assuming that posters use identity-independent pure strategies is not essential and is made only to simplify the notation. This follows from Proposition 1 which shows that a seller's optimal mechanisms must be efficient.

constitute an equilibrium of submarket S if the following conditions hold:

(i) *seller mechanism optimality:* for any $c \in [0,1]$ and $M \in \mathcal{M}_s$, $\hat{V}(c|M_s^*(c), D^*(A_s^*)) \geq \hat{V}(c|M, D^*(A_s^{i*}, M))$, where A_s^* is the equilibrium mechanism profile which, given the distribution of seller types $G_s(.)$, can be represented as the following mapping from $[0,1]$ to the set $\{M_s^*(c)\}_{x\in[0,1]} \cup M^0$: $A^*(x) = M^*(G_s^{-1}(x))$ for $x \in [0, \kappa_s]$, $A^*(x) = M^0$ for $x \in (\kappa_s, 1]$, and (A_s^{i*}, M) is a mechanism profile obtained from A_s^* by replacing the mechanism $M^*(c)$ of our fixed seller with cost c with mechanism M.

(ii) *buyers' best response under market utility assumption:*
Buyers' visiting strategy profile $\{D_v^*\}_{v\in[0,1]}$ satisfies:
(a) $D_v^*(A_s^*)(M^*(c)) > 0$ for some $c \in [0,1]$ only if
$\hat{U}(v|M^*(c), D^*(A_s^*)) \geq \max_{c\in[0,1]} \hat{U}(v|M^*(c), D^*(A_s^*)) = u(v)$.
(b) Let (A_s^{-i*}, M_i) be a mechanism profile such that seller i offers mechanism M_i and other sellers offer equilibrium mechanism profile A_s^{-i*}. Then $D_v^*(A_s^{-i*}, M_i)(M_i) > 0$ only if $\hat{U}(v|M_i, D^*(A_s^{-i*}, M_i)) \geq u(v)$.

Note that part (a) of (ii) defines the buyers' market utility u, while part (b) reflects that a buyer visits any mechanism M_i posted by seller i only if her payoff there reaches the market utility level $u(v)$ which is unaffected by M_i.

3.2 Equilibrium Characterization for a Submarket

We start the characterization of the identity-independent equilibrium in the submarket S by deriving the sellers' best-response mechanisms. Under the market utility assumption, each seller takes the visitors' utility schedule as given. So a seller cannot affect the buyers' expected payoff through her choice of mechanism, but she can affect the queue length of the visiting buyers.

Thus, in the first step of our analysis, we solve a relaxed problem of an individual seller assuming that she can achieve any desired buyer participation rate as long as she offers the market utility $u(v)$ to visiting buyer type v. So, in the relaxed program the seller directly chooses both the object allocation rule and the length of the queue of the visiting buyers of any type. The solution involves offering an efficient mechanism.

In the second step, we establish that the buyers' best response strategies induce the same queue at the seller that solves this seller's relaxed program. In combination, these two steps yield Proposition 1 establishing that offering an efficient mechanism, such as an auction, is a seller's unique best response *to any profile of mechanisms offered* by the other sellers. Then in Proposition 2 we provide a complete characterization of the unique equilibrium outcome in market S.

To state the first result, let $\lambda(v) \geq 0$ be the queue i.e., the expected number, of buyers of type v visiting a given seller's mechanism.

Proposition 1 *Suppose that the buyers' market payoff schedule u is continuous, increasing and convex with $u(1) < 1 - c$.*[11]

Then any optimal mechanism for a seller with cost $c \in [0, 1]$ is efficient i.e., assigns the object to the highest visiting buyer type if the latter is at least c, and retains the good otherwise, which can be implemented by offering a second price auction with a reservation price equal to c.

The buyers' unique equilibrium visiting strategies induce a queue schedule λ^* at this mechanism such that $\lambda^* = 0$ for all $z < \widehat{z}(c)$ and $\exp^{-\int_z^1 \lambda^*(x)dx} = u'(z)$ for almost all $z \geq \widehat{z}(c)$ where $\widehat{z}(c) = c$ if $u(c) = 0$ and $\widehat{z}(c) = \sup\{z | u(z) > (z - c)u'_-(z)\} \in (c, 1)$ otherwise.

To understand Proposition 1, suppose that a seller with cost c offers a mechanism that provides utility $u(v)$ to a visiting buyer of type v and implements allocation rule (a buyer's expected probability of trading) $q(v)$. Attracting type v requires the mechanism to provide a payoff at least $u(v)$ to her. In the proof of Proposition we show that providing a payoff strictly exceeding $u(v)$ to this type is suboptimal. Hence, the seller's expected profit is given by:

$$\pi(q, \lambda) = \int_c^1 (x - c)q(x)\lambda(x)dx - \int_c^1 u(x)\lambda(x)dx. \tag{1}$$

In the proof of Proposition 1, we first solve the relaxed problem of maximizing (1) assuming that the seller can choose both q and the queue λ under the feasibility conditions $0 \leq q(v) \leq 1$ and $\int_v^1 q(x)\lambda(x)dx \leq 1 - e^{-\int_v^1 \lambda(x)dx}$. The latter inequality says that the probability of allocating the good to type $v \in [x, 1]$ cannot exceed the probability that such type visits our seller (see Border (1991)).

Lemma 2 in the Appendix establishes that in the optimal mechanism the good must be allocated efficiently, i.e. to the highest visiting type. This result stems from the fact that $\pi(q, \lambda)$ depends on q only through the total surplus $\int_c^1 (x - c)q(x)\lambda(x)dx$. This implies that, for given queue λ, $q(v) = e^{-\int_v^1 \lambda(x)dx}$ for all $v > c$. Finally, characterizing

[11]If $u(1) \geq 1 - c$, then convexity of $u(.)$ and $u'(1) \leq 1$ imply that $u(v) \geq v - c$ for all v, and so a seller of type c will not be willing to attract any buyer types.

the optimal queue length λ^*, we show in Lemma 3 that, whenever $\lambda^* > 0$ and $u'(v) > 0$ it must satisfy the first-order condition $e^{-\int_v^1 \lambda^*(x)dx} = u'(v)$. Intuitively, this condition says that, at the optimum, an increase in the seller's profits from a marginal increase in the buyers' queue must be equal to the marginal increase in the payoff that the seller must offer to the participating buyers.

To complete the proof, we show that the seller's optimal queue λ^* characterized in Lemma 3 is consistent with unique buyers' optimal participation strategies. Recall that given that our seller offers an efficient mechanism, the probability that a buyer type v trades with her is equal to the probability that no higher types visits this seller, $e^{-\int_v^1 \lambda^*(x)dx}$. The incentive compatibility of the buyers' participation strategy implies that this probability must be equal to the marginal payoff of this type in the mechanism, $u'(v)$ (envelope condition). That is, we have $e^{-\int_v^1 \lambda^*(x)dx} = u'(v)$. But this is exactly the same condition that yields the optimal seller queue, as explained above. Thus, the seller's optimal queue λ^* will, in fact, be induced by the unique buyers' equilibrium participation strategies in this mechanism.

Importantly, Proposition 1 also applies when $u'(1) < 1$. In this case the market payoff schedule violates efficiency at the top since the probability of winning for the highest buyer type $v = 1$ is equal to $u'(1) < 1$. Encountering this problem, McAfee (1993) restricted consideration to the case where $u'(1) = 1$. However, we show that when $u'(1) < 1$, a seller's best response is to attract an atom of highest-value buyer of measure $\Lambda^*(1) = -\log(u'(1))$, which she can still achieve by posting an efficient mechanism. Yet, this cannot occur in an equilibrium, since otherwise all sellers would want to attract an atom of the highest value buyers. So, $u'(1) = 1$ in any equilibrium.

The following Proposition builds on Proposition 1 and provides a detailed characterization of the unique equilibrium allocation in the one-sided market.

Proposition 2 *In every identity-independent equilibrium in submarket S, all sellers offer efficient mechanisms, and a buyer with value v visits a seller with cost c if $v \geq \widehat{z}(c)$, where $\widehat{z}(c)$ is a unique solution to the following equation with initial condition $\widehat{z}(0) = 0$:*

$$\widehat{z}'(c) = \frac{G_S(c)}{g_B(\widehat{z}(c))(\widehat{z}(c) - c)}. \tag{2}$$

Each buyer randomizes uniformly among all sellers that she visits, and the queue of buyers with value v at a seller with cost c is equal to $\lambda^*(v) = \frac{g_B(v)}{G_S(\widehat{z}^{-1}(v))}$ if $v \geq \widehat{z}(c)$, *and zero otherwise. The expected equilibrium payoff of buyer type $v \in [0,1]$ is equal to $U(v) = \int_0^v e^{-\int_x^1 \frac{g_B(y)}{G_S(\widehat{z}^{-1}(y))}dy} dx$.*

Proposition 2 establishes that an identity-independent equilibrium in submarket S is essentially unique. The only non-unique aspect is that a seller may offer different versions of an efficient mechanism. However, the equilibrium outcome is unique since all such mechanisms result in the same allocation. So, henceforth we will without loss of generality consider that our equilibrium is unique, and in particular, the sellers offer second-price auctions with reservation prices equal to their costs.

Note that sellers with different cost levels trade with different probabilities solely due to the fact that the lowest visiting type $\widehat{z}(c)$ in an auction with reservation price c is increasing in c. This follows from the equilibrium property that a type v buyer randomizes uniformly among all sellers whose reservation prices c' are such that $\widehat{z}(c') \leq v$. Proposition 2 provides a quantitative characterization of the threshold function $\widehat{z}(c)$ via (2), and the buyers' random participation strategies and their expected payoffs are also characterized explicitly.

In particular, we can obtain the closed form solution in the following

Example 1. *Suppose that the distributions of buyers and seller in submarket S are uniform on the support $[0,1]$ with $G_i(c) = \kappa_i c$ for all $c \in [0,1]$ i.e., the total masses of buyers and sellers are equal to κ_b and κ_s, respectively.*

Then letting $r = \frac{\kappa_s}{\kappa_b}$, equation (2) can be solved to yield $\widehat{z}(c) = \frac{1+\sqrt{1+4r}}{2}c$, and we may also compute: $\lambda^*(v) = \frac{1+\sqrt{1+4r}}{2rv}$ and $U(v) = v^{\frac{1+\sqrt{1+4r}+2r}{2r}} \frac{2r}{1+\sqrt{1+4r}+2r}$.

Our quantitative characterization of the equilibrium and the sellers' cost heterogeneity is new to the literature on mechanism design in one-sided posting markets. In particular, McAfee (1993) shows that, with homogenous sellers, all sellers post second-price auctions with reservation prices equal to their common cost, or allocationally equivalent mechanisms, in the unique symmetric equilibrium of a large market under competitive assumptions. Peters and Severinov (1997) generalize this analysis to a fully strategic setting where a seller's mechanism affects the buyers' payoffs at other sellers and show that McAfee's result holds in the limit as the market grows large.

An important difference between the markets with homogeneous and heterogeneous sellers is that in the latter the distribution of reservation prices is non-degenerate and in fact has full support in equilibrium. So, when a seller deviates, In particular, by posting a reserve price different from her true cost, her queue of attracted buyers changes continuously in the magnitude of her deviation. In contrast, when the sellers are homogeneous and all post reserve prices equal to the common cost, a single seller's deviation to a different reserve price causes a discontinuous jump (up or down depending on the direction of deviation) in the queue of the buyers visiting her. This happens because the auction of a seller deviating from the common reserve price down/up becomes significantly more/significantly less profitable for the buyers with values close to the reserve price. Such a discontinuity provides each seller with an incentive to set its reservation price equal to its cost similar to the case of a standard Bertrand competition with equal marginal costs.

Peters (1997) shows that, when all sellers are restricted to offer second-price auctions and the market is competitive, it is optimal for a seller to set reservation price equal to her cost.[12] Our setup is different from Peters (1997) in competitive assumptions and equilibrium concept. In particular, we allow sellers to offer any direct mechanisms and establish that the equilibrium outcome is unique because an efficient mechanisms is a seller's best response to any profile of mechanisms, and not just auctions.

4 Analysis of the Bilateral Posting Market

4.1 Equilibrium conditions

In this subsection we turn to the characterization of identity-independent equilibria of the whole market. We will rely on the results of the previous section (In particular, Proposition 2 for submarket S and analogous result for submarket B that must hold by symmetry), which show that a posting trader must offer an efficient mechanism and characterize the visiting traders' equilibrium participation strategies.

Our first step is to provide the equilibrium conditions for the choice between posting and visiting. To this end, let us introduce the following notation. Let $\beta_b(v)/\beta_s(c) \in [0,1]$ denote the probability that a buyer with value v/seller with cost c posts a mechanism. For visiting strategies, let $\tau_v^b(c) \geq 0$ denote the visiting density of a buyer with value v at a seller posting a second-price auction with reservation price c or an equivalent mechanism. Likewise, let $\tau_c^s(v) \geq 0$ denote the visiting density of a seller with cost c at a buyer v posting a reverse auction with reservation price v.

[12] Virag (2010), building on Burguet and Sakovics (1999), explicitly considers finite markets where sellers are restricted to post auctions, and shows that under intuitive conditions the reservation prices converge to the sellers' costs as the market becomes large, without imposing any competitive assumptions on the model.

To rule out trivial equilibria in which one of the markets, B or S, is inactive due to coordination failures (e.g., market S is inactive because all buyers post and all sellers visit), we adopt an assumption that a trader chooses to visit if her payoff from posting is zero. Since visiting is at least weakly better for such a trader than posting or not participating in the market at all, this assumption constitutes a weak equilibrium refinement. As we will see below, the consequence of this refinement is that both markets are active in equilibrium, because low valuation buyers and high cost sellers cannot attract any visitors when posting, and therefore they visit with probability 1.

Technically, this refinement implies the following balance conditions for buyers and sellers respectively: $\beta_b(v)+\int_0^1 \tau_v^b(c)dc = 1$ for all v, and $\beta_s(c)+\int_0^1 \tau_c^s(v)dv = 1$ for all c.

The equilibrium visiting strategy profile (τ_v^b, τ_c^s) is characterized in Proposition 2. Specifically, a seller posting an auction with reservation price c is visited by all buyers with valuations in $[\widehat{z}(c), 1]$ and each buyer randomizes among all sellers that she visit uniformly, which implies that

$$\tau_v^b(c) = \tau_v^{b*}(c, \beta_b, \beta_s) \equiv \begin{cases} \frac{(1-\beta_b(v))f_s(c)\beta_s(c)}{\int_0^{\widehat{z}^{-1}(v))} f_s(x)\beta_s(x)dx}, & \text{if } v \geq \widehat{z}(c), \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

where by (2) $\widehat{z}'(c) = \frac{\int_0^c f_s(x)\beta_s(x)dx}{f_b(\widehat{z}(c))(\widehat{z}(c)-c)(1-\beta_b(\widehat{z}(c))}$.

Similarly, letting $\widehat{w}(v)$ denote the highest cost type that visits buyer v posting in market B, it follows that

$$\tau_c^s(v) = \tau_c^{s*}(v, \beta_b, \beta_s) \equiv \begin{cases} \frac{(1-\beta_s(c))f_b(v)\beta_b(v)}{\int_{\widehat{w}^{-1}(c))}^1 f_b(x)\beta_b(x)dx}, & \text{if } c \leq \widehat{w}(v) \\ 0, & \text{otherwise,} \end{cases} \tag{4}$$

where by (2) $\widehat{w}'(v) = \frac{\int_v^1 f_b(x)\beta_b(x)dx}{f_s(\widehat{w}(v))(v-\widehat{w}(v))(1-\beta_s(\widehat{w}(v))}$.

Note that the equilibrium visiting strategy profile (τ_v^b, τ_c^s) is determined by the posting strategies (β_b, β_s) via (3) and (4), respectively. So we next focus on the posting strategy profile (β_b, β_s). In equilibrium, it should be sequentially rational for every trader type given the posting strategy profile (β_b, β_s) itself, the probability distributions of the buyers' values and the sellers costs, F_b and F_c, respectively, and the continuation equilibrium strategies in the submarkets B and S.

To characterize the equilibrium profile (β_b, β_s), we first need to derive the traders' expected payoffs in markets B and S. To this end, recall that $\lambda_c(v)$ is the queue of type v buyers in the continuation equilibrium of market S at a seller posting an auction with the reservation price c. Then, using (3) yields for $v \geq \widehat{z}(c)$:

$$\lambda_c(v) = \frac{f_b(v)\tau_v^b(c)}{f_s(c)\beta_s(c)} = \frac{(1-\beta_b(v))f_b(v)}{\int_0^{\widehat{z}^{-1}(v))} f_s(x)\beta_s(x)dx}. \tag{5}$$

Similarly, let $\lambda_v(c)$ denote the queue of type c sellers in a continuation equilibrium of market B at a buyer posting an auction with the reservation price v. Using (4) we obtain for $c \leq \widehat{w}(v)$:

$$\lambda_v(c) = \frac{f_s(c)\tau_c^s(v)}{f_b(v)\beta_b(v)} = \frac{(1-\beta_s(c))f_s(c)}{\int_{\widehat{w}^{-1}(c))}^1 f_b(x)\beta_b(x)dx}. \tag{6}$$

The queue of buyers with a valuation of at least v at an auction with reservation price c is equal to $\Lambda_c(v) = \int_v^1 \lambda_c(x)dx$, while the queue of sellers with a cost at most c at a buyer's auction with reservation price v is then equal to $\Lambda_v(c) = \int_0^c \lambda_v(x)dx$.

As shown above, the number of buyers with values at least v in an auction with reservation price c is distributed according to a Poisson distribution with parameter $\Lambda_c(v)$. Similarly, the number of sellers with costs at most c at an auction with reservation price v is distributed according to a Poisson distribution with parameter $\Lambda_v(c)$. So, the probability that a seller of type c posting a second-price auction with reservation price c in market S trades and the probability that a buyer of type v wins this auction are equal to, respectively:

$$\pi_S^s(c) = 1 - e^{-\Lambda_c(\widehat{z}(c))}, \tag{7}$$

$$\pi_S^b(v) = e^{-\Lambda_c(v)}. \tag{8}$$

This seller makes a positive profit only if at least two buyers with valuations no less than c visit her mechanism. The probability of this event is:

$$G_c(v) = 1 - e^{-\Lambda_c(v)}(1 + \Lambda_c(v)).$$

Let $g_c(v) = -\frac{\partial G_c(v)}{\partial v} = \lambda_c(v)\Lambda_c(v)e^{-\Lambda_c(v)}$. Then the payoff of this seller is:

$$V_S(c) = \int_c^1 g_c(v)(v-c)dv. \tag{9}$$

On the visitors' side, let $U_S(v)$ denote the payoff of a buyer type v visiting an auction in market S with reservation price c. The probability that in this auction the highest visitor type is less than v is equal to $H_c(v) = e^{-\Lambda_c(v)}$, that has density $h_c(v) = \frac{\partial H_c(v)}{\partial v}$. So the payoffs of a buyer type v from visiting this auction and from visiting an optimally chosen auction are equal to, respectively:

$$\widehat{U}_S(v,c) = H_c(c)(v-c) + \int_c^v h_c(x)(v-x)dx;$$
$$U_S(v) = \max_c \widehat{U}_S(v,c) \tag{10}$$

By Proposition 2, all buyer types visit the most efficient seller in the equilibrium of market S. Therefore, $U_S(v) = \widehat{U}_S(v,0)$ for all v.

Similar arguments apply to market B where the buyers post mechanisms. Specifically, a buyer who posts a second-price reverse auction with a reservation price equal to her value v makes a positive profit only if at least two sellers participate in her mechanism. With the number of visiting sellers following a Poisson distribution, the probability that at least two sellers with costs below c visit this auction is given by:

$$G_v(c) = 1 - e^{-\Lambda_v(c)}(1+\Lambda_v(c)).$$

With $g_v(c) = \frac{\partial G_v(c)}{\partial c}$, the payoff of buyer type v from her auction is equal to:

$$U_B(v) = \int_0^v g_v(c)(v-c)dc. \tag{11}$$

On the visitors' side in market B, a visiting seller gets the good if and only if no seller with a lower cost visits the same auction. The probability that the lowest cost among the sellers visiting a buyer's auction with reservation price v is at least c is given by $H_v(c) = e^{-\Lambda_v(c)}$. Then $h_v(c) = -\frac{\partial H_v(c)}{\partial c}$ is the density of the lowest cost among the sellers visiting such auction. Then the payoffs of a type c seller visiting the auction with reservation price v and from visiting an optimal chosen auction are equal to:

$$\widehat{V}_B(c,v) = H_v(v)(v-c) + \int_c^v h_v(x)(x-c)dx,$$
$$V_B(c) = \max_v \widehat{V}_B(c,v) = \widehat{V}_B(c,1) \tag{12}$$

The very last equality holds because by Proposition 2 every visiting seller type will visit the most efficient buyer type 1. The probabilities of trading in this market for a buyer type v and a seller type c are equal to $\pi_B^b(v) = 1 - e^{-\Lambda_c(\widehat{w}(v))}$ and $\pi_B^s(c) = e^{-\Lambda_v(c)}$, respectively.

The above steps together with Proposition 2 yield following Lemma:

Lemma 1 *A posting strategy profile (β_b, β_s) uniquely determines the equilibrium payoff functions (U_S, V_S, U_B, V_B), the visiting probabilities $(\tau_v^b(c), \tau_c^s(v))$ and the probabilities of trading $(\pi_S^b, \pi_S^s, \pi_B^b, \pi_B^s)$ in the identity-independent continuation equilibrium of submarkets B and S via equations (2)- (12).*

Relying on Lemma (3) we can now state the conditions on an equilibrium strategy profile:

Definition 1 *Let (U_B, V_S, U_S, V_B) be the expected payoff profile induced by the strategy profile $(\beta_b, \beta_s, \tau_b, \tau_s)$ via (2)- (12). Then $(\beta_b, \beta_s, \tau_b, \tau_s)$ is an identity-independent equilibrium strategy profile of the bilateral posting market if and only if:*
(1) $\beta_b(v) = 1$ if $U_B(v) > U_S(v)$; $\beta_b(v) = 0$ if $U_B(v) < U_S(v)$ or if $U_B(v) = 0$;
(2) $\beta_s(c) = 1$ if $V_S(c) > V_B(c)$; $\beta_s(c) = 0$ if $V_S(c) < V_B(c)$ or if $V_S(c) = 0$;
(3) $\tau_v^b(c) = \tau_v^{b*}(c, \beta_b, \beta_s)$;
(4) $\tau_c^s(v) = \tau_c^{s*}(v, \beta_b, \beta_s)$,
where $\tau_v^{b*}(c, \beta_b, \beta_s)$ and $\tau_c^{s*}(v, \beta_b, \beta_s)$ are given by (3) and (4), respectively.

4.2 Equilibrium existence and constrained efficiency

An important benchmark for the decentralized markets with search frictions is the planner's problem maximizing the welfare when the planner is capable of matching the buyers and sellers optimally subject to the constraint that all traders of the same type get the same, possibly random, allocation. Such allocation is customarily referred to as constrained efficient in the directed search literature. Thus, the constrained efficient allocation in our bilateral posting environment solves the following problem:

$$\max_{(\beta_s, \beta_b, \tau_c^s, \tau_v^b) \in [0,1]^4} \int_0^1 \beta_s(c) f_s(c) \int_c^1 (v-c) dH_c(v) dc + \int_0^1 \beta_b(v) f_b(v) \int_0^v (v-c) d(1-H_v(c)) dv, \tag{13}$$

where $H_c(v) = e^{-\Lambda_c(v)}$, and $H_v(c) = e^{-\Lambda_v(c)}$. This problem is continuous in the topology of uniform convergence, so its solution, a constrained efficient allocation, exists. Furthermore, the next Proposition presents one of our central results that the equilibrium in our market is unique and coincides with the constrained-efficient allocation.

Proposition 3 *The unique constrained-efficient allocation $(\beta_b, \beta_s, \tau_s, \tau_b)$ constitutes the unique identity-independent equilibrium in the bilateral posting market.*

This Proposition provides a new insight about the properties of two-sided posting markets with heterogeneity of both buyers and sellers, and therefore it differs substantially from the efficiency results in the literature. In particular, is establishes a novel phenomenon that a trader in a two-sided market prefers visiting over posting if and only if she generates a higher social surplus by visiting.

To tackle the technical challenge of Proposition 3 and solve a complex posting and visiting problem with heterogeneous populations of buyers and sellers, we first analyze the model with discrete types. Then we take a limit to approximate the original economy with a continuum of types. The proof relies on two key intermediate results. First, Lemma 8 shows that a strategy vector $(\beta_b, \beta_s, \tau_b, \tau_s)$ forms an equilibrium in our market and only if it satisfies the first-order conditions of the constrained welfare maximization program. This is so because all the externalities that a trader exerts on the other traders balance out when a trader uses a strategy maximizing her payoff. So thereby, she also maximizes her contribution to total welfare. Second, Lemma 9 shows that the constrained welfare function is strictly concave, and hence the first-order conditions characterize a unique welfare maximum. This result is a notable technical contribution to the analysis of complex directed search problems.

Notably, splitting our market further and creating additional submarkets would not increase social welfare because our markets B and S feature one-to-many matching technology, which we view as the central feature of moderns decentralized trading platforms. Indeed, suppose that there were a total of three markets with sellers posting mechanisms in two of them. Then combining the latter two markets and reoptimizing with respect to the queue lengths will increase the total welfare since the combined sellers' posting market can achieve all the allocations that were possible to achieve in the two separate sellers' posting markets. We show this result formally in Online Appendix 2.[13]

4.3 Equilibrium participation

Next, we provide an important result stating that all types participate in equilibrium because the low-value buyer types get positive payoffs only in the market S. Likewise, high-cost seller types get positive payoffs only in the market B.

[13]This result has been shown for the case of two buyer types by Eeckhout and Kircher (2010). To the best of our knowledge, our more general result does not appear in the literature.

Proposition 4 *In the equilibrium of the two-sided posting market,* $U_S(v) > 0$ *and* $V_B(c) > 0$ *for any* $v > 0$ *and* $c < 1$*;* $U_B(v) = 0$ *when* v *is sufficiently small, and* $V_S(c) = 0$ *when* c *is sufficiently large. Consequently, both markets operate in equilibrium.*

To prove this Proposition, we first show that low-value buyers with types in some interval $[0, \widetilde{v}]$ cannot earn a positive payoff by posting in market B, so such buyers join the market S as visitors. We then establish the key step that the market opportunity arising from the ability to attract low value buyers is sufficiently appealing for low-cost seller types that they prefer posting. Intuitively, if no such seller posts in equilibrium, then it would be profitable for some seller with a very low cost to deviate to posting because she would then sell and make a positive profit with probability 1. But when low-cost sellers post with a positive probability, then all low value buyers can earn positive payoffs by visiting. So both markets are open, and no trader types stay out.

This result highlights the key difference between our market with bilateral posting and the standard competing mechanisms setting with one-sided positing. For concreteness, consider an example with uniformly distributed types of buyers and sellers[14] where sellers are limited to posting and buyers can only visit. The next Proposition characterizes the unique equilibrium in this one-sided market and shows that a substantial fraction of the sellers make zero payoffs and effectively stay out.

Proposition 5 *Consider a market in which the traders' types are distributed uniformly over* $[0, 1]$*, and sellers can only post auctions while buyers can only visit. In the unique equilibrium of this market all types of sellers with costs in* $[\frac{\sqrt{5}-1}{2}, 1]$ *do not attract any buyers and make zero profits, while all other seller types post efficient mechanisms with reservation prices equal to their costs. A buyer type* $v > 0$ *randomizes uniformly among the auctions of the sellers whose reservation prices do not exceed* $\frac{\sqrt{5}-1}{2}v$*. All buyers with strictly positive values make positive profits.*

Every buyer types $v \in (0, 1]$ *is strictly better off than her counterpart seller type* $c = 1 - v$ *i.e.,* $U(v) > V(1 - v)$*, whereas* $U(1) = V(0)$*.*

Propositions 4 and 5 identify a significant difference between the equilibrium outcomes with two-sided and one-sided posting. In the former all types participate and make positive profits, while in the latter this is not so as high-cost seller types cannot attract visitors and end up leaving the market.

[14] We use a uniform distribution to obtain a closed-form solution, but the argument would work for any continuous distribution.

Another important implication of Propositions 4 and 5 is that all types of visitor (buyers) are better off than their counterpart poster types (sellers) in the one-sided market, while in the bilateral market both sides are equally well off. Hence, providing an opportunity for the visitors to post not only causes some of them to switch to posting, but also makes the traders on the original posting side better off, as they can now visit. Naturally, the latter opportunity is especially attractive to those posters who could not attract any visitors, but earn a positive payoff visiting in the bilateral posting markets.

These conclusions suggest that a market with one-sided posting by sellers would be upset by efficient buyers moving to the other side and posting auctions and some sellers following them to become visitors. Thereby, bilateral posting promotes efficiency, and hence a marketplace with two-sided posting can increase equilibrium surplus and the volume of trade. Numerical computations in Section 4.3 provide quantitative estimates of associated efficiency gains.

It is also instructive to compare the properties of the bilateral posting equilibrium with a competitive outcome. The latter can be implemented via a double auction, a centralized mechanism standing in contrast to our decentralized market. Satterthwaite and Williams (1989) and Rustichini, Satterthwaite, and Williams (1994) have shown that the outcome of a double auction or a similar centralized mechanism converges to efficiency at the rate $O(\frac{1}{m})$ where m is the number of traders on each side of the market.[15] The outcome of a double auction with a continuum of traders is then fully efficient, with all trades executed at a single price p^* solving $F_s(p^*) = 1 - F_b(p^*)$. In contrast, the outcome of our bilateral posting market is not fully efficient due to search frictions, and the equilibrium price distribution has a full support on $(0, 1)$. The latter follows from the fact that the payoffs of all trader types except the most inefficient ones are positive (see Proposition 4), which is only possible if each price $p \in (0, 1)$ occurs with a positive probability in our market.

4.4 Visiting and Posting Patterns

The next issue that we address is how the valuations and costs of buyers and sellers affect their affinity toward becoming a mechanism designer (”posting”) or a bidder in a mechanism (”visiting”). Building on Proposition 4, we establish the following result:

[15] Peters and Severinov (2008) provide an efficiency result for a large double auction where the traders' values are interdependent.

Proposition 6 *In equilibrium, (i)* $\beta_b(v), \beta_s(c) < 1$ *for all* $v > 0$ *and* $c < 1$.

There exist cutoffs $\underline{v}, \overline{v}, \underline{c}, \overline{c} \in (0,1)$ *such that:*

(ii) $\beta_b(v) = \beta_s(c) = 0$ *for all* $v \in [0, \underline{v})$, $c \in (\overline{c}, 1]$*;*

(iii) $\beta_b(v) > 0$ *and* $\beta_s(c) > 0$ *for all* $v \in [\overline{v}, 1]$ *and all* $c \in [0, \underline{c}]$.

(iv) $\beta_s(0) = \min\{1, 2f_b(0)/f_s(0)\}$ *and* $\beta_v(1) = \min\{1, 2f_s(1)/f_b(1)\}$.

Thus, according to Proposition 6, inefficient types will only visit other mechanisms, while more efficient types will both post their mechanisms and visit other mechanisms. Inefficient trader types will visit the auctions posted by efficient counterparts, where they obtain bargains in the unlikely event that no other visitor arrives. Efficient types wish to post because they attract a large number of visitors and therefore trade with a high probability. However, they can also trade with a high probability by visiting mechanisms on the other side of the market and placing high bids. Therefore, efficient types randomize. Proposition 6 suggests that the overall number of visitors should exceed the number of posters. Our numerical analysis of the two-type version of the model in Section 4 confirms this conjecture.

The last part of Proposition 6 shows that posting probabilities of the most efficient types are determined by local densities. In particular, when $f_b(0)/f_s(0)$ is close to zero, then the most efficient seller type posts with a low probability, and posting probabilities are non-monotone in costs. However, posting probabilities are monotone under well-behaved distributions such as uniform, as the following results shows.

Proposition 7 *If valuations and cost are distributed uniformly on* $[0,1]$*, then* $\beta_s(c)$ *is monotonically decreasing in* c *and* $\beta_b(v)$ *is monotonically increasing in* v*. Moreover, there exists* $\overline{c} < 0.5$ *and* $\overline{v} > 0.5$ *such that* $\beta_s(c) > 0$ *if and only if* $c \leq \overline{c}$*, and* $\beta_b(v) > 0$ *if and only if* $v \geq \overline{v}$.

To highlight the reason why monotonicity of the posting probabilities fails for certain distributions, we provide the following simple example:

Example 1. Suppose that the traders are homogeneous: all buyers have value $v = 1$ and all sellers have cost $c = 0$. Now, let us introduce small masses of high-value, $v = 1 + \alpha$, and low-value, $v = 1 - \alpha$ buyers. The new equilibrium derived in the online Appendix is such that for any $\alpha > 0$, both high-value and the low-value buyers visit with probability 1, while medium value buyers with $v = 1$ post with a positive probability. Visiting is the optimal choice for the low- and high-value buyer types because their trading probabilities are more sensitive to valuations when visiting than when posting. In particular, all three types trade with almost the same probability when posting. On the other hand, since the middle types $v = 1$ constitute a large atom, the winning probabilities of the three types are very different when they visit. Visiting low-value buyers trade with a low probability and pay less, while high-value buyers buy with a higher probability but pay a higher price.

Propositions 6 and 7 and Example 1 imply that the posting and visiting pattern is sensitive to the relative weights of the efficient and inefficient traders in the population. Example 1 highlights that monotonicity tends to fail when there is a high density (or an atom) at some cost or value. In this case, the types just below and just above the type with the atom are both more likely to visit than the latter. In summary, a smooth and well-behaved distribution like the uniform features monotone posting probabilities in types but an atom in valuations or costs make it more likely that both types above and below this atom prefer visiting over posting.

5 Posting and Visiting Decisions in a Two-type Case

To illustrate which types are more likely to post or visit, in this subsection we will consider a simple two-type version of out model.[16]

So, suppose that a seller's cost is either $c_1 = 0$ or $c_2 = \alpha$, and a buyer's value is either $v_1 = 1 - \alpha$ or $v_2 = 1$. Let π denote the probability that a buyer/a seller has a low value/a high-cost. There is a mass 1 of buyers and an equal mass of sellers.

Since the equilibrium outcome coincides with the constrained-efficient allocation, it is sufficient to derive the latter. In this symmetric setup, the constrained efficient allocation is also symmetric,[17] and so we only need to find four probabilities maximizing constrained welfare: $\underline{\beta} = \Pr(v_1 \text{ posts}) = \Pr(c_2 \text{ posts})$, $\overline{\beta} = \Pr(v_2 \text{ posts}) = \Pr(c_1 \text{ posts})$, $\underline{\rho} = \Pr(v_1 \text{ visits } c_1 \mid v_1 \text{ visits}) = \Pr(c_2 \text{ visits } v_2 \mid c_2 \text{ visits})$, and $\overline{\rho} = \Pr(v_2 \text{ visits } c_1 \mid v_2 \text{ visits}) = \Pr(c_1 \text{ visits } v_2 \mid c_1 \text{ visits})$. In words, $\underline{\beta}$ ($\overline{\beta}$) is the probability that inefficient (efficient) buyer and seller post, and $\underline{\rho}$ ($\overline{\rho}$) is the probability that an inefficient (efficient) trader type visits an efficient type conditional on visiting.

[16] The result that the unique equilibrium is a constrained welfare maximum applies to this set-up by a similar proof as in Proposition 3.

[17] This follows from the fact that the constrained welfare function is strictly concave. Therefore, an asymmetric allocation can be improved upon by using a convex combination of the two resulting asymmetric allocations.

Given $\underline{\beta}$, $\overline{\beta}$, $\underline{\rho}$, $\overline{\rho}$, the following queue lengths arise at a posting buyer, where the subscript denotes the type of the posting buyer and the superscript denotes the type of the visiting seller: $\lambda_1^1 = \frac{(1-\pi)(1-\overline{\beta})(1-\overline{\rho})}{\pi\underline{\beta}}$, $\lambda_1^2 = \frac{\pi(1-\underline{\beta})(1-\underline{\rho})}{\pi\underline{\beta}}$, $\lambda_2^1 = \frac{(1-\pi)(1-\overline{\beta})\overline{\rho}}{(1-\pi)\overline{\beta}}$, $\lambda_2^2 = \frac{\pi(1-\underline{\beta})\underline{\rho}}{(1-\pi)\overline{\beta}}$.

Total constrained welfare W is then twice the welfare that is generated in the market B where buyers post. Therefore, the planner's problem can be stated as follows:

$$\max_{\underline{\beta},\overline{\beta},\underline{\rho},\overline{\rho}} W/2 = \pi\underline{\beta}[(1-\alpha)(1-e^{-\lambda_1^1}) + e^{-\lambda_1^1}(1-e^{-\lambda_1^2})(1-2\alpha)]+$$
$$+ (1-\pi)\overline{\beta}[(1-e^{-\lambda_2^1}) + e^{-\lambda_2^1}(1-e^{-\lambda_2^2})(1-\alpha)]. \tag{14}$$

We have solved the problem (14) numerically using Matlab, varying π between 0.05 and 0.95, and varying α between 0.05 and 0.45. The results are presented in Tables 1-2. Table 1 provides the equilibrium values of ($\underline{\beta}$, $\overline{\beta}$, $\underline{\rho}$, $\overline{\rho}$). Table 2 provides the percentage difference between the welfare levels under two-sided and one-sided posting, respectively. Table 3 presents additional results with posting costs as explained below.

A few observations regarding testable implications of our model are in order:

1. Table 2 reveals that the total welfare is higher under bilateral posting, since in this case efficient buyer types post and inefficient seller types visit with a higher probability. This reduces the inefficiency from miscoordination, and allows high types to trade with a higher probability than under one-sided posting. This intuition is confirmed by the observation that the welfare difference is more significant under large type heterogeneity (e.g. $\alpha = 0.45$) and/or when the type distribution is close to uniform (i.e. $\pi = 0.5$).

2. Table 1 shows that typically $\overline{\beta} > \underline{\beta}$ i.e., more efficient types post more often than less efficient types. This is also highlighted in Proposition 6 in case of continuously distributed types. However, if one type if much more common than another, e.g. the probability π is either small or large, then the more common type mixes between posting and visiting, and the less common type only visits. This observation was highlighted in Example 1 as a source of non-monotonicity in equilibrium posting/visiting.

3. If both types randomize in equilibrium, then $\underline{\rho} > \overline{\rho}$ (In particular, this occurs when π is in the range between 0.4 and 0.8). This means that inefficient types, compared to efficient types, are relatively more likely to visit efficient trading partners. This result is similar to the one in Proposition 2 showing that buyers with low valuations are more likely to visit sellers with low reservation prices.

Next, we consider the effects of trading costs, as introduced by Shi and Delacroix

(2018), on posting and visiting patterns in the two-type case to identify the effects of such costs in our model with heterogeneous traders.

The effect of posting costs. First, let us consider the effect of a fixed cost $\gamma > 0$ incurred by a trader posting a mechanism. This costs diminishes the attractiveness of posting, but the magnitude of this effect depends on the trader's type and, In particular, the payoffs that they get by visiting. In particular, in the following example we focus on buyers without loss of generality, and show that inefficient buyers switch to visiting more frequently, while efficient traders are more negatively affected by larger congestion and therefore may even post with a higher probability, as posting costs increase.

To illustrate this formally, observe that the difference between the visiting payoffs of an efficient buyer, $U(1)$, and an inefficient buyer, $U(1-\alpha)$, is given by $U(1)-U(1-\alpha) = (1-\pi_2(\gamma))\alpha$, where $\pi_2(\gamma)$ is the probability that an efficient seller with cost $c = 0$ is visited by some efficient buyer. Then, as posting cost γ increases, the number of visitors also grows, so $\pi_2(\gamma)$ increases and $U(1) - U(1-\alpha)$ decreases in γ, respectively. Thus, congestion of efficient buyers in the visitor's market hurts efficient types more than inefficient types, and the former would reduce their posting probabilities by less than the inefficient buyers. Specifically, there exists $\widehat{\gamma} \in (0, 1)$ such that for all $\gamma \geq \widehat{\gamma}$, the inefficient types get a negative utility from positing and do not post at all.

Confirming this intuition, we illustrate the effects of posting costs with numerical results in Table 3. It shows that not only the posting probability of the inefficient type decreases more significantly, but also that efficient types may post more frequently as posting costs increase. This occurs in a large region of parameter values under which inefficient types post with a significant probability in the absence of posting costs. So, the withdrawal of inefficient types from posting creates a countervailing effect on efficient types via crowding in the sellers' posting market. which may outweigh the increased posting costs and cause efficient types to post more.

The larger negative effect of posting costs on the payoffs of more efficient visitors than on the less efficient ones remains straightforwardly true regardless of the number of types, and so we expect that it also holds with a continuum of types.

Entry costs. Suppose now that each trader type incurs an entry cost $e > 0$ in our two-type model. When e is small it acts us a sunk cost, and has no effect on the equilibrium. When e reaches a high level satisfying $e^* = (1-\alpha)e^{-\lambda^*}$ where λ^* solves $e^x = 2 + x$, the inefficient traders drop out from the market. Interestingly, we have:

Proposition 8 *There exists a cost $\widetilde{e} < e^*$, such that for all costs $e \in [\widetilde{e}, e^*)$, the traders of inefficient type participate with a positive probability and post with probability zero.*

The proof of this Proposition is provided in the Online Appendix. The intuition behind it derives from Example 1: less common types tend to visit, and low type traders enter less and become less common as entry costs increase.

Theorem 3 of Shi and Delacroix (2018) show that in their model with one-sided heterogeneity in valuations, and differences in the entry margin between the two sides (elastic vs inelastic side), there is a parameter region where high types post and low types visit mechanisms. Their result hinges on differences between the two sides with respect to the entry margin, while our effect is based on incentive effects for both visitors and posters. Moreover, the interpretation of our result is also different because our posters use optimal auction mechanisms, which lead to different pooling patterns between visitors. Crucially, price posting forces the same winning probability on all visitors, but the latter face different winning probabilities in auctions. Therefore, different types of a trader are more likely to remain on the same side of the market when auction mechanisms can be posted used. In contrast, price posting induces agents of different types to post or visit different mechanisms given their different willingness to trade.

6 Conclusions

This paper contributes to the competing mechanism design and directed search literatures by endogenizing the traders' functions in the economy. To this end, we study a model of a search market where agents can decide whether they want to post a mechanism or respond to mechanisms posted by the others. We have established three main results. First, the posting and directed search equilibrium decentralizes the constrained efficient allocation. Second, less efficient types exclusively respond to mechanisms offered by others, while more efficient types randomize between posting their own mechanisms and visiting. We show that under plausible assumptions the equilibrium features monotone posting probabilities. Third, we have showed via a numerical analysis that the equilibrium in our market with endogenous bargaining roles differs considerably from the equilibrium in the market with fixed roles and, in particular, the welfare created in the former substantially exceeds the welfare created in the latter.

This last result suggests that decentralized marketplaces should allow agents to self-select into different roles in the marketplace. Future research should shed additional light on how other traders' characteristics determine such self-selection. Finally, as a technical contribution to the literature, we have shown that the welfare function is strictly concave in a matching model with significant demand and cost heterogeneity. This concavity property, and the associated techniques of proving the existence and characterizations may be extended to other directed search models and aid in proving existence of equilibrium and studying its welfare properties.

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Appendix 1: Proofs of Propositions 1 and 2

Proof of Proposition 1. Recall that by assumption, there is a positive measure of buyers in market S, whose valuations are distributed with strictly positive density f_b.[18]

Fix a seller with cost c. Since the sellers are restricted to post direct mechanisms which treat buyers symmetrically, the mechanism offered by our seller with can be represented allocation rule $\widetilde{Pr}(v|M)$ that specifies the probability that buyer type v gets the good for all $v \in [0,1]$, and the transfer $\widetilde{t}(v|M)$ that she pays to the mechanism. Then $\tilde{u}(v) = v\widetilde{Pr}(v|M) - \widetilde{t}(v|M)$ is the expected payoff of type v in M, where we omit the dependence of $\tilde{u}(v)$ on M for brevity. So mechanism M can be represented by a pair $(\widetilde{Pr}(.|M), \tilde{u}(.))$ that must be incentive compatible and consistent with λ and market utility schedule u.

By assumption, a buyer of type v gets payoff $u(v)$ by optimally choosing among the mechanisms offered by other sellers in market S, that does not depend on the mechanism M offered by our seller.

Let $\lambda(v) \geq 0$ be the queue (expected number) of buyers of type $v \in [0,1]$ at our seller. The queue length of types of at least v is $\Lambda(v) = \int_v^1 \lambda(s)ds$. The seller's expected profit $\Pi(M,\lambda)$ in the mechanism M with buyers' queue λ can be expressed as a difference between the welfare generated by the mechanism and the total expected surplus earned by the buyers as follows:

$$\Pi(M,\lambda) = \int_0^1 \widetilde{Pr}(x|M)(x-c)\lambda(x)dx - \int_0^1 \tilde{u}(x)\lambda(x)dx \tag{15}$$

The seller's best response M^* must be sequentially rational, i.e. maximize $\Pi(M,\lambda^*)$ given the buyers' queue schedule λ^*. M^* and In particular the allocation rule $\widetilde{Pr}(.|M)$, must also be feasible given λ^*. At the same, λ^* must be generated by the buyers' optimal visiting strategies given M^* and market utility schedule u.

To derive M^* and λ^*, we will first consider the following relaxed program in which the seller chooses both the mechanism and the buyers' queue:

$$\max_{M,\lambda} \Pi(M,\lambda) \tag{16}$$

[18] Our argument can be adjusted to handle an arbitrary density function by considering a sequence of such density functions with limit equal to zero on a measurable subset of $[0,1]$, and show that by continuity the optimal mechanism for a seller is still a second-price auction even if some buyer types are not present.

subject to the following constraints on M and λ: (i) $\widetilde{Pr}(v|M) \leq 1$ for all v; (ii) $\int_v^1 \widetilde{Pr}(x|M)\lambda(x)dx \leq 1 - \exp^{-\Lambda(v)}$; (iii) $\tilde{u}(.)$ is continuous; (iv) $\lambda(v) = 0$ if $\tilde{u}(v) < u(v)$; (v) $\lambda(v) = \infty$ if $\tilde{u}(v) > u(v)$.

Constraint (i) and (ii) are feasibility constraints on the mechanism: (i) requires the probability of assigning the good to any buyer-type not to exceed 1. (ii) says that the ex-ante probability that the good ends up with a buyer type in $[v, 1]$ does not exceed the probability $1 - \exp^{-\Lambda(v)}$ that a buyer with such type visits our seller. It is a version of the well-known constraint of Border (1991) which he shows to be necessary and sufficient for the implementation of reduced-form symmetric auctions. Here it is stated for a continuum of types and participants. The combination of (i) and (ii) are weaker than the feasibility constraint requiring that at most one unit of the good is allocated to the visiting buyers ex-post. However, we will demonstrate that the latter is satisfied by the solution to the relaxed program. Constraint (iii) must hold by buyers' incentive compatibility.

Constraints (iv) and (v) must hold by buyers' individual rationality. Indeed, if $\tilde{u}(v) < u(v)$, no buyer of type v will choose our seller resulting in $\lambda(v) = 0$. If $\tilde{u}(v) > u(v)$, any visiting buyer of type v will choose our seller with probability 1, and so will all types close to v by continuity of u, which will result in $\lambda(v) = \infty$.

Note that in our relaxed program we omit the incentive constraints on the mechanism that $\widetilde{Pr}(v|M)$ is increasing in v and the envelope condition $\tilde{u}'(v) = \widetilde{Pr}(v|M)$ holds. Later we will check that the solution (M^*, λ^*) is such that these constraints hold and M^* is incentive compatible. We will also show that the queue λ^* is induced by the buyers' unique equilibrium visiting strategies, so that (M^*, λ^*) is an equilibrium outcome given the utility schedule u.

To begin solving the relaxed program, suppose that the seller offers mechanism M s.t. $\tilde{u}(x) > u(x)$ for some x. Then by continuity of $\tilde{u}$ and u, $\tilde{u}(y) > u(y) \geq 0$ and hence $\lambda(y) = \infty$ for all $y \in (x - \varepsilon, x + \varepsilon)$ for some $\varepsilon > 0$. Using this and constraint (ii) in (15) yields:

$$\Pi(M, \lambda) = \int_0^1 \widetilde{Pr}(x|M)(x-c)\lambda(x)dx - \int_0^1 \tilde{u}(x)\lambda(x)dx \leq 1 - c - \int_{x-\varepsilon}^{x+\varepsilon} \tilde{u}(x)\lambda(x)dx = -\infty$$

So, such mechanism M is not optimal and we must have $\tilde{u}(x) \leq u(x)$ for all x, with equality when $\lambda(x) > 0$. That is, our seller has to offer exactly the market utility $u(v)$ to any type v that she attracts to her mechanism. Therefore, $\int_0^1 \tilde{u}(x)\lambda(x))dx = \int_0^1 u(x)\lambda(x))dx$, and hence we can omit the constraints (iii)-(v) and rewrite the objective (15) as follows:

$$\max_{M,\lambda} \Pi(M,\lambda) = \int_0^1 \widetilde{Pr}(x|M)(x-c)\lambda(x)dx - \int_0^1 u(x)\lambda(x)dx. \tag{17}$$

So a seller's relaxed problem boils down to the choice of an optimal allocation rule $\widetilde{P}(.)$ and the queue $\lambda(.)$ subject to (i)-(ii). This observation leads to the following result.

Lemma 2 *If (M^*,λ^*) is a solution to the problem $\max_{M,\lambda}\Pi(M,\lambda)$ in (17) subject to constraints (i)- (ii), then the mechanism M^* must assign the good efficiently i.e., to the visiting buyer with the highest valuation, provided the latter is at least c.*

Proof of Lemma 2: Suppose that (M^*,λ^*) is the solution to this problem. Then, given $\lambda^*(x)$, M^* must solve $\max_M \int_0^1 \widetilde{Pr}(x|M^*)(x-c)\lambda^*(x)dx$ subject to constraints (i)-(ii). Then $\widetilde{Pr}(x|M^*)=0$ for all $x<c$, for otherwise the value of $\Pi(M,\lambda)$ in (17) can be increased by setting $\widetilde{Pr}(x|M^*)=0$ for $x<c$, relaxing (i) and (ii) at the same time.

Now, let us show that constraint (ii) is binding i.e., $\int_v^1 \widetilde{Pr}(x|M^*)\lambda(x)dx = 1-\exp^{-\Lambda(v)}$ for all $v\in[c,1]$. The proof is by contradiction. So, suppose not. Then, by continuity, there exist $v_1,v_2\in[c,1]$ s.t. $v_1<v_2$ and $\int_v^1 \widetilde{Pr}(x|M^*)\lambda(x)dx < 1-\exp^{-\Lambda(v)}$ for all $v\in(v_1,v_2)$.The last inequality implies that $\widetilde{Pr}(x|M^*)<\exp^{-\Lambda(x)}$ for all $v\in[v_3,v_2]$ for some $v_3\in(v_1,v_2)$. If $v_1=v_3=c$, we can increase $\widetilde{Pr}(x|M^*)$ for all $x\in(c,v_2)$ by some $\epsilon>0$ without violating constraints (i)-(ii). This modification increases (17).

Now suppose that $v_1>c$ and so $\int_{v_1}^1 \widetilde{Pr}(x|M^*)\lambda(x)dx = 1-\exp^{-\Lambda(v)}$. Then there exists $v_4,v_5\in(v_1,v_3)$ s.t. $v_4<v_5$ and $\widetilde{Pr}(x|M^*)>\exp^{-\Lambda(x)}$ for all $x\in[v_4,v_5]$. Then let $\Delta_1=\int_{v_3}^{v_2}\lambda(x)dx$, $\Delta_2=\int_{v_4}^{v_5}\lambda(x)dx$, and let ϵ_1 be such that $0<\epsilon_1\leq \exp^{-\Lambda(x)}-\widetilde{Pr}(x|M^*)$ for all $x\in(v_3,v_2)$ and ϵ_2 be such that $0<\epsilon_2\leq\widetilde{Pr}(x|M^*)-\exp^{-\Lambda(x)}$ for all $x\in(v_4,v_5)$. Finally, let $\delta=\min\{\epsilon_1\Delta_1,\epsilon_2\Delta_2\}$. Then the seller can attain a higher profit with queue λ and mechanism M' which differs from mechanism M^* only for $x\in(v_4,v_5)\cup(v_3,v_2)$ as follows: $\widetilde{Pr}(x|M')=\widetilde{Pr}(x|M^*)+\frac{\delta}{\Delta_1}$ for $v\in(v_3,v_2)$; $\widetilde{Pr}(x|M')=\widetilde{Pr}(x|M^*)-\frac{\delta}{\Delta_2}$ for $v\in(v_4,v_5)$. Indeed, the difference in the values of (17) in the mechanism M' and M^* is:

$$\delta\left(\int_{v_3}^{v_2}(x-c)\frac{\lambda(x)}{\Delta_1}dx - \int_{v_4}^{v_5}(x-c)\frac{\lambda(x)}{\Delta_2}dx\right) > \delta(v_3-v_5)>0.$$

This completes the proof of the claim that in an optimal mechanism M^*, $\int_v^1 \widetilde{Pr}(x|M^*)\lambda(x)dx = 1-\exp^{-\Lambda(v)}$ for all $v\in[c,1]$, which implies that $\widetilde{Pr}(v|M^*)=\exp^{-\Lambda(v)}$. Note that

$\exp^{-\Lambda(v)}$ is the probability that no buyer of type higher than v visits our seller. So the allocation rule in the mechanism M^* must assign the good to the visiting buyer with the highest type v, when $v \geq c$, and any allocation rule that differs from the efficient one on a set of types of a positive measure is not optimal. *Q.E.D.*

The next Lemma derives a key property of the optimal queue function.

Lemma 3 *If* (M^*, λ^*) *is a solution to the problem* $\max_{M,\lambda} \Pi(M, \lambda)$ *in (17) subject to constraints (i)- (ii) then: (a)* $\lambda^*(x) = 0$ *for all* $x \in [0, c)$*; (b)* $\int_c^z e^{-\Lambda^*(x)}dx \leq u(z)$ *for all* $z \geq c$*, with equality at all* z *s.t.* $\lambda^*(z) > 0$*.*

Proof of Lemma 3. By Lemma 2, $\widetilde{Pr}(x|M^*) = 0$ for all $x \in [0, c)$, so maximizing (15) requires setting $\lambda(x) = 0$ for all $x \in [0, c)$ s.t. $u(x) > 0$. It is also optimal to set $\lambda(x) = 0$ for all $x \in [0, c)$ s.t. $u(x) = 0$.

Since by Lemma 2 M^* must assign the good efficiently, a buyer of type $x \geq c$ will get it with probability $e^{-\Lambda(x)}$ where $\Lambda(x) = \int_x^1 \lambda(y)dy$. Therefore, the seller's expected profit (17) can be rewritten as follows:

$$\Pi(M^*, \lambda) = \int_c^1 \left((x - c)e^{-\Lambda(x)} - u(x)\right) \lambda(x)dx. \tag{18}$$

To find λ^* maximizing (18), we use optimal control method. The Hamiltonian for this problem is:

$$H(x, u(x), \Lambda(x), \lambda(x), \mu) = \left((x - c)e^{-\Lambda(x)} - u(x)\right) \lambda(x) + \mu(x)(-\lambda(x)) \tag{19}$$

where $\Lambda(x)$ is a state variable, $\lambda(x)$ is control, and μ is a costate variable associated with the evolution equation $\Lambda'(x) = -\lambda(x)$.

The Hamiltonian (19) is linear in the control variable. Pontyagrin's Maximum principle applies to the problems of this class and requires that the optimal control λ maximize the Hamiltonian. Since λ must be nonnegative, we therefore have:

$$\begin{aligned}
&(x - c)e^{-\Lambda(x)} - u(x) - \mu(x) < 0 \Rightarrow \lambda(x) = 0 \\
&(x - c)e^{-\Lambda(x)} - u(x) - \mu(x) = 0 \Rightarrow \lambda(x) \geq 0 \\
&(x - c)e^{-\Lambda(x)} - u(x) - \mu(x) > 0 \Rightarrow \lambda(x) = \infty
\end{aligned} \tag{20}$$

By (20) the control variable λ takes non-zero values only on intervals of x where $(x - c)e^{-\Lambda(x)} - u(x) - \mu(x)$ vanishes. Such intervals are called *singular arcs.*

The costate equation is:

$$\dot{\mu} = -\frac{\partial H}{\partial \Lambda} = (x-c)e^{-\Lambda(x)}\lambda(x) \tag{21}$$

The transversality condition for the fixed 'initial time' c with free value $\Lambda(c)$ is $\mu(c) = 0$, using which and (21) yields:

$$\mu(x) = \int_c^x \mu'(s)ds = \int_c^x (z-c)e^{-\Lambda(z)}\lambda(z)dz = (x-c)e^{-\Lambda(x)} - \int_c^x e^{-\Lambda(z)}dz \tag{22}$$

Using (22), we can rewrite (20) as follows:

$$\begin{aligned}
&\int_c^x e^{-\Lambda(z)}dz < u(x) \Rightarrow \lambda(x) = 0 \\
&\int_c^x e^{-\Lambda(z)}dz = u(x) \Rightarrow \lambda(x) \geq 0 \\
&\int_c^x e^{-\Lambda(z)}dz > u(x) \Rightarrow \lambda(x) = \infty
\end{aligned} \tag{23}$$

To complete the proof of the Lemma, let us now show that $\int_c^x e^{-\Lambda(z)}dz \leq u(x)$ for all $x > c$. The argument is by contradiction so suppose that $\int_c^x e^{-\Lambda(z)}dz > u(x)$ for some $x \in (c, 1]$. Then there exists $x_1 \in (c, 1)$ s.t. $\int_c^{x_1} e^{-\Lambda(z)}dz > u(x_1)$. Indeed, if $\int_c^1 e^{-\Lambda(z)}dz > u(1)$ but $\int_c^x e^{-\Lambda(z)}dz \leq u(x)$ for all $x \in [c, 1)$ then, since u is non-decreasing, $\lim_{x\to 1} \int_x^1 e^{-\Lambda(z)}dz > 0$, but this contradicts the fact that $e^{-\Lambda(x)} \leq 1$ for all x, which follows from $\Lambda(x) \geq 0$.

Next, since $\int_c^x e^{-\Lambda(z)}dz$ and u are continuous, there exists $x_2 \in (x_1, 1]$ s.t. $\int_c^y e^{-\Lambda(z)}dz > u(y)$ and hence $\lambda(y) = \infty$ for all $y \in [x_1, x_2]$. Hence, $e^{-\Lambda(x)} = 0$ for all $x \in [c, x_2)$ implying that $\int_c^{x_1} e^{-\Lambda(z)}dz \leq u(x_1)$, which contradicts our earlier assumption. *Q.E.D.*

Lemma 3 has two notable implications. First, since $u(x) = \tilde{u}(x)$ for all x s.t. $\lambda^*(x) > 0$, we have $\tilde{u}(x) = \int_c^z e^{-\Lambda^*(x)}dx$ for such x. Second, the inequality $\int_c^x e^{-\Lambda(z)}dz \leq u(x)$ for all $x > c$ implies that following. If there exists $x > c$ s.t. $u(x) = 0$, then $\int_c^y e^{-\Lambda(z)}dz = 0$ and hence $\Lambda(y) = \infty$ for all $y \in (c, x)$. So, if $\bar{x} = \max\{x : u(x) = 0\}$, then $\lambda(y) = \infty$ for all $y \in (\bar{x} - \epsilon, \bar{x})$ for some $\epsilon > 0$.

The next two Lemmas build on Lemma 3 to complete the characterization of the unique optimal queue solving the relaxed program. Let $u'_-(.)$ ($u'_+(.)$) be the left-hand (right-hand) side derivative of $u(.)$, respectively. Then we have:

Lemma 4 *Suppose that $u(c) > 0$.*

If $\frac{u(z)}{z-c} > \frac{u(1)}{1-c}$ *for all* $z \in (c,1)$*, then the optimal queue in the solution to the relaxed program in (17) subject to constraints (i)-(ii) is* $\lambda^*(z) = 0$ *for all* $z < 1$*, and* $\Lambda^*(1) = -\log\left(\frac{u(1)}{1-c}\right)$.

If $\frac{u(z)}{z-c} \leq \frac{u(1)}{1-c}$ *for some* $z \in (c,1)$*, let*

$$\widehat{z}(c) = \sup\{z|u(z) > (z-c)u'_-(z)\} \in (c,1) \tag{24}$$

Then $\lambda^*(x) = 0$ *for all* $x \in [0, \widehat{z}(c))$*, and*

$$\Lambda^*(x) = \begin{cases} -\log(\frac{u(\widehat{z}(c))}{\widehat{z}(c)-c}) & \textit{if } x \in [0, \widehat{z}(c)], \\ -\log(u'_-(x)) & \textit{if } x \in (\widehat{z}(c), 1]. \end{cases} \tag{25}$$

So, $\int_c^x e^{-\Lambda^*(s)}ds < u(x)$ *for all* $x \in [0, \widehat{z}(c))$ *and* $\int_c^x e^{-\Lambda^*(s)}ds = u(x)$ *for all* $x \geq \widehat{z}(c)$.[19]

Proof of Lemma 4. Let us define $\widehat{z}(c) = \inf\{z \in [c,1] | \int_c^z e^{-\Lambda(x)}dx = u(z)\}$. To confirm that such $\widehat{z}(c)$ exists, suppose otherwise. Then, $\int_c^z e^{-\Lambda(x)}dx < u(z)$ for all $(c,1]$ since $u(c) > 0 = \int_c^c e^{-\Lambda(x)}dx$, and both $u(z)$ and $\int_c^z e^{-\Lambda(x)}dx$ are continuous. Hence by Lemma 3, $\lambda(x) = \Lambda(x) = 0$ for all $x \in [c,1]$, and so $\int_c^1 e^{-\Lambda(x)}dx = 1 - c > u(1)$ where the inequality holds by assumption. A contradiction.

Next, let us show that $u(z) = \int_c^z e^{-\Lambda(x)}dx$ for all $z \in [\widehat{z}(c), 1]$ if $\widehat{z}(c) < 1$. By Lemma 3, we only need to rule out $u(z) > \int_c^z e^{-\Lambda(x)}dx$ for some $z \in (\widehat{z}(c), 1]$. The proof is by contradiction, so suppose that the last inequality holds at some $z > \widehat{z}(c)$. Let $\bar{z} = \inf\{z | z \in [\widehat{z}(c), 1],\ u(z) > \int_c^z e^{-\Lambda(x)}dx\}$. So, $u(\bar{z}) = \int_c^{\bar{z}} e^{-\Lambda(x)}dx$. If $u(z) > \int_c^z e^{-\Lambda(x)}dx$ for all $z \in (\bar{z}, 1]$, then $\lambda(z) = 0$ and hence $e^{-\Lambda(z)} = 1$ for all $z \in (\bar{z}, 1]$ by Lemma 3. But $u'_+(z) \leq 1$ for all z, and so $\int_c^z e^{-\Lambda(x)}dx \geq u(z)$ for all $z \in (\bar{z}, 1]$. A contradiction. So there exists $z_2 \in (\bar{z}, 1]$ s.t. $z_2 = \inf\{z \in (\bar{z}, 1] | \int_c^z e^{-\Lambda(x)}dx = u(z)\}$. But then $\int_c^z e^{-\Lambda(x)}dx < u(z)$ for all $z \in (\bar{z}, z_2)$ and hence $e^{-\Lambda(z_2)} \geq u'_-(z_2)$. At the same time, $\lambda(z) = 0$ for all $z \in (\bar{z}, z_2)$ and so $e^{-\Lambda(z)} = e^{-\Lambda(z_2)} \geq u'_-(z_2) \geq u'_+(z)$ for all $z \in (\bar{z}, z_2)$. The last inequality holds by convexity of u. Hence, for all $z \in (\bar{z}, z_2)$, $\int_c^z e^{-\Lambda(x)}dx \geq u(z)$, contradicting the earlier conclusion that $\int_c^z e^{-\Lambda(x)}dx < u(z)$.

Now, let us now show that $\widehat{z}(c) = 1$ if and only if $\frac{u(z)}{z-c} > \frac{u(1)}{1-c}$ for all $z \in [c,1]$. If $\widehat{z}(c) = 1$, then $(1-c)e^{-\Lambda(1)} = u(1)$, $\lambda(z) = 0$ and $u(z) > \int_c^z e^{-\Lambda(x)}dx$ for all $z < 1$. Thus, $u(z) > \int_c^z e^{-\Lambda(x)}dx = (z-c)e^{-\Lambda(1)} = (z-c)\frac{u(1)}{1-c}$, and so $\frac{u(z)}{z-c} > \frac{u(1)}{1-c}$ for all $z \geq c$. Note that $\Lambda(1) = -\log(\frac{u(1)}{1-c})$, establishing the first claim of the Lemma.

[19] Note that we have an atom at 1 i.e., $\Lambda^*(1) > 0$ if $u'_-(x) < 1$.

To prove the claim in the opposite direction, suppose that $\widehat{z}(c) < 1$. Since $\lambda(x) = 0$ for all $x < \widehat{z}(c)$, $u(\widehat{z}(c)) = \int_c^{\widehat{z}(c)} e^{-\Lambda(x)}dx = (\widehat{z}(c) - c)e^{-\Lambda(\widehat{z}(c))}$, while $u(1) = \int_c^1 e^{-\Lambda(x)}dx \geq \int_c^1 e^{-\Lambda(\widehat{z}(c))}dx = (1-c)e^{-\Lambda(\widehat{z}(c))}$. Combining the last two inequalities yields $\frac{u(1)}{1-c} \geq \frac{u(\widehat{z}(c))}{\widehat{z}(c)-c}$.

So, let us now suppose that $\frac{u(z)}{z-c} \leq \frac{u(1)}{1-c}$ for some $z \in [c, 1)$, and hence $\widehat{z}(c) < 1$. Then $\int_c^z e^{-\Lambda^*(x)}dx \leq u(z)$ for all $z \in [c, 1]$, with equality iff $z \in [\widehat{z}(c), 1]$, and therefore $\Lambda^*(z) = \Lambda^*(\widehat{z}(c))$ for all $z \in [0, \widehat{z}(c))$.

Note that $u(.)$ must be strictly increasing on $[\widehat{z}(c), 1]$. For suppose $u(.)$ is constant on some $[a, b]$ s.t. $\widehat{z}(c) < b$. Since $\int_c^z e^{-\Lambda(x)}dx = u(z)$ for all $z \geq \widehat{z}(c)$, it follows that $e^{-\Lambda(z)} = 0$ for all $z \in [a, b)$. But then $e^{-\Lambda(z)} = 0$ for all $z < b$ since $\Lambda(z)$ is nonnegative and decreasing. So, $\int_c^{\widehat{z}(c)} e^{-\Lambda(x)}dx = 0 < u(\widehat{z}(c))$, contradicting the definition of $\widehat{z}(c)$.

Since $u(.)$ is continuous, and increasing, it is almost everywhere differentiable and at every x possesses left-hand and right-hand side derivatives $u'_-(x) > 0$ and $u'_+(x) > 0$ s.t. $u'_-(x) \leq u'_+(x)$, with equality when $u(.)$ is differentiable at x. From $u(z) - u(z') \leq (z - z')$ it follows that $u(.)$ is uniformly continuous and, by the Fundamental Theorem of Calculus, $u(z) = \int_0^z u'(x)dx + u(0)$. So, $\int_0^z u'(x)dx + u(0) = \int_c^z e^{-\Lambda^*(x)}dx$ for all $z \in [\widehat{z}(c)), 1]$. Therefore, we must have $e^{-\Lambda^*(x)} = u'(x)$ at all points of differentiability of $u(.)$ (i.e. almost everywhere on $[\widehat{z}(c), 1]$), and $u'_-(x) \leq e^{-\Lambda^*(x)} \leq u'_+(x)$ when $u(.)$ is not differentiable at $x \in (\widehat{z}(c), 1]$. Also, $e^{-\Lambda^*(\widehat{z}(c))} \leq u'_+(\widehat{z}(c))$ since otherwise we cannot have $e^{-\Lambda^*(x)} = u'(x)$ a.e. on $(\widehat{z}(c), 1]$. Also, $u'_-(\widehat{z}(c)) \leq e^{-\Lambda^*(\widehat{z}(c))}$ since otherwise there exists $\epsilon > 0$ s.t. for all $z \in [\widehat{z}(c) - \epsilon, \widehat{z}(c)]$ we have $\int_c^z e^{-\Lambda^*(x)}dx = (z - c)e^{-\Lambda^*(\widehat{z})} = u(\widehat{z}(c)) - (\widehat{z}(c) - z)e^{-\Lambda^*(\widehat{z})} > u(z)$, which contradicts the definition of $\widehat{z}(c)$. So we can set $e^{-\Lambda^*(x)} = u'_-(x)$ at all points of non-differentiability of $u(.)$ on $(\widehat{z}(c), 1]$, as this choice does not affect $\int_c^z e^{-\Lambda^*(x)}dx$ and so $u(z) = \int_c^z e^{-\Lambda^*(x)}dx$ at any $z \in (\widehat{z}(c), 1]$. This establishes the second line in (25).

The first line in (25) holds because, since $\lambda(x) = 0$ for all $x < \widehat{z}(c)$,

$$\int_c^{\widehat{z}(c)} e^{-\Lambda^*(x)}dx = (\widehat{z}(c) - c)e^{-\Lambda^*(\widehat{z}(c))} = u(\widehat{z}(c)). \tag{26}$$

Next, let us show that $\widehat{z}(c)$ is well-defined and satisfies (24) i.e., $\widehat{z}(c) = \bar{z}$ where $\bar{z} = \sup\{x \in [c, 1] | u'_-(x)(x - c) - u(x) < 0\}$. To this end, first note that $u'_-(z)(z - c) - u(z)$ is increasing in z on $[c, 1]$ because $u(.)$ is convex. Since $u(c) > 0$, $u'_-(z)(z-c) - u(z) < 0$ for z sufficiently close to c. So, $\bar{z} > c$.

Since $\frac{u(1)}{1-c} - \frac{u(z)}{z-c} = \int_z^1 \frac{u'_-(x)(x-c)-u(x)}{(x-c)^2}dx$ and $\frac{u(1)}{1-c} - \frac{u(z)}{z-c} \geq 0$ for some $z \in (c, 1)$ in the case under consideration, $u'_-(x)(x - c) - u(x) \geq 0$ for some $x \in [z, 1)$. So, $\bar{z} < 1$.

Finally, let us show that $\widehat{z}(c) = \bar{z}$. If $\widehat{z}(c) > \bar{z}$, then $\int_c^{\bar{z}} e^{-\Lambda(x)}dx = (\bar{z}-c)e^{-\Lambda(\widehat{z}(c))} = (\bar{z}-c)\frac{u(\widehat{z}(c))}{\widehat{z}(c)-c} \geq u(\bar{z})$. The last inequality contradicts the definition of $\widehat{z}(c)$ and holds because $\frac{u(z)}{z-c}$ is increasing for all $z \in [\bar{z}, 1]$ which follows from $u'_{-}(z)(z-c) - u(z) \geq 0$.

On the other hand, if $\widehat{z}(c) < \bar{z}$, then by definition $u'_{-}(z)(z-c) - u(z) < 0$ for all $z \in [\widehat{z}(c), \bar{z})$, and $\int_c^{\widehat{z}} e^{-\Lambda(x)}dx = (\widehat{z}-c)e^{-\Lambda(\widehat{z}(c))} = u(\widehat{z}(c))$. So, $e^{-\Lambda(\widehat{z}(c))} > u'_{-}(\widehat{z}(c))$, and by continuity of $u(.)$ and $e^{-\Lambda(z)}$, and the fact that $u'_{-}(z)(z-c) - u(z) < 0$ for all $z \in [\widehat{z}(c), \bar{z})$, it follows that there exists $\epsilon > 0$ s.t. $\int_c^{z} e^{-\Lambda(x)}dx > u(z)$ for $z \in (\widehat{z}(c), \widehat{z}(c)+\epsilon)$. This contradicts the fact that, as shown above in this proof, $\int_c^{z} e^{-\Lambda(x)}dx = u(z)$ for all z s.t. $z > \widehat{z}(c)$.

Q.E.D.

The next Lemma deals with the case $u(c) = 0$.

Lemma 5 *Suppose that $u(c) = 0$ and let $\widetilde{v} = \max\{v | u(v) = 0\}$. Then the solution (M^*, λ^*) to the relaxed problem in (17) subject to constraints (i)-(ii) is such that $\int_c^x e^{-\Lambda^*(s)}ds = u(x)$ for all $x \in [0,1]$. In particular, $\lambda^*(x) = 0$ if $x \in [0, c]$ and*

$$\Lambda^*(x) = \begin{cases} \infty & \text{if } x \in [0, \tilde{v}] \\ -\log(u'_{-}(x)) & \text{if } x \in (\tilde{v}, 1], \end{cases} \tag{27}$$

Proof of Lemma 5: First, $\lambda^*(x) = 0$ for all $x \in [0, c]$ since by Lemma 2 the mechanism solving the relaxed program must be efficient. Further, by Lemma 3, $\int_c^x e^{-\Lambda^*(s)}ds = 0$ for all $x \in (c, \tilde{v}]$. So, $\Lambda^*(x) = \infty$ for all $x \leq \widetilde{v}$. At the same time, $\lambda^*(x) < \infty$ for almost all $x > \tilde{v}$ for otherwise $\Pi(M^*, \lambda^*) = -\infty$ in (15). So we must have $\lambda^*(x) = \infty$ for all $x \in (\widetilde{v}-\epsilon, \widetilde{v}]$ for some $\epsilon \in (0, \tilde{v}-c)$.

For $x > \widetilde{v}$, the argument from Lemma 4 can be used verbatim, after replacing $\widehat{z}(c)$ with $\tilde{v}$, to show that $\Lambda^*(x) = -\log(u'_{-}(x))$ for all $x \in (\tilde{v}, 1]$. *Q.E.D.*

To complete the description of the optimal mechanism M^*, let us provide the buyers payoff function $\tilde{u}(.)$ in M^* that follow from the previous Lemmas. Again, we need to consider two cases. First, suppose that $u(c) = 0$. Then $\tilde{u}(v) = u(v) = \int_c^x e^{-\Lambda^*(s)}ds$ for all $v \in [c, 1]$ by Lemma 5. Now, suppose that $u(c) > 0$. Then $\tilde{u}(v) = u(v) = \int_c^x e^{-\Lambda^*(s)}ds$ for all $v \in [\widehat{z}(c), 1]$ by Lemma 3. By the same Lemma, if $v < \widehat{z}(c)$, then $\lambda^*(x) = 0$ and so we can set $\tilde{u}(v)$ on $[0, \widehat{z}(c))$ arbitrarily as long as $\tilde{u}(v) \leq u(v)$. So, let us set $\tilde{u}(v) = 0$ for all $v < c$, and $\tilde{u}(v) = \int_c^v e^{-\Lambda^*(s)}ds = (v-c)e^{-\Lambda^*(\widehat{z}(c))} < u(v)$ for all $v \in [c, \widehat{z}(c))$, where the last inequality holds because $(\widehat{z}(c)-c)e^{-\Lambda^*(\widehat{z}(c))} = u(\widehat{z}(c))$, and $u(v) = \int_c^x e^{-\Lambda^*(s)}ds$ is strictly convex for all $v \in [\widehat{z}(c), 1]$.

Finally, it is immediate that the second-price auction with reservation price c implements the mechanism M^*, since the second-price auction is efficient and incentive compatible (in dominant strategies) and delivers the payoff $\tilde{u}(v)$ to a buyer-type v under the queue λ^*. In particular, given the queue $\lambda^*(.)$, the probability that a buyer of type $v \geq \widehat{z}(c)$ gets the good in the second-price auction is $e^{-\Lambda^*(v)}$, which is increasing in v and implies that $\tilde{u}(v) = \int_c^v e^{-\Lambda^*(x)}dx$ holds. Also, with an arbitrary tie-breaking rule in the second price auction, no more than one unit of the good is allocated to the buyers ex-post i.e. it is feasible.

The final Lemma in this proof shows that λ^* is a.e. unique queue consistent with the buyers' optimal visiting strategies:

Lemma 6 *If a seller offers the second price auction with reserve price c, then the buyers' queue at this seller induced by the buyers' optimal participating strategies is equal to* λ^* *given in (25) and (27) a.e. on* $[0, 1]$.

Proof. We will provide the proof for the case $u(c) > 0$. The proof for the case $u(c) = 0$ is analogous. By construction, $\int_c^v e^{-\Lambda^*(x)}dx = u(v)$ for $v \geq \widehat{z}(c)$. So, such buyer type is indifferent between visiting our seller and some other seller, and hence visiting our seller with any probability is optimal. Since there is a positive mass of buyers with valuations distributed with density $g_B(.) > 0$ over $[0, 1]$, there are sufficiently many buyers to generate the queue $\lambda^*(v)$ at our seller. In particular, if all buyers with value v visit our seller with probability 1, they would generate a queue $\lambda(v) = \infty$. On the other hand, if all buyers with value v visit our seller with probability zero, they would generate a queue $\lambda(v) = 0$ there. So, there exist buyers' optimal participating strategies inducing queue λ^*, and under this queue a buyer of type $v \geq \widehat{z}(c)$ earns payoff $\int_c^v e^{-\Lambda^*(x)}dx$ at this seller.

Further, the payoff of a buyer $v \in [c, \widehat{z}(c))$ in this seller' auction does not exceed $\int_c^v e^{-\Lambda^*(x)}dx = (v - c)e^{-\Lambda^*(\widehat{z}(c))}$ which is strictly less than $u(v)$ for all $v \in [c, \widehat{z}(c))$ because $(\widehat{z}(c) - c)e^{-\Lambda^*(\widehat{z}(c))} = u(\widehat{z}(c))$ by construction and $u(v)$ is strictly convex. So, such buyer would not visit our seller resulting in $\lambda^*(v) = 0$, as required.

Now suppose that the buyers' optimal participation strategies in our seller's second price auction induce some queue λ different from λ^*. Let $\Lambda(v) = \int_v^1 \lambda(x)dx$. Then the payoff of buyer type v is equal to $\int_c^v e^{-\Lambda(x)}dx$. The optimality of the buyers' participation strategies requires that $\lambda(v) = 0$ if $\int_c^v e^{-\Lambda(x)}dx < u(v)$, $\lambda(v) = \infty$ if $\int_c^v e^{-\Lambda(x)}dx > u(v)$. and $\lambda(v) > 0$ only if $\int_c^v e^{-\Lambda(x)}dx \geq u(v)$. So we cannot have

$\int_c^v e^{-\Lambda(x)}dx > u(v)$ for some v, for in this case this inequality also holds and $\lambda = \infty$ in some right neighborhood of v.

Thus, $\int_c^v e^{-\Lambda(x)}dx = 0 \le u(v)$ for all v. So, the argument of Lemma 4 can be used verbatim to establish the existence of $\widetilde{z}$ s.t. $\int_c^v e^{-\Lambda(x)}dx = u(v)$ for all $v \in [\widetilde{z}, 1]$ and $\lambda(v) = 0$, and $\int_c^v e^{-\Lambda(x)}dx = (v-c)e^{-\Lambda(\widetilde{z})} < u(v)$ for all $v \in [0, \widetilde{z}]$. Then $e^{-\Lambda(v)} = u'(v)$ and hence $\lambda(v) = \lambda^*(v)$ for almost all $v \in [\widetilde{z}, 1]$. Hence $\widetilde{z} = \widehat{z}(c)$, implying that $\lambda(v) = \lambda^*(v)$ a.e. completing the proof of the Lemma.

This completes the proof of Proposition 1. *Q.E.D.*

Proof of Proposition 2.

By Proposition 1, in market S a seller's unique best response to any profile of the other sellers' mechanisms is to offer an efficient mechanism such as a second-price auction with a reservation price equal to her cost, and the buyers' have unique best response equilibrium participation strategies. Therefore, in every equilibrium the sellers offer efficient mechanisms, equivalent to second-price auctions with reserve prices equal to their respective cost. In the rest of this proof we will characterize the equilibrium allocation and establish its uniqueness.

To this end, note that if a buyer type v visits sellers who offer second-price auctions with reserve prices c_2 and c_3 then by optimality, a buyer must get the same payoff at every seller that she visits i.e.,

$$\int_{c_2}^{v} Prob.[x\ wins\ at\ c_2]dx = \int_{c_3}^{v} Prob.[x\ wins\ at\ c_3]dx. \tag{28}$$

By Proposition 1 there exists $\widehat{z}(c) \in [c, 1]$ such that a buyer of type v visits a seller with reservation price type c if and only if $v \ge \widehat{z}(c)$. So, assuming without loss of generality that $c_3 > c_2$, the equality (28) has to hold for all $v \ge \widehat{z}(c_3)$, and hence we must have $Prob.[v\ wins\ at\ c_2] = Prob.[v\ wins\ at\ c_3]$ for all $v \in [\widehat{z}(c_3), 1]$. Since c_2 and c_3 were chosen arbitrarily, this implies that each buyer type v must win with the same probability at each seller that she visits.

From this it follows that a buyer must randomize uniformly between all sellers that she visits, and so buyers of every type must generate the same queue at all sellers that they visit. In particular, the equilibrium queue that buyers of type v form at seller with cost c such that $v \ge \widehat{z}(c)$ is equal to $\lambda^*(v) = \frac{g_B(v)}{G_S(\widehat{z}^{-1}(v))}$. Therefore, the equilibrium queue of buyers with valuations at least v at a seller with reservation price c, $v \ge \widehat{z}(c)$, is

$$\Lambda^*(v) = \int_{\max\{v,\widehat{z}(c)\}}^{1} \lambda^*(v)dx = \int_{\max\{v,\widehat{z}(c)\}}^{1} \frac{g_B(x)}{G_S(\widehat{z}^{-1}(x))}dx,$$

Then buyer v's equilibrium probability of trading at a seller with cost c is equal to $\exp^{-\Lambda^*(v)} = \exp^{-\int_{\max\{v,\widehat{z}(c)\}}^{1} \frac{g_B(x)}{G_S(\widehat{z}^{-1}(x))}dx}$. Also, in equilibrium, the market payoff $u(v)$ must be equal to buyer v's payoff from participating in a trading mechanism $U(v)$, i.e.

$$u(v) = U(v) = \int_0^v \exp^{-\Lambda^*(y)} dy = \int_0^v \exp^{-\int_y^1 \frac{g_B(x)}{G_S(\widehat{z}^{-1}(x))}dx} dy.$$

Thus, to complete equilibrium characterization we need to characterize the equilibrium cutoff type function $\widehat{z}(c)$. This is done in two steps below. In Step 1, we derive the differential equation for $\widehat{z}(c)$. In Step 2, we establish its uniqueness.

Step 1. The buyers' equilibrium cutoff function $\widehat{z}(c)$ satisfies the following condition:

$$\lambda(\widehat{z}(c))\widehat{z}'(c)(\widehat{z}(c) - c) = 1 \tag{29}$$

The expected payoff of buyer type v visiting a seller holding a second price auction with reservation price c s.t. $v \geq \widehat{z}(c)$ is equal to

$$\int_{\widehat{z}(c)}^{v} e^{-\Lambda^*(x)}dx + (\widehat{z}(c) - c)e^{-\Lambda^*(\widehat{z}(c))} \tag{30}$$

As shown above, this buyer type must get the same payoff by visiting the second price auction with another reservation price c' s.t. $v > \widehat{z}(c')$. Therefore, the derivative of (30) with respect to c must be zero. Differentiating (30) and setting the derivative to zero yields (29).

Since $\lambda^*(x) = \frac{g_B(x)}{G_S(\widehat{z}^{-1}(x))}$, (29) can be rewritten as follows:

$$\widehat{z}'(c) = \frac{G_S(c)}{g_B(\widehat{z}(c))(\widehat{z}(c) - c)}. \tag{31}$$

Step 2. In this step we establish the uniqueness of the equilibrium schedule $\widehat{z}(.)$.

Note that $\widehat{z}(c) \geq c$ by buyer optimality. This and (31) imply that $\widehat{z}(.)$ must be increasing. Therefore, $\widehat{z}(0) = 0$. For suppose not i.e., $\widehat{z}(0) = \epsilon > 0$. Then $\widehat{z}(c) \geq \epsilon$ for all $c \in [0, 1]$. So, a buyer type $v \in [0, \epsilon)$ would not visit any seller, despite being present in the market. But this participation strategy is suboptimal, since such buyer would get a positive expected payoff by visiting any sellers with reservation price $c < v$.

Now suppose that there are two different schedules (31) $\widehat{z}_1(.)$ and $\widehat{z}_2(.)$ satisfying (31) and $\widehat{z}_1(0) = \widehat{z}_2(0) = 0$. Let $c_h \in (0, 1)$ be such that $\widehat{z}_1(c_h) = 1$. c_h exists because by (31) $\widehat{z}_1(.)$ is increasing and $\widehat{z}_1(c) > c$ for all $c \in [0, 1]$.

If $\widehat{z}_2(c_h) = 1$ also, then $\widehat{z}_1(c) = \widehat{z}_2(c)$ for all $c \in (0, \infty)$ because the right-hand side of (31) is Lipshitz continuous on $(0, \infty)$, contradicting that $\widehat{z}_1(.) \neq \widehat{z}_2(.)$. So, without loss of generality, assume that $\widehat{z}_2(c_2) = 1$ where $c_2 < c_h$. It follows that $\widehat{z}_1(c_2) < \widehat{z}_2(c_2)$, and so by the fundamental theorem of ordinary differential equations $\widehat{z}_1(c) < \widehat{z}_2(c)$ for all $c \in (0, c_2)$. By construction, for all $c, v \in (0, 1]$, $\Lambda^*(v|\widehat{z}_1) > \Lambda^*(v|\widehat{z}_2)$ where $\Lambda^*(v|\widehat{z}_i)$ is the value of Λ^* when $\widehat{z} = \widehat{z}_i$, $i \in \{1, 2\}$. Therefore, we have:

$$G_B(1) = \int_0^{c_h} g_S(c)\Lambda_1(c|\widehat{z}_1)dc > \int_0^{c_2} g_S(c)\Lambda_2(c|\widehat{z}_2)dc = G_B(1).$$

The contradiction established in the above inequality implies that the equilibrium cutoff schedule $\widehat{z}$ is unique. *Q.E.D.*

Appendix 2

Proof of Proposition 3.

Consider a market with N buyer and N seller types. Let $v_1, v_2, ..., v_N$ be the buyer' types (values) and $c_1, c_2, ..., c_N$ be the seller' types (costs), with $0 < v_i < v_{i+1} \leq 1$ and $0 \leq c_i < c_{i+1}$ for all $i \in \{1, ..., N-1\}$. All traders' types are distributed independently.

Let π_B^i denote the probability that a buyer's type is v_i and β_B^i denote the probability that a buyer of type v_i posts. Likewise, let π_S^j denote the probability that a seller's type is c_j, and β_S^j denote the probability that a seller of type c_j posts. Next, let τ_{Sj}^i denote the probability that buyer type v_i visits a seller of type c_j, with $\sum_{j=1}^{N} \tau_{Sj}^i = 1 - \beta_B^i$, and τ_{Bi}^j denote the probability that seller type c_j visits a buyer of type v_i, with $\sum_{i=1}^{N} \tau_{Bi}^j = 1 - \beta_S^j$.

Then the queue of type c_j sellers visiting buyer type v_i, s.t. $\beta_B^i > 0$, satisfies:

$$\lambda_{Bi}^j = \frac{\pi_S^j \tau_{Bi}^j}{\pi_B^i \beta_B^i}. \tag{32}$$

Likewise, the queue of type c_j sellers visiting buyer type v_i, s.t. $\beta_S^j > 0$, satisfies:

$$\lambda_{Sj}^i = \frac{\pi_B^i \tau_{Sj}^i}{\pi_S^j \beta_S^j}. \tag{33}$$

Without loss of generality, we can assume that $\lambda_{Bi}^j > 0$ only is $c_j < v_i$, while $\lambda_{Sj}^i > 0$ only if $c_j < v_i$ because trade cannot occur if $c_j > v_i$.

Let U_B^i be the expected utility of a buyer type i who posts, and U_{Sj}^i be the utility of a buyer with type i who visits a seller with type j. Similarly, let V_S^j be the utility of a seller type j when posting, and V_{Bi}^j the utility of a seller with type j who visits a buyer with type i. The total welfare in the economy can be expressed as the sum of the expected welfare measures generated at each posting agent type:

$$W = \sum_{i=1}^{N} \left(\pi_B^i \beta_B^i (U_B^i + \sum_{j=1}^{N} \lambda_{Bi}^j V_{Bi}^j) \right) + \sum_{j=1}^{N} \left(\pi_S^j \beta_S^j (V_S^j + \sum_{i=1}^{N} \lambda_{Sj}^i U_{Sj}^i) \right). \tag{34}$$

Let $\beta_B = (\beta_B^1, ..., \beta_B^N)$ and $\beta_S = (\beta_S^1, ..., \beta_S^N)$ denote the posting probability vectors for each type. Also, let $\tau_B = (\tau_{B1}^1, ..., \tau_{BN}^N)$ and $\tau_S = (\tau_{S1}^1, ..., \tau_{SN}^N)$ denote the vectors of visiting probabilities. Given the profiles $(\beta_B, \beta_S, \tau_S, \tau_B)$, the utilities $U_B^i, V_{Bi}^j, V_S^j, U_{Sj}^i$ and the queue lengths $\lambda_{Bi}^j, \lambda_{Sj}^i$ are uniquely defined when the corresponding posting probabilities β_S^j and β_B^i are strictly positive. However, if $\beta_B^i = 0$, then the corresponding component of the welfare function is zero, and so the queue lengths λ_{Sj}^i can be defined arbitrarily.

Thus, the constrained efficient allocation $(\beta_B, \beta_S, \tau_S, \tau_B)$ solves

$$\max_{\beta_B, \beta_S, \tau_S, \tau_B \in [0,1]^{2N+2N^2}} W$$

$$s.t. \ \sum_{j=1}^{N} \tau_{Sj}^i = 1 - \beta_B^i, \ \ \sum_{i=1}^{N} \tau_{Bi}^j = 1 - \beta_S^j.$$

The first-order conditions are the standard Kuhn-Tucker conditions. First, for all i and $\beta_B^i \in (0, 1)$,

$$\frac{\partial W}{\partial \beta_B^i} = \max_j \frac{\partial W}{\partial \tau_{Sj}^i}, \tag{35}$$

Further, if $\beta_B^i = 0$, then $\frac{\partial W}{\partial \beta_B^i} \le \max_j \frac{\partial W}{\partial \tau_{Sj}^i}$. If $\beta_B^i = 1$, then $\frac{\partial W}{\partial \beta_B^i} \ge \max_j \frac{\partial W}{\partial \tau_{Sj}^i}$.

Also, if $\tau_{Sj}^i > 0$, then

$$\frac{\partial W}{\partial \tau_{Sj}^i} \ge \max \left\{ \frac{\partial W}{\partial \beta_B^i}, \max_k \frac{\partial W}{\partial \tau_{Sk}^i} \right\}. \tag{36}$$

If $\tau_{Sj}^i = 0$ and $\beta_B^i < 1$, then there exists k such that

$$\frac{\partial W}{\partial \tau_{Sj}^i} \le \frac{\partial W}{\partial \tau_{Sk}^i}. \tag{37}$$

Similar conditions apply to the sellers' choice variables.

It is clear that W is continuous in all choice variables whenever the posting probabilities β are positive. Moreover, if $\beta_B^i = 0$, then U_B^i and V_{Bi}^j can be set arbitrarily without violating the continuity of W at such β_B^i.[20] Since W is a continuous function defined on the compact set $[0,1]^{2N+2N^2}$, an optimum exists. Moreover, W is differentiable at all points such that the posting probabilities are positive, so the optimum can be found via standard methods.

To complete the proof of the Proposition we need the following three Lemmas the proofs of which are relegated to Appendix 3 below. First, Lemma 7 states that the effect of changing a choice variable on welfare is confined to the corresponding type who is making the change:

Lemma 7 *For all j and l, $\frac{\partial W}{\partial \beta_B^l} = \pi_B^l U_B^l$, $\frac{\partial W}{\partial \tau_{Sj}^l} = \pi_B^l U_{Sj}^l$, $\frac{\partial W}{\partial \beta_S^l} = \pi_S^l V_S^l$ and $\frac{\partial W}{\partial \tau_{Bi}^l} = \pi_S^l V_{Bi}^l$.*

Lemma 7 implies that a trader's choice does not produce any externalities on the other traders. Therefore, by maximizing his own utility each type also maximizes his contribution to welfare. This intuition is formalized in the next Lemma:

Lemma 8 *In any game with a discrete type-space, a strategy vector $(\beta_B, \beta_S, \tau_S, \tau_B)$ forms an equilibrium if and only if it satisfies the first-order conditions of the welfare maximization program (35)-(37).*

The second part of proof is to show that the welfare functions are strictly concave. For this, we need the following Lemma:

Lemma 9 *In any discrete-type economy as well as in the economy where types are distributed continuously according to distribution functions F_s and F_b, the welfare function W is strictly concave.*

Combining Lemmas 8 and 9 we can now complete the proof of Proposition 3 and show that our game possesses a unique equilibrium which maximizes the welfare W.

[20] Our notion of the equilibrium for the continuous-type game also respects continuity at $\beta = 0$, see Section 4.1. This makes it possible to concentrate on strategy vectors with $\beta > 0$ and approximate the $\beta = 0$ case as a limit of $\beta \to 0$ both for the equilibrium conditions and the welfare function.

First, as we argued above, there exists an optimum for the welfare program of the economy with a continuum of possible types. Let a sequence of discrete-type economies be indexed by $k \in \{1, 2, ...\}$, and suppose that this sequence converges in probability to our continuous-type economy, that is, π_B^k converges to F_b and π_S^k converges to F_s in probability.

Consider the unique sequence of strategies $(\beta_{Bk}^*, \beta_{Sk}^*, \tau_{Sk}^*, \tau_{Bk}^*)$ that maximize the welfare in economy k. By the maximum theorem, $(\beta_B^*, \beta_S^*, \tau_S^*, \tau_B^*) = \lim_{k\to\infty}(\beta_{Bk}^*, \beta_{Sk}^*, \tau_{Sk}^*, \tau_{Bk}^*)$ is the (unique) welfare maximum in the limiting continuous-type economy. By Lemma 8, each strategy vector $(\beta_{Bk}^*, \beta_{Sk}^*, \tau_{Sk}^*, \tau_{Bk}^*)$, by virtue of being a global maximum, forms an equilibrium in economy k. Given that the set of equilibria is upper hemi-continuous by continuity of the payoffs in the probabilities of different types, it follows that $(\beta_B^*, \beta_S^*, \tau_S^*, \tau_B^*)$ forms an equilibrium in the original game with the continuous type-space. Finally, take any non welfare maximizing allocation of types, that is, an allocation $(\beta_B, \beta_S, \tau_S, \tau_B) \neq (\beta_B^*, \beta_S^*, \tau_S^*, \tau_B^*)$ in the original game with continuous types. Then taking appropriate limits of the results of Lemmas 7 and 8,[21] we obtain that the first-order conditions for equilibrium would be violated by the vector $(\beta_B, \beta_S, \tau_S, \tau_B)$, so it cannot be an equilibrium in the limiting, continuous type game. *Q.E.D.*

Appendix 3

Proof of Lemma 7.

Rearranging terms in W as defined in (34), and using (32) yield:

$$W = \sum_{i=1}^{N} \pi_B^i \beta_B^i U_B^i + \sum_{i=1}^{N}\sum_{j=1}^{N} \pi_B^i \beta_B^i \lambda_{Bi}^j V_{Bi}^j + \sum_{j=1}^{N} \pi_S^j \beta_S^j V_S^j + \sum_{j=1}^{N} \pi_S^j \beta_S^j \lambda_{Sj}^l U_{Sj}^l + \sum_{j=1}^{N}\sum_{i\neq l} \pi_S^j \beta_S^j \lambda_{Sj}^i U_{Sj}^i$$

$$= \sum_{i=1}^{N} \pi_B^i \beta_B^i U_B^i + \sum_{i=1}^{N}\sum_{j=1}^{N} \pi_S^j \tau_{Bi}^j V_{Bi}^j + \sum_{j=1}^{N} \pi_S^j \beta_S^j V_S^j + \sum_{j=1}^{N} \pi_B^l \tau_{Sj}^l U_{Sj}^l + \sum_{j=1}^{N}\sum_{i\neq l} \pi_S^j \beta_S^j \lambda_{Sj}^i U_{Sj}^i.$$

This last expression shows that W depends on β_B^l directly, i.e., not through the utilities (U and V), only through the first sum. The "direct" derivative is then $\pi_B^l U_B^l$, while the derivative through the utilities can be calculated using from the above:

$$\frac{\partial W}{\partial \beta_B^l} = \pi_B^l U_B^l + \sum_{i=1}^{N} \pi_B^i \beta_B^i \left(\frac{\partial U_B^i}{\partial \beta_B^l} + \sum_{j=1}^{N} \lambda_{Bi}^j \frac{\partial V_{Bi}^j}{\partial \beta_B^l} \right) + \sum_{j=1}^{N} \pi_S^j \beta_S^j \left(\frac{\partial V_S^j}{\partial \beta_B^l} + \sum_{i=1}^{N} \lambda_{Sj}^i \frac{\partial U_{Sj}^i}{\partial \beta_B^l} \right).$$

[21] In particular, the limiting version of the result of Lemma 8 establishes that the first-order conditions for the welfare maximum and the equilibrium conditions coincide in the continuous type model.

To prove our result, we only need to establish that the last two terms are zero, i.e.,

$$\sum_{i=1}^{N} \pi_B^i \beta_B^i \left(\frac{\partial U_B^i}{\partial \beta_B^l} + \sum_{j=1}^{N} \lambda_{Bi}^j \frac{\partial V_{Bi}^j}{\partial \beta_B^l} \right) + \sum_{j=1}^{N} \pi_S^j \beta_S^j \left(\frac{\partial V_S^j}{\partial \beta_B^l} + \sum_{i=1}^{N} \lambda_{Sj}^i \frac{\partial S_{Sj}^i}{\partial \beta_B^l} \right) = 0.$$

In what follows, we show that for all i,

$$\frac{\partial U_B^i}{\partial \beta_B^l} + \sum_{j=1}^{N} \lambda_{Bi}^j \frac{\partial V_{Bi}^j}{\partial \beta_B^l} = 0, \tag{38}$$

and for all j,

$$\frac{\partial V_S^j}{\partial \beta_B^l} + \sum_{i=1}^{N} \lambda_{Sj}^i \frac{\partial U_{Sj}^i}{\partial \beta_B^l} = 0. \tag{39}$$

First, let us consider (38). Note that buyer i's payoff from posting depends only on who visits him, and is therefore independent of the probability that type l, $l \neq i$, buyer posts. Therefore, $\frac{\partial U_B^i}{\partial \beta_B^l} = 0$. Likewise, the payoff of type j seller when visiting type i buyer does not depend on the probability with which buyer-type l posts. Therefore, $\sum_{j=1}^{N} \lambda_{Bi}^j \frac{\partial V_{Bi}^j}{\partial \beta_B^l} = 0$. Hence, (38) holds. Similarly, $\frac{\partial V_S^j}{\partial \beta_B^l} = \frac{\partial U_{Sj}^i}{\partial \beta_B^l} = 0$ for all $i \neq l$ because V_S^j and U_{Sj}^i depend only on $(\tau_{Sj}^1, ..., \tau_{Sj}^N)$ and β_S^j but not on β_B^l by construction; see subsection (2) for the relevant queue lengths when a seller of type j posts an auction.

Next, set $i = l$ and note that the utility of a posting buyer with type i is equal to the surplus generated at that buyer minus the rents of the visiting sellers. Therefore,

$$U_B^i = W_{Bi} - \sum_{j=1}^{N} \lambda_{Bi}^j V_{Bi}^j \tag{40}$$

where

$$\begin{aligned} W_{Bi} &= \left(1 - e^{-\lambda_{Bi}^1}\right)(v_i - c_1) + e^{-\lambda_{Bi}^1}\left(1 - e^{-\lambda_{Bi}^2}\right)(v_i - c_2) + ... \\ &\quad + e^{-\lambda_{Bi}^1 - \lambda_{Bi}^2 - ... - \lambda_{Bi}^{N-1}}(1 - e^{-\lambda_{Bi}^N})(v_i - c_N). \end{aligned} \tag{41}$$

Expression (41) reflects that the welfare generated at the posting buyer of type v_i is equal to $v_i - c_k$ if the most efficient seller visiting her has cost c_k, which occurs with probability $e^{-\lambda_{Bi}^1 - \lambda_{Bi}^2 - ... - \lambda_{Bi}^{k-1}}(1 - e^{-\lambda_{Bi}^k})(v_i - c_k)$. Note, that (41) assumes that $v_i \geq c_N$.

We do so for ease of exposition but the proof would be essentially the same if this assumption did not hold.[22]

Rearranging (40) and taking a derivative yields

$$\frac{\partial U_B^i}{\partial \beta_B^l} + \sum_{j=1}^{N} \lambda_{Bi}^j \frac{\partial V_{Bi}^j}{\partial \beta_B^l} + \sum_{j=1}^{N} V_{Bi}^j \frac{\partial \lambda_{Bi}^j}{\partial \beta_B^l} = \frac{\partial W_{Bi}}{\partial \beta_B^l}. \tag{42}$$

By (41), W_{Bi} depends on β_B^l only through the queue lengths, and thus we have:

$$\frac{\partial W_{Bi}}{\partial \beta_B^l} = \sum_{j=1}^{N} \frac{\partial W_{Bi}}{\partial \lambda_{Bi}^j} \frac{\partial \lambda_{Bi}^j}{\partial \beta_B^l}. \tag{43}$$

Using (42), and (43), we can rewrite condition (38) as follows:

$$\sum_{j=1}^{N} V_{Bi}^j \frac{\partial \lambda_{Bi}^j}{\partial \beta_B^l} = \sum_{j=1}^{N} \frac{\partial W_{Bi}}{\partial \lambda_{Bi}^j} \frac{\partial \lambda_{Bi}^j}{\partial \beta_B^l}.$$

Therefore, it is sufficient to prove that for all i, j, $i \neq j$, we have:

$$\frac{\partial W_{Bi}}{\partial \lambda_{Bi}^j} = V_{Bi}^j. \tag{44}$$

We next show that (44) holds, thus establishing (38). From (41) we obtain:

$$\frac{\partial W_{Bi}}{\partial \lambda_{Bi}^j} = e^{-\lambda_{Bi}^1 - \lambda_{Bi}^2 - \ldots - \lambda_{Bi}^j}(v_i - c_j) - \sum_{k=j}^{N-1} e^{-\lambda_{Bi}^1 - \lambda_{Bi}^2 - \ldots - \lambda_{Bi}^k}(1 - e^{-\lambda_{Bi}^{k+1}})(v_i - c_{k+1}). \tag{45}$$

On the other hand,

$$\begin{aligned} V_{Bi}^j =& e^{-\lambda_{Bi}^1 - \lambda_{Bi}^2 - \ldots - \lambda_{Bi}^N}(v_i - c_j) + e^{-\lambda_{Bi}^1 - \lambda_{Bi}^2 - \ldots - \lambda_{Bi}^{N-1}}(1 - e^{-\lambda_{Bi}^N})(c_N - c_j) + \ldots + \\ & + e^{-\lambda_{Bi}^1 - \lambda_{Bi}^2 - \ldots - \lambda_{Bi}^j}(1 - e^{-\lambda_{Bi}^{j+1}})(c_{j+1} - c_j) = e^{-\lambda_{Bi}^1 - \lambda_{Bi}^2 - \ldots - \lambda_{Bi}^N}(v_i - c_j) + \\ & \sum_{k=j}^{N-1} e^{-\lambda_{Bi}^1 - \lambda_{Bi}^2 - \ldots - \lambda_{Bi}^k}(1 - e^{-\lambda_{Bi}^{k+1}})(v_i - c_j) - \sum_{k=j}^{N-1} e^{-\lambda_{Bi}^1 - \lambda_{Bi}^2 - \ldots - \lambda_{Bi}^k}(1 - e^{-\lambda_{Bi}^{k+1}})(v_i - c_{k+1}). \end{aligned} \tag{46}$$

Using the identity

$$e^{-\lambda_{Bi}^1 - \lambda_{Bi}^2 - \ldots - \lambda_{Bi}^j} = e^{-\lambda_{Bi}^1 - \lambda_{Bi}^2 - \ldots - \lambda_{Bi}^N} + \sum_{k=j}^{N-1} e^{-\lambda_{Bi}^1 - \lambda_{Bi}^2 - \ldots - \lambda_{Bi}^k}(1 - e^{-\lambda_{Bi}^{k+1}}),$$

[22] Particularly, if $v_i < c_{k'}$ for some $k' \in \{, \ldots, N\}$, then modify (41) by keeping only the terms with c_k such that $v_i \geq c_k$.

we obtain from (46) that

$$V_{Bi}^{j} = \quad e^{-\lambda_{Bi}^{1}-\lambda_{Bi}^{2}-\ldots-\lambda_{Bi}^{j}}(v_i - c_j) - \sum_{k=j}^{N-1} e^{-\lambda_{Bi}^{1}-\lambda_{Bi}^{2}-\ldots-\lambda_{Bi}^{k}}(1-e^{-\lambda_{Bi}^{k+1}})(v_i - c_{k+1}). \quad (47)$$

A comparison of (47) and (45) shows that $\frac{\partial W_{Bi}}{\partial \lambda_{Bi}^{j}} = V_{Bi}^{j}$, and thus (44) and (38) hold.

The proof of (39) follows a similar argument. In particular, it is sufficient to establish the following condition similar to (44):

$$\frac{\partial W_{Sj}}{\partial \lambda_{Sj}^{i}} = U_{Sj}^{i}. \quad (48)$$

The proof of (48) is analogous to the proof of (44), and is thus omitted. The above establishes that

$$\frac{\partial W}{\partial \beta_B^l} = \pi_B^l U_B^l. \quad (49)$$

The next argument shows that

$$\frac{\partial W}{\partial \tau_{Sj}^l} = \pi_B^l U_{Sj}^l. \quad (50)$$

For ease of exposition, let us rewrite the welfare formula as follows:

$$W = \sum_{i=1}^{N} \left(\pi_B^i \beta_B^i (U_B^i + \sum_{j=1}^{N} \lambda_{Bi}^j V_{Bi}^j) \right) + + \sum_{j=1}^{N} \left(\pi_S^j \beta_S^j (V_S^j + \sum_{i=1}^{N} \lambda_{Sj}^i U_{Sj}^i) \right).$$

The only term that depends directly on τ_{Sj}^l is $\pi_S^j \beta_S^j \lambda_{Sj}^l U_{Sj}^i = \pi_B^i \tau_{Sj}^l U_{Sj}^l$. Therefore,

$$\frac{\partial W}{\partial \tau_{Sj}^l} = \pi_B^l U_{Sj}^l + \sum_{i=1}^{N} \pi_B^i \beta_B^i \left(\frac{\partial U_B^i}{\partial \tau_{Sj}^l} + \sum_{j=1}^{N} \lambda_{Bi}^j \frac{\partial V_{Bi}^j}{\partial \tau_{Sj}^l} \right) + \sum_{j=1}^{N} \pi_S^j \beta_S^j \left(\frac{\partial V_S^j}{\partial \tau_{Sj}^l} + \sum_{i=1}^{N} \lambda_{Sj}^i \frac{\partial U_{Sj}^i}{\partial \tau_{Sj}^l} \right).$$

Again, we can show that the terms in the last two lines add up to zero. To see this, note that the whole argument that lead to (44) did not depend on the variable being β_B^l, as the same sufficient condition holds for any other variable, including τ_{Sj}^l. Therefore, $\frac{\partial W}{\partial \tau_{Sj}^l} = \pi_B^l U_{Sj}^l$ indeed holds. The proof of the last two statements can be completed by following analogous steps. *Q.E.D.*

Proof of Lemma 8. We will provide a formal proof for the buyers' side. The argument for the sellers' side is identical. First, we show that a buyer of type i prefers posting over visiting at $(\beta_B, \beta_S, \tau_S, \tau_B)$ if and only if $\partial W/\partial\beta_B^i \geq \max_j \partial W/\partial\tau_{Sj}^i$. Similar observations apply to *which* posters are visited; that is, to the τ variables. Thus, an allocation $(\beta_B, \beta_S, \tau_S, \tau_B)$ satisfies the first-order conditions (35)- (37) of welfare maximization if and only if it constitutes an equilibrium.

Take a vector $(\beta_B, \beta_S, \tau_S, \tau_B)$ that satisfies the first-order conditions (35)-(37), and suppose that $\beta_B^l \in (0,1)$. Then by (35) $\frac{\partial W}{\partial \beta_B^l} = \max_j \frac{\partial W}{\partial \tau_{Sj}^l}$. In this case, (49) and (50) imply that $U_B^l = \max_j U_{Sj}^l$, which means that the buyer type l is indifferent between posting and visiting optimally.

Second, suppose that $\beta_B^l = 0$. Then the first-order conditions imply that $\frac{\partial W}{\partial \beta_B^l} = \pi_B^l U_B^l \leq \max_j \pi_B^l U_{Sj}^l$, so $\beta_B^l = 0$ is an optimal choice. A similar argument applies to the case $\beta_B^l = 1$.

Let us also show that whenever $(\beta_B, \beta_S, \tau_S, \tau_B)$ satisfies the first-order conditions of the welfare program, then the corresponding visiting probabilities are optimal for the buyers. Indeed, suppose that $\tau_{Sj}^l > 0$. Then, by (36), we have $\frac{\partial W}{\partial \tau_{Sj}^l} \geq \frac{\partial W}{\partial \tau_{Sk}^l}$ for all k which, by (50), is equivalent to $U_{Sj}^l \geq U_{Sk}^l$. So it is indeed optimal for a buyer type l to visit a seller with type j.

In the other direction, we need to show that if a strategy vector $(\beta_B, \beta_S, \tau_S, \tau_B)$ forms an equilibrium, then it satisfies the first-order conditions for the welfare maximization. The proof is by counterpoint. So suppose that the vector $(\beta_B, \beta_S, \tau_S, \tau_B)$ constitutes an equilibrium but does not satisfy the first-order conditions. First, suppose that $\tau_{Sj}^l > 0$ but $\frac{\partial W}{\partial \tau_{Sj}^l} < \frac{\partial W}{\partial \tau_{Sk}^l}$. Then (50) implies that $U_{Sj}^l < U_{Sk}^l$, and thus it is not optimal for buyer type l to visit seller j, a contradiction. The same argument applies if $\beta_B^l > 0$, as in this case the violation of the first-order condition for maximizing W would mean that $U_B^l < \max_j U_{Sj}^l$, which implies that it is not optimal for such a type to post. Q.E.D.

Proof of Lemma 9:

First, we prove the Lemma for the discrete-type case. Then we show the result in the original continuous-type game by taking a limit of the type distribution and using convexity of the value function.

Strict concavity of W holds if the Hessian of W is negative definite. Using Lemma 7, we show that the Hessian of W is block-diagonal. In particular, the only variables where the cross-partial is non-zero is for variables $(\beta_B^i, \tau_{Bi}^1, ..., \tau_{Bi}^N)$. This follows because the queue lengths λ_{Bi}^{k} with $k = 1, 2, ..., N$ determine the utility levels at buyer i (see (40) and (41)), so for example $\frac{\partial U_B^i}{\partial \lambda_{Bj}^l} = 0$ for any $j \neq i$, and $\frac{\partial U_B^i}{\partial \lambda_{Sj}^l} = 0$ for any i, j, l. Next, notice that by (32) λ_{Bi}^{k} only depends on β_B^i, τ_{Bi}^k, which implies that for any $j \neq i$,

$$\frac{\partial^2 W}{\partial \beta_B^i \partial \lambda_{Bj}^l} = \pi_B^i \frac{\partial U_B^i}{\partial \lambda_{Bj}^l} = 0.$$

A similar argument can be made for the other cross-partials that do not belong in the same block. Similar block matrices can be constructed with the second derivatives of W with respect to variables where sellers post and buyers visit. The jth such block contains variables $(\beta_S^j, \tau_{Sj}^1, ..., \tau_{Sj}^N)$, and the argument is entirely symmetric.

With such a block-diagonal matrix, negative definiteness of the Hessian is equivalent to each of the N block matrices being negative definite. Let us call the ith such block-matrix that contains the second derivatives with respect to $(\beta_S^j, \tau_{Sj}^1, ..., \tau_{Sj}^N)$ as H_i. Formally, if for all $i = 1, 2, ..., N$ matrix H_i is negative definite, then the Hessian of W is also negative definite.

Using Lemma 7, we have the following second-derivatives for any $j = 1, 2, ..., N$:

$$\frac{\partial^2 W}{\partial \left(\beta_B^i\right)^2} = \pi_B^i \frac{\partial U_B^i}{\partial \beta_B^i},$$

$$\frac{\partial^2 W}{\partial \left(\tau_{Bi}^j\right)^2} = \pi_S^j \frac{\partial V_{Bi}^j}{\partial \tau_{Bi}^j},$$

and

$$\frac{\partial^2 W}{\partial \beta_B^i \partial \tau_{Bi}^j} = \pi_B^i \frac{\partial U_B^i}{\partial \tau_{Bi}^j} = \pi_S^j \frac{\partial V_{Bi}^j}{\partial \beta_B^i}.$$

Finally,

$$\frac{\partial^2 W}{\partial \tau_{Bi}^l \partial \tau_{Bi}^j} = \pi_S^j \frac{\partial V_{Bi}^j}{\partial \tau_{Bi}^l} = \pi_S^l \frac{\partial V_{Bi}^l}{\partial \tau_{Bi}^j}.$$

The following calculations, using (32), transform the problem into derivatives with respect to queue lengths; we also record the signs of these derivatives:

$$\frac{\partial^2 W}{\partial \left(\beta_B^i\right)^2} = \pi_B^i \frac{\partial U_B^i}{\partial \beta_B^i} = \pi_B^i \sum_{l=1}^{N} \frac{\partial U_B^i}{\partial \lambda_{Bi}^l} \frac{\partial \lambda_{Bi}^l}{\partial \beta_B^i} = -\pi_B^i \sum_{l=1}^{N} \frac{\lambda_{Bi}^l}{\beta_B^i} \frac{\partial U_B^i}{\partial \lambda_{Bi}^l} < 0, \tag{51}$$

$$\frac{\partial^2 W}{\partial \left(\tau_{Bi}^j\right)^2} = \pi_S^j \frac{\partial V_{Bi}^j}{\partial \tau_{Bi}^j} = \pi_S^j \frac{\partial V_{Bi}^j}{\partial \lambda_{Bi}^j} \frac{\partial \lambda_{Bi}^j}{\partial \tau_{Bi}^j} = \pi_S^j \frac{\lambda_{Bi}^j}{\tau_{Bi}^j} \frac{\partial V_{Bi}^j}{\partial \lambda_{Bi}^j} < 0, \tag{52}$$

and
$$\frac{\partial^2 W}{\partial\beta_B^i\partial\tau_{Bi}^j}=\pi_S^j\frac{\partial V_{Bi}^j}{\partial\beta_B^i}=\pi_S^j\sum_{l=1}^N\frac{\partial V_{Bi}^j}{\partial\lambda_{Bi}^l}\frac{\partial\lambda_{Bi}^l}{\partial\beta_B^i}=-\pi_S^j\sum_{l=1}^N\frac{\lambda_{Bi}^l}{\beta_B^i}\frac{\partial V_{Bi}^j}{\partial\lambda_{Bi}^l}>0. \tag{53}$$
Also,
$$\frac{\partial^2 W}{\partial\beta_B^i\partial\tau_{Bi}^j}=\pi_B^i\frac{\partial U_B^i}{\partial\tau_{Bi}^j}=\pi_B^i\frac{\partial\lambda_{Bi}^j}{\partial\tau_{Bi}^j}\frac{\partial U_B^i}{\partial\lambda_{Bi}^j}=\pi_B^i\frac{\lambda_{Bi}^j}{\tau_{Bi}^j}\frac{\partial U_B^i}{\partial\lambda_{Bi}^j}>0. \tag{54}$$
Finally,
$$\frac{\partial^2 W}{\partial\tau_{Bi}^l\partial\tau_{Bi}^j}=\pi_S^j\frac{\partial V_{Bi}^j}{\partial\tau_{Bi}^l}=\pi_S^j\frac{\lambda_{Bi}^l}{\tau_{Bi}^l}\frac{\partial V_{Bi}^j}{\partial\lambda_{Bi}^l}, \tag{55}$$
and, by symmetry of the Hessian,
$$\frac{\partial^2 W}{\partial\tau_{Bi}^l\partial\tau_{Bi}^j}=\pi_S^l\frac{\lambda_{Bi}^j}{\tau_{Bi}^j}\frac{\partial V_{Bi}^l}{\partial\lambda_{Bi}^j}. \tag{56}$$

Using (51), and (54),
$$\frac{\partial^2 W}{\partial\left(\beta_B^i\right)^2}=-\sum_{l=1}^N\frac{\tau_{Bi}^l}{\beta_B^i}\frac{\partial^2 W}{\partial\tau_{Bi}^l\partial\tau_{Bi}^j}, \tag{57}$$
and, by (52) and (53),
$$\frac{\partial^2 W}{\partial\beta_B^i\partial\tau_{Bi}^j}=-\sum_{l=1}^N\frac{\tau_{Bi}^l}{\beta_B^i}\frac{\partial^2 W}{\partial\tau_{Bi}^l\partial\tau_{Bi}^j}. \tag{58}$$

From (51) and (52) one can see that the main diagonal of the Hessian is negative. On the other hand, (57) and (58) imply that the Hessian matrix is singular as the first-column (where the derivatives with respect to β_B^i appear) is generated by using the same weights (weight $\frac{\tau_{Bi}^l}{\beta_B^i}$ for column l) in each row. Notice, that (57) and (58) imply that the weights that yield the null vector are $(\beta_B^i,\tau_{Bi}^1,..,\tau_{Bi}^N)$ because
$$\beta_B^i\frac{\partial^2 W}{\partial\left(\beta_B^i\right)^2}+\sum_{l=1}^N\tau_{Bi}^l\frac{\partial^2 W}{\partial\tau_{Bi}^l\partial\tau_{Bi}^j}=0. \tag{59}$$
By Euler's Theorem, this implies that the Hessian is homogenous of degree zero.[23]

The following important result, one of our main technical contributions, is useful for completing the proof of Lemma 9:

[23] This also follows from how the welfare was defined by observing that queue lengths are homogeneous of degree zero, and thus welfare is homogeneous of degree one.

Lemma 10 *In any game with a discrete type space, the function W is strictly concave on the constraint set, and hence has a unique maximum.*

Proof of Lemma 10.

The proof of this Lemma proceeds by establishing two claims. First, we show that the function W is strictly concave on the constraint set if matrix $\widetilde{H}_i$ is negative definite for all $i = 1, 2, ..., N$, where $\widetilde{H}_i$ is a matrix obtained by deleting the first row and column of the Hessian of block i.[24] Then we establish the negative definiteness of $\widetilde{H}_i$ for all $i = 1, 2, ..., N$.

Claim 1. The function W is concave on the constraint set if $\widetilde{H}_i$ is negative definite on this set for all i.

Proof of Claim 1. If $\widetilde{H}_i$ is negative definite and $\det H_i < 0$ holds then the result follows from determinant-based test of negative definiteness of a symmetric matrix. However, $\det H_i = 0$ as we have shown above, so we need a further step to prove this Claim.

Let $x = (x_1, x_2, ..., x_N)$ with $x_i = (\beta_B^i, \tau_{Bi}^1, .., \tau_{Bi}^N)$ be a starting point, and consider a change $d = (d_1, d_2, ..., d_N)$. The rest of the proof argues that given any starting point x, the directions of change in which the quadratic form disappears, that is, for which $d_i' H_i d_i = 0$ are such that they contradict the constraints, which implies negative definiteness of H_i on the constraint set. It is clear from (57) and (58) that- under the assumption that $\widetilde{H}_i$ is negative definite, and thus non-singular- the directions where the quadratic form disappears is such that (59) holds, that is, $H_i d_i = 0$ with $d_i = \kappa(\beta_B^i, \tau_{Bi}^1, .., \tau_{Bi}^N) = \kappa x_i$ for some constant κ. Therefore, by (59), the ratio $\frac{\tau_{Bi}^l}{\beta_B^i}$ is constant in the direction of the change. Consequently, the queue length $\lambda_{Bi}^l = \frac{\pi_S^l \tau_{Bi}^l}{\pi_B^i \beta_B^i}$ is also constant for all i, l. Then take an $d = (d_1, d_2, ..., d_N)$ such that $H_i d_i = 0$ for all i, and recall that all the queue lengths $(\lambda_{B1}^1, ..., \lambda_{BN}^N)$ are pinned down. Since $(\lambda_{B1}^1, ..., \lambda_{BN}^N)$ are one-to-one with the strategies, it follows that there is just a unique strategy vector $\{(\beta_B^i, \tau_{Bi}^1, .., \tau_{Bi}^N)\}_{i=1}^N = \{t_i\}_{i=1}^N$ that satisfies $H_i t_i = 0$ for all i. This provides a contradiction, as the proposed direction of change means no change at all in the strategies, which concludes the proof of Claim 1.

Claim 2. The matrix $\widetilde{H}_i$ is negative definite for all i.

Proof of Claim 2. To use (55), we need to calculate $\frac{\partial V_{Bi}^k}{\partial \lambda_{Bi}^j}$. Using (46), it directly

[24]The Hessian $\widetilde{H}_i$ contains the elements $\left\{ \frac{\partial^2 W}{\partial \tau_{Bi}^l \partial \tau_{Bi}^j} \right\}_{l,j=1,2,...,N}$

follows that for all $j \leq k$,

$$\frac{\partial V_{Bi}^k}{\partial \lambda_{Bi}^j} = -V_{Bi}^k. \tag{60}$$

Given this, the matrix $\widetilde{H}_i$ has the following form. Let $j \leq k$ without loss of generality. Then by (56) and (60),

$$\frac{\partial^2 W}{\partial \tau_{Bi}^k \partial \tau_{Bi}^j} = \pi_S^k \frac{\lambda_{Bi}^j}{\tau_{Bi}^j} \frac{\partial V_{Bi}^k}{\partial \lambda_{Bi}^j} = -\pi_S^k \frac{\lambda_{Bi}^j}{\tau_{Bi}^j} V_{Bi}^k.$$

Let

$$\alpha_m = -\pi_S^m V_{Bi}^m < 0, \quad \text{and} \quad \delta_m = \frac{\lambda_{Bi}^m}{\tau_{Bi}^m} > 0.$$

Then for all $j \leq k$, we have $\frac{\partial^2 W}{\partial \tau_{Bi}^k \partial \tau_{Bi}^j} = \alpha_k \delta_j$.

Given this structure, matrix $\widetilde{H}_i$ has the following form:

$$\begin{bmatrix} \alpha_1\delta_1 & \alpha_2\delta_1 & \alpha_3\delta_1 & \alpha_4\delta_1 & . & . & \alpha_N\delta_1 \\ \alpha_2\delta_1 & \alpha_2\delta_2 & \alpha_3\delta_2 & \alpha_4\delta_2 & . & . & \alpha_N\delta_2 \\ \alpha_3\delta_1 & \alpha_3\delta_2 & \alpha_3\delta_3 & \alpha_4\delta_3 & . & . & \alpha_N\delta_3 \\ \\ \\ \\ \alpha_N\delta_1 & . & . & . & . & . & \alpha_N\delta_N \end{bmatrix}$$

Notice, that any two adjacent rows are "almost" collinear except for the proportion changing from α_{k+1}/α_k to δ_{k+1}/δ_k at the main diagonal. Using this observation, we perform a simple transformation that does not change the determinant of the matrix: we deduct δ_k/δ_{k+1} times row $k+1$ from row k. This way it is easy to see that the transformed matrix is (lower) triangular. The determinant of a triangular matrix is just the product of its entries on the main diagonal. Therefore, for our purposes it is sufficient to show that the main diagonal has all strictly negative entries. Let ω_k denote the kth entry on the main diagonal of the transformed matrix. By construction,

$$\omega_k = \alpha_k \delta_k - \frac{\delta_k}{\delta_{k+1}} \alpha_{k+1} \delta_k.$$

Since $\delta_k > 0$, it follows that

$$\omega_k < 0 \Longleftrightarrow \alpha_k < \frac{\delta_k}{\delta_{k+1}} \alpha_{k+1},$$

and since $\alpha_{k+1} < 0$, this becomes

$$\omega_k < 0 \Longleftrightarrow \frac{\alpha_k}{\alpha_{k+1}} > \frac{\delta_k}{\delta_{k+1}}. \tag{61}$$

Upon using the definitions of α, δ and (32), (61) is equivalent to

$$\frac{V_{Bi}^{k}}{V_{Bi}^{k+1}} > 1, \tag{62}$$

which holds since $c_k < c_{k+1}$, and a buyer of type i runs a second price auction with a reservation price. Q.E.D.

To complete the proof of Lemma 9, we need to establish strict concavity of the welfare function for the continuous type game as well. By taking the appropriate limit, and using the strict concavity of the welfare function in the discrete games along the sequence, it follows that the welfare function of the continuous type game is concave. To show *strict* concavity of the welfare function in the continuous type game, one only needs to show that a continuous type space version of (62) still holds with a strict inequality. In particular, the argument for strict concavity of the welfare function in the original continuous type game still goes through if function V is strictly decreasing in c in the continuous type game.[25] But strict monotonicity of the value function V holds in the original continuous type game by construction, which completes our proof. Q.E.D.

Appendix 4: The Proofs of Proposition 4 and Proposition 6, parts (i)-(iii).

Proof of Proposition 4. To prove the Proposition, we will show that $\widetilde{v} = \sup\{v : U_S(v) = 0\} = 0$, so any buyer type $v > 0$ earns a positive profit in equilibrium, and will not stay out. The proof is by contradiction. So, suppose that $\widetilde{v} > 0$.

Step 0. *There exists a cutoff* $v_1 > 0$ *such that* $U_B(v) = 0$ *for all* $v \leq v_1$, and $U_B(v) > 0$ *for all* $v > v_1$,

[25] If V is strictly decreasing in c, then any discrete approximation would have a value function with strictly negative slope at $c < 1$ when the approximation is close to the continuous limit because V is convex by construction (consequently $V'(c) < 0$ for all $c < 1$). But then the relevant determinants for the discrete approximations would all be bounded away from zero when close to the continuous limit, and could not converge to zero in the limit, except perhaps at $c = 1$. Therefore, strict concavity of the welfare function must hold in the limit.

Let $\varepsilon > 0$ be small. Note that participating in an auction of a buyer type $v = \varepsilon$ is individually rational only for a seller with type $c \leq \varepsilon$.

Next, we claim that $\min\{V_S(0), V_B(0)\} > 0$ in equilibrium. To prove this claim we argue by contradiction. So suppose that $V_S(0) = V_B(0) = 0$. Then by monotonicity of the seller's payoff function, we must have $V_S(c) = V_B(c) = 0$ for all $c \in [0, 1]$ (since otherwise i.e., if $\min\{V_S(c), V_B(c)\} > 0$ for some $c \in (0, 1]$, a seller with cost 0 could obtain a positive payoff by imitating a seller with cost c, which would contradict $V_S(0) = V_B(0) = 0$). But then it follows that all buyers choose to be in one market and all sellers choose to be in the other market. However, this cannot be an equilibrium since a seller of any type in $[0, 1)$ can deviate profitably by choosing the same market as the one chosen by all buyers and either posting an auction (if all buyers choose S market) or visiting one of the buyer's auction (if all buyers choose B market).

Given that $\min\{V_S(0), V_B(0)\} > 0$, by continuity of the value function there exists $\varepsilon > 0$ s.t. $\min\{V_S(\varepsilon), V_B(\varepsilon)\} > \varepsilon$. Therefore, in equilibrium no seller type visits a buyer of type v, $v \leq \varepsilon$. So, we have $U_B(v) = 0$ for any $v \in [0, \varepsilon]$.

Next, let us show that $U_B(1) > 0$. First, note that by an argument similar to the one in the previous paragraph there exists $\check{c} \in [0, 1)$ s.t. $V_S(c) = 0$ for all $c \in [\check{c}, 1]$. Therefore, every seller with cost in $[\check{c}, 1]$ chooses to be in the market B.

The rest of the proof is by contradiction, so suppose that $U_B(1) = 0$. Then the probability with which a buyer posting the "reservation price" 1 is visited by at least two sellers is equal to 0. Given our independence assumptions on visiting probabilities, if the probability of at least two visits is zero, then the probability of a single visit is also zero. So each seller type visits an auction of a buyer with reservation price 1 with probability 0. But this cannot be an equilibrium because a seller of type $c \in [\check{c}, 1]$, who participates in B market as shown above, has a profitable deviation: by visiting such buyer, this deviating seller would attain the highest possible payoff $1 - c$ that strictly exceeds her equilibrium payoff from visiting buyers whose reservation price is below 1. A contradiction.

Finally, since $U_B(1) > 0$ and $U_B(\varepsilon) = 0$ for some $\varepsilon > 0$, by continuity and monotonicity of the payoff function $U(.)$, there must therefore exists a unique $v_1 \in (0, 1)$ such that $U_B(v) > 0$ if and only if $v > v_1$.

Step 1: *For (almost) all* $c, c < \widetilde{v}$, $\beta_s(c) = 0$. *For any* $\varepsilon > 0$, $\int_{\widetilde{v}}^{\widetilde{v}+\varepsilon} \beta_s(c) F_s(c) dc > 0$.

The first statement holds because otherwise $U_S(\widetilde{v}) > 0$ would hold because a buyer of type $\widetilde{v}$ could visit a seller type $c < \widetilde{v}$ with a finite queue length, and win with a positive probability, contradicting that $\widetilde{v} = \arg\max\{v : U_S(v) = 0\}$. The second statement holds because otherwise $U_S(\widetilde{v}+\varepsilon) = 0$, which contradicts the definition of $\widetilde{v}$. These two statements imply that seller type $c = \widetilde{v}$ is indifferent between posting and visiting, and thus equilibrium conditions imply that $V_S(\widetilde{v}) = V_B(\widetilde{v})$.

Case 1: $\widetilde{v} \leq v_1$. This case is handled in Steps 2 and 3.

Step 2: *For all* $c \leq \widetilde{v}$, $V_S'(c) = -1$.

Using the envelope theorem, the slope of the utility function is equal to the negative of the probability of trading after posting an auction in market S. The assumption that for all $v \leq \widetilde{v}$, $U_B(v) = U_S(v) = 0$ implies that any seller who posts an auction with a reservation price below $\widetilde{v}$ is visited by all the buyers with valuations not exceeding $\widetilde{v}$. Therefore, such seller will trade with probability 1. So in this large economy a seller setting a reservation price r such that $r \leq \widetilde{v}$ sells for sure.

Step 3: $V_S(0) > V_B(0)$.

By Step 1

$$V_S(\widetilde{v}) = V_B(\widetilde{v}), \tag{63}$$

and by Step 2,

$$V_S(0) - V_S(\widetilde{v}) = \widetilde{v}. \tag{64}$$

Using the envelope theorem, $V_B(0) - V_B(\widetilde{v}) = \int_0^{\widetilde{v}} \Pr_{B,c}(c)dc$ where $\Pr_{B,c}(c)$ is the probability that a seller with type c wins by participating in market B, that is by visiting rather than posting. This probability is well defined because Proposition 2 implies that type c wins with the same probability in every auction that she visits. Note, that for all $c > 0$, $\Pr_{B,c}(c) < 1$ because otherwise monotonicity of the trading probabilities across different seller types would be violated. Therefore,

$$\widetilde{v} > V_B(0) - V_B(\widetilde{v}). \tag{65}$$

Combining (63)-(65) establishes Step 3. However, Steps 1 and 3 contradict each other, because in equilibrium $\beta_s(0) = 0$ implies that $V_S(0) \leq V_B(0)$, which completes Case 1.

Case 2: $\widetilde{v} > v_1$. This case is handled in Steps 4 and 5.

Step 4: $V_S(0) > v_1$.

A buyer of type v s.t. $v \leq v_1$ participates in market S with probability 1 since for this type $U_B(v) = U_S(v) = 0$. Therefore, any seller who deviates and offers an auction with a reservation price less than v_1 would sell for sure at a price that is at least v_1, because such seller will be visited by all the buyers with valuations not exceeding $\widetilde{v}$.

Since this strategy is feasible for a seller of type 0, it follows that $V_S(0) > v_1$. This inequality is strict because type v_1 wins with a probability strictly less than 1.[26]

Step 5: *There exists* v', $v' > v_1$, *such that* $U_B(v) = 0$ *for all* $v \leq v'$.

By Step 4, seller type $c = 0$ does not visit any buyer's auction with reservation price below $V_S(0)$. Recall that the buyers set reservation prices equal to their valuations. So, Proposition 2 implies that no seller type visits any buyer type with valuation less than $V_S(0)$. Therefore, $v' \geq V_S(0)$. Since $V_S(0) > v_1$, the result follows. The result of Step 5 contradicts the definition of v_1, which completes the proof of the Proposition.

Proof of Proposition 6, (i)-(iii) : First, we show that (ii) holds. To see this, note that by Proposition 4 all types $v > 0$ either visit or post but none stays out. Next, we argue that inefficient types (v close to zero or c close to one) cannot profitably post and therefore they must visit with probability 1. Note that inefficient types cannot attract any visitors if they post mechanisms with reservation prices equal to their types. To see this, consider a buyer of type $v > 0$ close to zero. If he posts an auction, then only a type c such that $c < v$ may wish to visit him. Moreover, such a type c would earn a profit less than v by visiting this buyer. However, as v is arbitrary small, an efficient seller with c close to zero would make a profit close to zero, which cannot occur in equilibrium. Therefore, in equilibrium inefficient types visit with probability 1. Next, note that Proposition 2 and (ii) imply (i). In particular, since by (ii) any type $v < \underline{v}$ does visiting with a positive probability, the Lemma implies that any buyer type greater than v also does visiting with a positive probability. Taking v to zero then implies (i). Finally, to prove (iii) note that if, for example, such $\overline{v} < 1$ did not exist, then $V_B(c) = 0$ for some $c < 1$, which contradicts Proposition 4.

[26]Otherwise, if $V_S(0) = v_1$, then the transaction price at that auction would be close to v_1 with probability 1, and a bid of $v_1 + \varepsilon$ would win for sure. But then the monotonicity of the winning probability in buyer valuations would be violated.

Table 1: Equilibrium Strategy Profile $\underline{\beta}$, $\overline{\beta}$, $\underline{\rho}$, $\overline{\rho}$ in Bilateral Posting Market

	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.15$
$\pi = 0.05$	0,0.489,1,1	0,0.488,1,1	0,0.487,1,1
$\pi = 0.1$	0,0.515,1,1	0,0.512,1,1	0,0.51,1,1
$\pi = 0.2$	0,0.576,1,1	0,0.569,1,1	0,0.563,1,1
$\pi = 0.3$	0,0.654,1,1	0,0.642,1,1	0,0.631,1,1
$\pi = 0.4$	0.066,0.716,0.946,0.942	0.012,0.732,1,0.989	0,0.72,1,1
$\pi = 0.5$	0.258,0.659,0.732,0.718	0.222,0.676,0.778,0.753	0.177,0.698,0.831,0.797
$\pi = 0.6$	0.388,0.571,0.512,0.495	0.365,0.591,0.554,0.519	0.334,0.614,0.603,0.549
$\pi = 0.7$	0.48,0.425,0.289,0.275	0.466,0.447,0.321,0.291	0.449,0.474,0.361,0.312
$\pi = 0.8$	0.549,0.135,0.062,0.058	0.542,0.162,0.079,0.07	0.533,0.196,0.103,0.084
$\pi = 0.9$	0.518,0,0,0	0.518,0,0,0	0.519,0,0,0
$\pi = 0.95$	0.49,0,0,0	0.491,0,0,0	0.491,0,0,0

	$\alpha = 0.2$	$\alpha = 0.25$	$\alpha = 0.3$
$\pi = 0.05$	0,0.485,1,1	0,0.484,1,1	0,0.483,1,1
$\pi = 0.1$	0,0.507,1,1	0,0.504,1,1	0,0.502,1,1
$\pi = 0.2$	0,0.557,1,1	0,0.551,1,1	0,0.544,1,1
$\pi = 0.3$	0,0.62,1,1	0,0.609,1,1	0,0.598,1,1
$\pi = 0.4$	0,0.702,1,1	0,0.684,1,1	0,0.666,1,1
$\pi = 0.5$	0.121,0.723,0.891,0.857	0.047,0.753,0.96,0.941	0,0.758,1,1
$\pi = 0.6$	0.299,0.642,0.661,0.589	0.25,0.676,0.73,0.643	0.183,0.719,0.815,0.724
$\pi = 0.7$	0.426,0.507,0.41,0.338	0.395,0.547,0.471,0.373	0.352,0.599,0.55,0.422
$\pi = 0.8$	0.521,0.237,0.134,0.102	0.504,0.29,0.178,0.126	0.479,0.358,0.239,0.158
$\pi = 0.9$	0.52,0,0,0	0.521,0,0,0	0.523,0,0,0
$\pi = 0.95$	0.491,0,0,0	0.491,0,0,0	0.492,0,0,0

	$\alpha = 0.35$	$\alpha = 0.4$	$\alpha = 0.45$
$\pi = 0.05$	0,0.482,1,1	0,0.48048,1,1	0,0.47925,1,1
$\pi = 0.1$	0,0.499,1,1	0,0.49625,1,1	0,0.49364,1,1
$\pi = 0.2$	0,0.538,1,1	0,0.53231,1,1	0,0.52634,1,1
$\pi = 0.3$	0,0.587,1,1	0,0.57621,1,1	0,0.566,1,1
$\pi = 0.4$	0,0.648,1,1	0,0.63127,1,1	0,0.615,1,1
$\pi = 0.5$	0,0.73,1,1	0,0.70315,1,1	0,0.677,1,1
$\pi = 0.6$	0.086,0.775,0.92,0.857	0,0.80221,1,1	0,0.761,1,1
$\pi = 0.7$	0.289,0.666,0.653,0.497	0.188,0.757,0.794,0.634	0,0.877,1,1
$\pi = 0.8$	0.441,0.449,0.327,0.203	0.377,0.576,0.46,0.276	0.253,0.759,0.676,0.428
$\pi = 0.9$	0.526,0,0,0	0.524,0.037,0.017,0	0.452,0.393,0.212,0.088
$\pi = 0.95$	0.493,0,0,0	0.494,0,0,0	0.498,0,0,0

Table 2: Welfare Difference (in percents) Between Bilateral And One-Sided Posting

	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.15$	$\alpha = 0.2$	$\alpha = 0.25$	$\alpha = 0.3$	$\alpha = 0.35$	$\alpha = 0.4$	$\alpha = 0.45$
$\pi = 0.05$	0.6978	0.8253	0.9455	1.0574	1.1594	1.2498	1.3265	1.3869	1.4278
$\pi = 0.1$	0.8119	1.052	1.2812	1.4972	1.6969	1.8758	2.0313	2.1559	2.2434
$\pi = 0.2$	0.9692	1.3836	1.7971	2.1942	2.5704	2.9194	3.2307	3.4936	3.6909
$\pi = 0.3$	1.0067	1.5	2.0335	2.588	3.1293	3.6256	4.0854	4.49	4.8143
$\pi = 0.4$	0.8937	1.3215	1.8658	2.5071	3.2144	3.9237	4.5265	5.0587	5.5123
$\pi = 0.5$	0.7618	1.0234	1.3747	1.8541	2.5221	3.4424	4.4245	5.1615	5.7056
$\pi = 0.6$	0.68	0.8285	1.0253	1.2944	1.6765	2.2436	3.1301	4.5205	5.4319
$\pi = 0.7$	0.6551	0.755	0.8687	1.0046	1.18	1.4322	1.8451	2.6238	4.3168
$\pi = 0.8$	0.6953	0.8249	0.9491	1.0622	1.1566	1.2267	1.2824	1.407	2.0136
$\pi = 0.9$	0.7363	0.9249	1.1301	1.3492	1.5727	1.7739	1.8802	1.6808	1.051
$\pi = 0.95$	0.6823	0.8168	0.9712	1.1492	1.3544	1.5879	1.8381	2.0367	1.7915

Table 3: Posting and visiting strategies (probabilities) β_1, β_2, ρ_1, ρ_2, with $\pi = 0.4$ and posting costs $\gamma > 0$.

	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.15$
$\gamma = 0.001$	0.0629,0.7176,0.9482,0.9448	0.009,0.733,0.993,0.9919	0.0003,0.7196,0.9998,0.9997
$\gamma = 0.002$	0.0608,0.7183,0.9499,0.9465	0.0073,0.7335,0.9943,0.9934	0.0003,0.7191,0.9998,0.9997
$\gamma = 0.003$	0.0588,0.7191,0.9515,0.9483	0.0058,0.7339,0.9954,0.9947	0.0003,0.7186,0.9998,0.9997
$\gamma = 0.004$	0.0568,0.7199,0.9532,0.95	0.0046,0.7341,0.9964,0.9958	0.0003,0.7181,0.9998,0.9998
$\gamma = 0.005$	0.0547,0.7206,0.9548,0.9518	0.0037,0.7341,0.9971,0.9966	0.0002,0.7176,0.9998,0.9998
$\gamma = 0.006$	0.0527,0.7214,0.9565,0.9535	0.003,0.734,0.9977,0.9973	0.0002,0.7171,0.9998,0.9998
$\gamma = 0.007$	0.0507,0.7221,0.9581,0.9553	0.0024,0.7338,0.9981,0.9978	0.0002,0.7166,0.9998,0.9998
$\gamma = 0.008$	0.0486,0.7229,0.9597,0.957	0.002,0.7335,0.9984,0.9981	0.0002,0.7161,0.9998,0.9998
$\gamma = 0.009$	0.0466,0.7236,0.9614,0.9588	0.0018,0.7332,0.9986,0.9984	0.0002,0.7156,0.9998,0.9998
$\gamma = 0.01$	0.0446,0.7244,0.963,0.9605	0.0015,0.7328,0.9988,0.9986	0.0002,0.7151,0.9998,0.9998

	$\alpha = 0.2$	$\alpha = 0.25$	$\alpha = 0.3$
$\gamma = 0.001$	0.0001,0.7013,0.9999,0.9999	0.0001,0.6832,1,0.9999	0,0.6654,1,1
$\gamma = 0.002$	0.0001,0.7008,0.9999,0.9999	0.0001,0.6827,1,0.9999	0,0.6649,1,1
$\gamma = 0.003$	0.0001,0.7003,0.9999,0.9999	0.0001,0.6823,1,0.9999	0,0.6645,1,1
$\gamma = 0.004$	0.0001,0.6998,0.9999,0.9999	0.0001,0.6818,1,0.9999	0,0.664,1,1
$\gamma = 0.005$	0.0001,0.6993,0.9999,0.9999	0.0001,0.6813,1,0.9999	0,0.6635,1,1
$\gamma = 0.006$	0.0001,0.6988,0.9999,0.9999	0.0001,0.6808,1,0.9999	0,0.663,1,1
$\gamma = 0.007$	0.0001,0.6983,0.9999,0.9999	0.0001,0.6803,1,0.9999	0,0.6625,1,1
$\gamma = 0.008$	0.0001,0.6979,0.9999,0.9999	0.0001,0.6798,1,0.9999	0,0.6621,1,1
$\gamma = 0.009$	0.0001,0.6974,0.9999,0.9999	0.0003,0.6792,0.9998,0.9997	0,0.6616,1,1
$\gamma = 0.01$	0.0001,0.6969,0.9999,0.9999	0.0001,0.6788,1,0.9999	0,0.6611,1,1

	$\alpha = 0.35$	$\alpha = 0.4$	$\alpha = 0.45$
$\gamma = 0.001$	0,0.6479,1,1	0,0.6308,1,1	0,0.6141,1,1
$\gamma = 0.002$	0,0.6475,1,1	0,0.6304,1,1	0,0.6137,1,1
$\gamma = 0.003$	0.0001,0.6469,0.9999,0.9999	0,0.6299,1,1	0,0.6132,1,1
$\gamma = 0.004$	0,0.6465,1,1	0,0.6294,1,1	0,0.6128,1,1
$\gamma = 0.005$	0,0.6461,1,1	0,0.629,1,1	0,0.6123,1,1
$\gamma = 0.006$	0,0.6456,1,1	0,0.6285,1,1	0,0.6119,1,1
$\gamma = 0.007$	0,0.6451,1,1	0,0.6281,1,1	0,0.6114,1,1
$\gamma = 0.008$	0,0.6446,1,1	0,0.6276,1,1	0,0.611,1,1
$\gamma = 0.009$	0,0.6442,1,1	0,0.6271,1,1	0,0.6105,1,1
$\gamma = 0.01$	0,0.6437,1,1	0,0.6267,1,1	0,0.6101,1,1

Online Appendix (Not For Publication)

6.1 Proof of result (iv) of Proposition 6

Recall that $\overline{c} < 1$ is defined by $\overline{c} = \sup_c \beta_s(c) > 0$. By Proposition 2, for any $c \in [0, \overline{c}]$ a seller with type c is visited by buyer types $v \in [\widehat{z}(c), 1]$ where $\widehat{z}$ is a function increasing in c. Moreover,

$$\widehat{z}(0) = 0, \widehat{z}(\overline{c}) = \widehat{v} < 1.$$

The fact that $\widehat{z}(\overline{c}) = \widehat{v} < 1$ means that all buyer types $v \in [\widehat{v}, 1]$ visit all posting seller types $c \in [0, \overline{c}]$.

By Proposition 2, there exists $c^* > 0$ such that $\forall c \in [0, c^*]$, $\beta_s(c) > 0$. Similarly, letting $\widetilde{c}(v)$ denote the highest cost seller type visiting a poster with type v,

$$\widetilde{c}(1) = 1, \widetilde{c}(\underline{v}) = \widehat{c} > 0.$$

Therefore, all types $c \in [0, \widehat{c}]$ visit all posting types $v \in [\underline{v}, 1]$. The last two displays hold because otherwise (that is, if $\widehat{v} = 1$ and $\widehat{c} = 0$ was true) the least efficient posting types ($\underline{v}$ and $\overline{c}$) would obtain zero queue lengths, and thus a zero equilibrium utility, which would contradict Proposition 2.

We can distinguish two cases depending on whether $\widehat{c} < \overline{c}$ or not. In the following computations we assume that $\widehat{c} < \overline{c}$ holds, which simplifies the notation slightly but does not change the argument given that we are interested in the equilibrium behavior of seller types that are close to 0. Take any $c \leq \overline{c}$, and let $\Pr_S(c)$ denote the probability that a type c seller sells when he is a poster. Then a type $\widehat{z}(c)$ buyer buys with probability $1 - \Pr_S(c)$ when he is a visitor because he buys if and only if no other buyer shows at seller c. If a buyer with type $\widehat{z}(c)$ visits a seller with type $x > c$, then he wins if and only a seller with type x cannot sell, which is with probability $1 - \Pr_S(x)$. The payment when this buyer wins is equal to the reservation price x. Thus incentive compatibility for type $\widehat{z}(c)$ requires

$$c = \arg\max_x (1 - Pr_S(x))(\widehat{z}(c) - x).$$

The corresponding first-order condition is

$$-Pr'_S(c)(\widehat{z}(c) - c) = 1 - Pr_S(c). \tag{66}$$

Let $\lambda_S(c)$ denote the queue length generated by type c when visiting, and let $\Pr_B(c)$ denote the probability that type c sells when he is visiting. By construction,

$$Pr_B(c) = e^{-\int_0^c \lambda_S(x)dx}.$$

Therefore,

$$Pr'_B(c) = -\lambda_S(c)Pr_B(c). \tag{67}$$

Since all types visit with a positive probability, and all types $c \leq \bar{c}$ post with a positive probability, we must have $\Pr_B(c) = \Pr_S(c)$, and thus $\Pr'_B(c) = \Pr'_S(c)$. Letting $\Pr(c) = \Pr_B(c) = \Pr_S(c)$, (66), and (67) then yield

$$\lambda_S(c) = \frac{1 - \Pr(c)}{\Pr(c)(\widehat{z}(c) - c)}. \tag{68}$$

From Proposition 2, we know that type c generates an equal queue length at each buyer he visits. Each type $c \leq \widehat{c}$ visits all posting types $v \in [\underline{v}, 1]$ with equal weight, so the queue length generated by a visiting type c is

$$\lambda_S(c) = \frac{(1 - \beta_s(c))\, f_c(c)}{\int_{\underline{v}}^1 \beta_b(v) f_v(v)dv} \tag{69}$$

where the numerator is the mass of type c sellers visiting, and the denominator is the mass of the buyers visited by type c. Let $\Lambda_S(c)$ denote the queue length at a seller with type c who posts an auction. This queue length is

$$\Lambda_S(c) = \int_{\widehat{z}(c)}^1 \lambda_B(v)dv$$

where $\lambda_B(v)$ is the queue length generated by a visiting buyer with type v. Then

$$\Lambda'_S(c) = -\widehat{z}'(c)\lambda_B(\widehat{z}(c)). \tag{70}$$

Moreover, the probability of selling is $\Pr_S(c) = 1 - e^{-\Lambda_S(c)}$, and thus

$$Pr'_S(c) = \Lambda'_S(c)\left(1 - Pr_S(c)\right).$$

Comparing the last display with (66) yields

$$\Lambda'_S(c) = -\frac{1}{\widehat{z}(c) - c}. \tag{71}$$

The fact that type $\widehat{z}(c)$ generates the same visiting queue length at any visited seller types on $[0, c]$ implies that

$$\lambda_B(\widehat{z}(c)) = \frac{(1-\beta_b(\widehat{z}(c)))\, f_v(\widehat{z}(c))}{\int_0^c \beta_s(x) f_c(x)dx}. \tag{72}$$

Then (70), (71) and (72) imply that

$$\widehat{z}'(c) = \frac{\int_0^c \beta_s(x) f_c(x)dx}{(1-\beta_b(\widehat{z}(c)))\, f_v(\widehat{z}(c))} \frac{1}{\widehat{z}(c) - c}. \tag{73}$$

For any c such that $\widehat{z}(c) < \underline{v}$, $\beta_b(\widehat{z}(c)) = 0$ by construction, and thus for all c small enough,

$$\widehat{z}'(c) = \frac{\int_0^c \beta_s(x) f_c(x)dx}{f_v(\widehat{z}(c))\,(\widehat{z}(c) - c)}. \tag{74}$$

Using the l'Hospital's rule,

$$\widehat{z}'(0) = \frac{\beta_s(0) f_c(0)}{f_v(0)\,(\widehat{z}'(0) - 1)}. \tag{75}$$

Let $\overline{\alpha} = \widehat{z}'(0) > 1$. Then the following steps complete the proof.

1. From (71),

$$\lim_{c\to 0} \frac{\Lambda_S(c)}{\log c} = \lim_{c\to 0} \frac{\Lambda'_S(c)}{1/c} = -\frac{1}{\overline{\alpha} - 1}. \tag{76}$$

2. Using $\Pr(c) = 1 - e^{-\Lambda_s(c)}$, we obtain from (76) that for some function $\kappa_1(c), \kappa_2(c)$ such that $\lim\limits_{c\to 0} \kappa_1(c) = \lim\limits_{c\to 0} \kappa_2(c) = 1$, for all $\overline{\alpha} \neq 2$,

$$\Pr{}'(0) = \lim_{c\to 0} \Pr{}'(c) = \lim_{c\to 0} \frac{-c\kappa_1(c)}{\overline{\alpha} - 1} e^{-\frac{\kappa_2(c) \log c}{\overline{\alpha} - 1}} = \frac{-1}{\overline{\alpha} - 1} \lim_{c\to 0} c^{1 - \frac{1}{\overline{\alpha} - 1}} = \tag{77}$$

$$= \frac{-1}{\overline{\alpha} - 1} \lim_{c\to 0} c^{\frac{2-\overline{\alpha}}{\overline{\alpha} - 1}},$$

and thus $\Pr'(0) = 0$ if $\overline{\alpha} < 2$, and $\Pr'(0) = -\infty$ if $\overline{\alpha} > 2$. When $\overline{\alpha} = 2$, the rate at which $1 - \kappa_2(c)$ converges to 0 determines the value of $\Pr'(0)$.

The following two cases are possible:

Case 1. $f_v(0)/f_c(0) > 1/2$

Suppose for the sake of contradiction that $\beta_s(0) < 1$. Then $\overline{\alpha} < 2$ from (75), and then (77) implies that $\Pr'(0) = 0$. Then (67) yields $\lambda_S(0) = 0$, and (69) implies that $\beta_s(0) = 1$, a contradiction.

Case 2. If $f_v(0)/f_c(0) < 1/2$, then we can show that

$$\beta_s(0) = 2f_v(0)/f_c(0) < 1.$$

Indeed, any other value of $\beta_s(0)$ would imply a contradiction:

a) If $\beta_s(0) > 2f_v(0)/f_c(0)$, then $\overline{\alpha} > 2$ by (75), and so by (77), $\Pr'(0) = -\infty$. Then (67) implies that $\lambda_S(0) = \infty$, which contradicts (69).

b) If $\beta_s(0) < 2f_v(0)/f_c(0)$, then (75) implies that $\overline{\alpha} < 2$. Hence, $\Pr'(0) = 0$ by (77), and (67) yields $\lambda_S(0) = 0$. But then (69) implies that $\beta_s(0) = 1$, a contradiction. *Q.E.D*

The Analysis of the Uniform Model -Proofs of Propositions 5 and 7

6.1.1 Proof of Proposition 5

Assume that $f_v(x) = f_c(x) = 1$ for all $x \in [0, 1]$, and we show the results of Proposition 5 where sellers post mechanisms and buyers visit mechanisms only. Let $U(v)$ denote the utility of type v in equilibrium, and denote $U(1) = u$. By Proposition 2, in equilibrium the sellers post second-price auctions with reservation prices equal to their costs. Note that $U(1)-U(v) = \int_v^1 Prob.(\text{buyer of type } z \text{ trades})dz \leq 1-v$. So, $U(v) \geq v-(1-u)$.

Hence, a seller can attract buyers only if her cost c satisfies $c \leq 1 - u$, because a buyer type v can earn at most $v - c$ by visiting an auction with a reservation price c, and so no buyer will visit an auction with reservation price above $1 - u$. Furthermore, by Proposition 2, seller type c is visited by an interval of types $[\widehat{z}(c), 1]$ where $\widehat{z}(c)$ satisfies: $\widehat{z}(1 - u) = 1$, $\widehat{z}(0) = 0$, and

$$\widehat{z}'(c) = \frac{c}{\widehat{z}(c) - c}. \tag{78}$$

We conjecture a linear solution to (78) and then confirm it. Then $\widehat{z}(c) = \alpha c$ since $\widehat{z}(0) = 0$. Plugging this into (78) yields $\alpha = \frac{1}{\alpha-1}$, from which we obtain: $\alpha = \frac{1+\sqrt{5}}{2} \approx 1.61$. Then $\widehat{z}(1 - u) = \alpha \times (1 - u) = 1$ implies that $\alpha = \frac{1}{1-u}$. Since we also have $\alpha = \frac{1}{\alpha-1}$, it follows that $u = 2 - \alpha = \frac{1}{1+\alpha} = \frac{3-\sqrt{5}}{2}$.

Now, let us compute the traders' payoffs in order to verify the solution and check that $U(1) = u$, and second, to show that posting sellers are relatively worse off than visiting buyers in this one-sided market.

To this end, recall that by Proposition 2, and In particular, its' result that the queue of buyers of type v at seller type c, $c < \widehat{z}^{-1}(v)$ is equal to $\lambda_c(v) = \frac{1}{\widehat{z}^{-1}(v)} = \frac{\alpha}{v}$. Also, the probability that a buyer with value v trades is the same at every seller that she visits and, thus, equal to the probability that a seller with cost $\widehat{z}^{-1}(v)$ has no other visitors, which is equal to $e^{-\int_v^1 \lambda_{\widehat{z}^{-1}(v)}(x)dx}$, and in the current case $\lambda_x(\widehat{z}^{-1}(v)) = \frac{\alpha}{x}$ for all $x \in [v, 1]$.

Therefore, since $U(0) = 0$, we obtain:

$$U(v) = \int_0^v e^{-\int_y^1 \lambda_{\widehat{z}^{-1}(y)}(x)dx} dy = \int_0^v e^{-\int_v^y \frac{\alpha}{x} dx} dy = \int_0^v y^\alpha dy = \frac{y^{\alpha+1}}{\alpha+1}. \tag{79}$$

Note that $U(1) = u$ since $u = \frac{1}{1+\alpha}$.

Similarly, using $V(1-u) = 0$ and the fact that probability that seller with cost c has no visitors is equal to $e^{-\int_{\widehat{z}(c)}^1 \lambda_c(x)dx}$, and hence she trades with probability $1 - e^{-\int_{\widehat{z}(c)}^1 \lambda_c(x)dx}$, her payoff is equal to:

$$\begin{aligned}
&V(y) = \int_y^{1-u} 1 - e^{-\int_{\widehat{z}(c)}^1 \lambda_c(x)dx} dc = (1-u-y) - \int_y^{1-u} e^{-\int_{\alpha c}^1 \frac{\alpha}{x} dx} dc = \\
&(1-u-y) - \int_y^{1-u} (\alpha c)^\alpha dc = (1-u-y) - \frac{\alpha^\alpha}{\alpha+1}\left((1-u)^{\alpha+1} - y^{\alpha+1}\right) = \\
&(1-u) - \frac{(1-u)}{\alpha+1} - y + \frac{\alpha^\alpha y^{\alpha+1}}{\alpha+1} = u - y + \frac{\alpha^\alpha y^{\alpha+1}}{\alpha+1},
\end{aligned} \tag{80}$$

where the last equality holds because $(1-u) - \frac{(1-u)}{\alpha+1} = (1-u)^2 = u$. Hence, $V(0) = U(1) = u$.

Next, by (79) and (80) we have:

$$\begin{aligned}
&\Delta(y) \equiv U(1-y) - V(y) = \frac{(1-y)^{\alpha+1}}{\alpha+1} - u + y - \frac{\alpha^\alpha y^{\alpha+1}}{\alpha+1}, \\
&\Delta'(y) = -(1-y)^\alpha + 1 - \alpha^\alpha y^\alpha, \\
&\Delta''(y) = \alpha(1-y)^{\alpha-1} - \alpha^{\alpha+1} y^{\alpha-1}.
\end{aligned}$$

By simple computation, $\Delta(0) = 0$, $\Delta(u) > 0$. Also, $\Delta'(0) = 0$. Further, $\Delta''(y) > 0$ for all $y \in [0, \frac{1}{1+\alpha^{\frac{\alpha}{\alpha-1}}}]$. and $\Delta''(y) < 0$ for all $y > \frac{1}{1+\alpha^{\frac{\alpha}{\alpha-1}}}$. Therefore, either $\Delta'(y) > 0$ for all $y \in (0, u)$, or there exists $\hat{y} \in (0, u)$ s.t. $\Delta'(y) > 0$ for all $y \in (0, \hat{y})$ and $\Delta'(y) < 0$ for all $y \in (\hat{y}, u)$. This, in combination with $\Delta(0) = 0$ and $\Delta(u) > 0$, implies that $\Delta(y) > 0$ for all $y \in (0, u]$. Thus, $V(y) < V(1-y)$ for all $y \in (0, 1]$ i.e., a seller of type $y \in (0, 1]$ gets a smaller payoff than her counterpart buyer type $1 - y$. *Q.E.D*

Setup and main results for the uniform case with two-sided posting - Proposition 7

Assume that $f_v(x) = f_c(x) = 1$ for all $x \in [0, 1]$. Let us show that there is a unique schedule $\beta_s(c)$ which satisfies the sufficient conditions for an equilibrium and this schedule is monotone. Thus the unique equilibrium is monotone. First, we construct an equilibrium where $\beta_s(c) > 0$ if and only if $c < \bar{c}$, and $\beta_s(c) > 0$ zero otherwise. Also, we will show that the necessary conditions for the equilibrium imply that $\beta_s'(c) < 0$ for all $c < \bar{c}$. We will also show that $\lim_{c \nearrow \bar{c}} \beta_s(c) > 0$, so there is a lower bound on the posting probabilities when they are positive.

Let us consider the differential equations characterizing an equilibrium in the proof of Proposition 7. Recall that $\widehat{z}(c)$ is the smallest buyer type that visits a seller with reservation price c, and $k(\bar{c}) = \int_0^{\bar{c}} \beta_S(c)dc$ is the mass of posting seller types, which by symmetry is also equal to the mass of posting buyer types.

Under the uniform type distribution the said differential equations can be rewritten as follows for all $c \leq \bar{c}$:

$$\widehat{z}'(c) = \frac{\int_0^c \beta_s(x)dx}{\widehat{z}(c) - c}, \tag{81}$$

$$Pr'(c) = -\frac{1 - Pr(c)}{\widehat{z}(c) - c}, \tag{82}$$

$$Pr'(c) = -\frac{Pr(c)}{k(\bar{c})}(1 - \beta_s(c)), \tag{83}$$

As we will show below, in equilibrium we must have $\widehat{z}(\bar{c}) < 1 - \bar{c}$.[27] So all buyer types who post with a positive probability (by symmetry, this set includes all buyers with values v s.t. $v \geq 1 - \bar{c}$) also visit all posting seller types with a positive probability. As established in the main body of the paper,

$$Pr_B(\bar{c}) = e^{-\int_0^{\bar{c}} \lambda_B(c)dc} \tag{84}$$

where $\lambda_B(c) = (1 - \beta_s(c))/k(\bar{c})$.

Since $k(\bar{c}) = \int_0^{\bar{c}} \beta_S(c)dc$, we have:

$$Pr_B(\bar{c}) = Pr(\bar{c}) = e^{-\int_0^{\bar{c}} \lambda_B(c)dc} = e^{-\int_0^{\bar{c}} \frac{(1-\beta_S(c))}{k(\bar{c})}dc} = e^{-\frac{\bar{c}-k(\bar{c})}{k(\bar{c})}}. \tag{85}$$

[27]The equilibrium that we construct possesses this property. Since our equilibrium is unique, this property must always hold.

It is also shown in the main body of the paper that $\Pr_S(\bar{c}) = 1 - e^{-\int_{\widehat{z}(\bar{c})}^{1} \lambda_S(v)dv}$ where $\lambda_S(v) = (1 - \beta_b(v))/k(\bar{c})$.[28] Since $\int_{\widehat{z}(\bar{c})}^{1} \beta_b(v)dv = \int_{1-\bar{c}}^{1} \beta_b(v)dv = k(\bar{c})$, it follows that $\Pr_S(\bar{c}) = 1 - e^{-\frac{1-\widehat{z}(c)-k(\bar{c})}{k(\bar{c})}}$. Then using (85), and $\Pr_S(\bar{c}) = \Pr_B(\bar{c}) = \Pr(\bar{c})$, we obtain

$$e^{-\frac{1-\widehat{z}(c)-k(\bar{c})}{k(\bar{c})}} + e^{-\frac{\bar{c}-k(\bar{c})}{k(\bar{c})}} = 1. \tag{86}$$

A solution of the system (81)-(83) with initial conditions (84), (85), (86), $\Pr(0) = 1$, and $\beta_s(0) = 1$ will be an equilibrium if the cutoff type $\bar{c}$ is indifferent between posting and visiting i.e.,

$$V_S(\bar{c}) = V_B(\bar{c}). \tag{87}$$

The system (81)-(86) implies that $\Pr_S(c) = \Pr_B(c)$ for all $c < \bar{c}$, and then the envelope theorem together with (87) implies $V_S(c) = V_B(c)$ for all $c < \bar{c}$.

Given the above, we have to establish the following:

1N. The system (81)-(86) possesses a solution with a monotone decreasing schedule $\beta_s(c)$.

2N. The indifference condition (87) is satisfied.

3N. $V_S(c) \leq V_B(c)$ for all $c > \bar{c}$.

If the properties (1N)-(3N) hold, then our solution constitutes a (unique) monotone equilibrium. So, the rest of this Appendix deals with properties (1N)-(3N). Note that properties (2N) and (3N) may fail for "irregular" distributions. In particular, we showed that the differential equations corresponding to differential equations (81) - (83) for general distributions imply that $\beta_s(0) = \min\{1, 2f_v(0)/f_c(0)\}$. Therefore, for distributions where $f_v(0)$ is close to zero (there are not a lot of buyers with valuations near zero), we will have $\beta_s(0)$ close to zero, and thus monotonicity would imply that $\beta_s(c)$ is very low for all values of c. This is obviously not welfare maximizing, and is thus not an equilibrium by Proposition 2 in the main text. In this case, the equilibrium is not monotone and, in fact, property (3N) would be violated if we follow the procedure here since posting would be more beneficial when other agents post with a very low probability (i.e., if $\beta_s(c)$ is low for all $c > 0$). The challenge to establish that the equilibrium is monotone is to prove property (3N). To tackle it we provide the following result:

[28]This is because a posting seller with cost $\bar{c}$ sells if he is visited by a buyer with type above $\widetilde{v}(\bar{c})$.

Lemma 11 *A solution to the system that consists of (81)-(87) with condition* $\Pr(0) = \beta_s(0) = 1$ *satisfies* $V_S(c) \leq V_B(c)$ *for all* $c > \bar{c}$.

Next, we show that if property (1N) holds for a range of different values of $\beta_s(\bar{c})$, then property (2N) holds when $\beta_s(\bar{c})$ is chosen appropriately:

Lemma 12 *Suppose that for all* $\overline{\beta_s} \in [0, 1)$ *there is a solution for an appropriate* $\bar{c} > 0$ *such that* $\beta_s(\bar{c}) = \overline{\beta_s} \in [0, 1)$ *and also (81) to (86) hold with* $\Pr(0) = \beta_s(0) = 1$. *Then there exists a value* β *such that if* $\overline{\beta_s} = \beta$, *then also (87) holds.*

At this point a few remarks are in order:

(i) First, given Lemmas 11 and 12, it is sufficient to prove the existence of an appropriate solution for all $\overline{\beta_s} \in [0, 1)$, which is the claim in point (1) for all $\overline{\beta_s} \in [0, 1)$.

(ii) As we show in the next Lemma, an appropriate solution exists for all $\overline{\beta_s} \in [0, 1)$. The reason is that as we change $\overline{\beta_s}$, the candidate for $\beta_s(\bar{c})$, the entire system changes and a different solution exists. In general, a solution can be run up to any $\bar{c}$ with a corresponding $\overline{\beta_s} \geq 0$ and the entire system can be solved. However, for a given value of $\bar{c}$ (or equivalently, $\overline{\beta_s}$) condition (87) does not typically hold. For example when $\bar{c}$ is small (and thus $\overline{\beta_s}$ is close to 1) the implied solution would have very few sellers posting mechanisms and thus supply/demand conditions imply that $V_S(\bar{c}) > V_B(\bar{c})$. When $\bar{c}$ is large (close to 0.5), the opposite holds. Then there is a solution for (87) holds. This is the logic of the proof of Lemma 12.

iii) The logic of our proof, and in particular Lemma 12, implies that a jump in the posting probabilities need to be allowed to satisfy (87). Moreover, the jump implies that the IC conditions of Lemma 11 hold in the vicinity of $\bar{c}$. Without such a jump the IC conditions may fail in that region.

The final result, given point i) after Lemma 12, is then as follows:

Lemma 13 *For all* $\overline{\beta_s} \in [0, 1)$ *and an appropriate* $\bar{c} > 0$, *there is a solution to the system (81)-(86) on* $[0, \bar{c}]$ *with initial conditions* $\Pr(0) = \beta_s(0) = 1$, *and* $\beta_s(\bar{c}) = \overline{\beta_s}$.

Given these results an existence of a monotone equilibrium follows. Since there is a unique equilibrium, so the monotone equilibrium is the unique equilibrium:

Proposition 9 *The unique directed search equilibrium is monotone when the values and costs are uniformly distributed on a common support.*

The proofs of these results are provided in the next sections.

Proof of Lemma 11

Let us take a candidate equilibrium where $\beta_s(c) > 0$ if and only if $c \le \overline{c}$, and β_s jumps to zero at $\overline{c}$. Let $\widehat{v} = \widehat{z}(\overline{c})$ denote the smallest buyer type that visits $\overline{c}$. By symmetry $\underline{v} = 1 - \overline{c}$ is the lowest buyer type that posts in equilibrium. In what follows, we construct an equilibrium where

$$\widehat{v} < \underline{v}. \tag{88}$$

Since the equilibrium is unique, this implies that there is no equilibrium where this does not hold. Let $\widehat{c} = 1 - \widehat{v} = 1 - \widehat{z}(\overline{c}) = \widetilde{c}(\underline{v})$ denote the seller with the largest cost who visits all posting buyer types in equilibrium. By (88) and the symmetry between buyers and sellers, we have:

$$\widehat{c} > \overline{c}.$$

Let c_1 denote the largest seller type who is visited with a positive probability when posting an auction. Since $U(v)$ is increasing and convex, we must have $c_1 = 1 - U(1) = 1 - V(0)$. Clearly,

$$c_1 > \overline{c}.$$

By the envelope theorem, $0 = V_S(c_1) = V(0) - \int_0^{c_1} Pr_S(c)dc > V(0) - c_1 = V(0) - (1 - V(0))$. Therefore,

$$V(0) < 1/2 < c_1.$$

Finally, let c^+ denote the largest seller type who is visited by all posting buyer types, $\widehat{z}(c^+) = \underline{v}$.

Thus, we have defined four cost cutoff levels $\overline{c}$, $\widehat{c}$, c^+, and c_1. By definition, $c^+ < c_1$.

Now, let us show that $c^+ < \widehat{c}$. Indeed, by definition $\widehat{z}(\overline{c}) + \widehat{c} = 1$ and $\widehat{z}(c^+) + \overline{c} = \underline{v} + \overline{c} = 1$. So,

$$\widehat{z}(\overline{c}) + \widehat{c} = \widehat{z}(c^+) + \overline{c}.$$

The latter equation together with $\widehat{z}' > 1$ implies that $c^+ - \overline{c} < \widehat{c} - \overline{c}$, or $c^+ < \widehat{c}$.

We also have $\overline{c} < c^+$ because the equilibrium which we construct has the property that $\widehat{z}(\overline{c}) < 1 - \overline{c} = \underline{v}$.

Collecting the above we obtain:

$$\overline{c} < c^+ < \min\{\widehat{c}, c_1\} \le \max\{\widehat{c}, c_1\} < 1.$$

As we argued before, for all $c \in [\overline{c}, c^+]$,

$$\widehat{z}'(c) = \frac{k(\overline{c})}{\widehat{z}(c) - c},$$

and for all $c \in (c^+, c_1]$,

$$\widehat{z}'(c) = \frac{k(\overline{c})}{(1 - \beta_b(\widehat{z}(c)))(\widehat{z}(c) - c)}.$$

Also, for all $c \in [\overline{c}, c_1]$ it holds that

$$Pr'_S = -\frac{1 - Pr_S}{\widehat{z}(c) - c},$$

and for all $c \in (\overline{c}, \widehat{c}]$

$$Pr'_B = -\frac{Pr_B}{k(\overline{c})}, \tag{89}$$

and for all $c \in [\widehat{c}, c_1]$,

$$Pr'_B = -\frac{Pr_B}{\int_{\widetilde{c}^{-1}(c)}^{1} \beta_b(v)dv}.$$

Using $V_B(\overline{c}) = V_S(\overline{c}) = V(\overline{c})$ and the envelope theorem yields that for $i = S, B$ and $c > \overline{c}$ we have:

$$V_i(c) = V(\overline{c}) - \int_{\overline{c}}^{c} Pr_i(x)dx, \tag{90}$$

Note that any type $c > \overline{c}$ posts with probability zero ($\beta_s(c) = 0$). However, we can calculate the off-equilibrium trading probabilities for these types if they post, which allows us to calculate $V_S(c)$ for all $c > \overline{c}$.

Given (90), $V_B(\overline{c}) = V_S(\overline{c}) = V(\overline{c})$, and $V_B(1) = V_S(1) = 0$, it is sufficient to show that $\Pr_S$ and $\Pr_B$ have the following simple single-crossing pattern on $[\overline{c}, c_1]$ for some appropriate $c^* \in [\overline{c}, c_1]$:

1. For all $c \in [\overline{c}, c^*]$, $\Pr_S(c) \geq \Pr_B(c)$.
2. For all $c \in [c^*, c_1]$, $\Pr_S(c) \leq \Pr_B(c)$.

Under these two conditions, by (90), $V_B(c) \geq V_S(c)$ holds for all $c \in [\overline{c}, c_1]$. To prove these two points, note that by construction $\Pr'_S(\overline{c}) = \Pr'_B(\overline{c})$ when we take the left-hand derivative. But because there is a jump that makes $\Pr_B$ steeper (more negatively sloped) at $c = \overline{c}$ (compare (83) with (89)) for the right-hand derivative it holds that $\Pr'_S(\overline{c}) > \Pr'_B(\overline{c})$.[29] Therefore, point 1. holds on an interval $[\overline{c}, c_2]$. If one can show that for all $c > c_2$, it holds that $\Pr_S \leq \Pr_B$, then the proof would be complete.

Let $\tilde{\Pr}(v)$ is the probability that a buyer with type v buys in equilibrium. By symmetry, $\tilde{\Pr}(v) = 1 - \Pr(1 - v)$. Note that $\Pr_S(c^+) = 1 - \tilde{\Pr}(\underline{v})$. To see this, note

[29] Because (82) holds for all $c < c_1$, there is no jump in $\Pr'_S$ at point $\overline{c}$ (or at any other point).

that type c^+ seller trades when posting if and only if she is visited by a buyer with type at least $\underline{v}$. At the same time, the buyer of type $\underline{v}$ trades when visiting c^+ when no other buyer with value above $\underline{v}$ visits this seller, which by definition, occurs with probability $\tilde{\Pr}(\underline{v})$. Hence $1 - \tilde{\Pr}(\underline{v})$ is the probability that type c^+ seller is visited by a buyer with type at least $\underline{v}$.

On the other hand, $\Pr_B(\widehat{c}) = 1 - \tilde{\Pr}(\underline{v})$. To see this, note that type $\widehat{c}$ seller trades when visiting type $\underline{v}$ if and only if there is no other seller visiting this buyer, which occurs with probability $1 - \tilde{\Pr}(\underline{v})$.

So, $\Pr_S(c^+) = \Pr_B(\widehat{c}) = 1 - \tilde{Pr}(\underline{v})$, and thus by monotonicity of $\Pr_i$ we have:

$$Pr_S(c) < Pr_B(c),\ \forall c \in [c^+, \widehat{c}].$$

If we show that $\Pr_S$ crosses only to go below $\Pr_B$ on interval $[\overline{c}, c^+]$, then we have that $Pr_S(c) \leq Pr_B(c)$ on $[c_2, \widehat{c}]$, and by construction $Pr_S(c) \geq Pr_B(c)$ on $[\overline{c}, c_2]$.

To summarize our results so far, we note that we need to establish two other claims to conclude the proof of the Lemma:

(a) $\Pr_S$ crosses only to go below $\Pr_B$ on interval $[\overline{c}, c^+]$, that is,

$$c \in [\overline{c}, c^+], Pr_S(c) = Pr_B(c) \Longrightarrow Pr'_S(c) < Pr'_B(c).$$

(b) For all $c \in [\widehat{c}, c_1]$, $Pr_S(c) < Pr_B(c)$.

We first prove Claim (a). As shown above,

$$\left(\frac{1 - Pr_S(c)}{\widehat{z}(c) - c}\right)' = \frac{(1 - \Pr_S(c))(2 - \widehat{z}'(c))}{\left(\widehat{z}(c) - c\right)^2} > 0.$$

Thus, since $\Pr'_B < 0$, it holds that

$$\left(\frac{1 - Pr_S}{(\widehat{z}(c) - c)\Pr_B}\right)' > 0. \tag{91}$$

As we discussed earlier, $\frac{1-Pr_S(\overline{c})}{\widehat{z}(\overline{c})-\overline{c}} < \frac{\Pr_B(\overline{c})}{k(\overline{c})}$. Now consider two possible cases:

1) $\frac{1-Pr_S(c^+)}{\widehat{z}(c^+)-c^+} \leq \frac{\Pr_B(c^+)}{k(\overline{c})}$

2) $\frac{1-Pr_S(c^+)}{\widehat{z}(c^+)-c^+} > \frac{\Pr_B(c^+)}{k(\overline{c})}$

In case 1), for all $c \in [\overline{c}, c^+]$, it holds that $\Pr'_S \geq \Pr'_B$ and no crossing can occur on the interval. We derive contradiction for this case. Note, that the visiting buyer type $\underline{v}$ wins at seller of type c^+ if and only if no competing buyer is present, that is, when a seller type c^+ does not sell as a poster. Therefore, $\Pr_S(c^+) = 1 - \Pr(\underline{v}) < 1/2$. We also know that $\widehat{z}'(c^+) = \frac{k(\overline{c})}{\widehat{z}(c^+)-c^+} > 1$, and thus $k(\overline{c}) > \widehat{z}(c^+) - c^+$. Also, if the crossing has not occurred, then it holds that $\Pr_S(c^+) \geq \Pr_B(c^+)$. Putting together, we have

$$\frac{1 - Pr_S(c^+)}{\widehat{z}(c^+) - c^+} > \frac{Pr_S(c^+)}{k(\overline{c})} \geq \frac{\Pr_B(c^+)}{k(\overline{c})}, \tag{92}$$

which contradicts the starting assumption of case 1).

Therefore, we can turn to case 2) where $\Pr'_S(c^+) < \Pr'_B(c^+)$, and a crossing might have occurred. However, two crossings of $\Pr_S$ and $\Pr_B$ cannot have occurred because after (and if) the first crossing occurred we stay in the region where $\frac{1-Pr_S(c)}{\widehat{z}(c)-c} > \frac{\Pr_B(c)}{k(\overline{c})}$, and thus $\Pr'_S(c) < \Pr'_B(c)$.[30] Therefore, a second crossing where $\Pr_S$ goes back above $\Pr_B$ is not possible in this region.

Now, we turn to point b). If $\widehat{c} \geq c_1$, then the proof is complete, so we assume that $\widehat{c} < c_1$. To show our result that $V_S(c) \leq V_B(c)$ for all relevant c (that is, for all $c > \overline{c}$ as needed; see page 1), note that our analysis above implies that it holds for all $c \leq \widehat{c}$. In general, $V_S - V_B$ attains its extrema at points where $\Pr_S = \Pr_B$. So, it is sufficient to show the following:

$$c > \widehat{c}, Pr_S(c) = Pr_B(c) \Rightarrow V_S(c) \leq V_B(c).$$

Let $r_B(c)$ and $r_S(c)$ denote the expected revenue conditional on selling. When $Pr_S(c) = Pr_B(c)$, it holds that $V_S(c) \leq V_B(c) \Longleftrightarrow r_S(c) \leq r_B(c)$, so it is sufficient to show that

$$c > \widehat{c}, Pr_S(c) = Pr_B(c) \Rightarrow r_S(c) \leq r_B(c).$$

By basic incentive compatibility conditions, functions r_B and r_S are increasing in c. Therefore, it is sufficient to show that $r_S(c_1) \leq r_B(\widehat{c})$. Now, $r_S(c_1) = c_1 = 1 - V(0)$ by construction. Also, $\widehat{c} = 1 - \widehat{z}(\overline{c})$, and note that $\widehat{z}(\overline{c})$ is the smallest buyer type that visits a seller with type $\overline{c}$. When visiting such a seller the buyer wins if and only if no other buyer is present and in this case his payment is $\overline{c}$. Therefore, using symmetry between buyers and sellers the revenue of a seller with type $\widehat{c}$ upon selling is $1 - \overline{c}$. Therefore, $r_S(c_1) \leq r_B(\widehat{c}) \Longleftrightarrow 1 - V(0) \leq 1 - \overline{c}$ or $V(0) \geq \overline{c}$.

[30]At the first crossing point c_2, $\Pr'_S(c_2) < \Pr'_B(c_2)$ or $\frac{1-Pr_S(c_2)}{(\widetilde{v}(c_2)-c_2)\Pr_B(c_2)} > \frac{1}{k(\overline{c})}$. By (91), for all $c \in (c_2, c^+)$, $\frac{1-Pr_S(c)}{(\widetilde{v}(c)-c)\Pr_B(c)} > \frac{1}{k(\overline{c})}$ and thus $\Pr'_S(c) < \Pr'_B(c)$.

In what follows, we assume $V(0) < \bar{c}$ to obtain a contradiction. By convexity of Pr[31] we obtain:

$$V(\bar{c}) = V(0) - \int_0^{\bar{c}} \Pr(c)dx < V(0) - \bar{c}\frac{1+\Pr(\bar{c})}{2}.$$

We are assuming $V(0) < \bar{c}$, so

$$V(\bar{c}) < \bar{c} - \bar{c}\frac{1+\Pr(\bar{c})}{2} = \bar{c}\frac{1-\Pr(\bar{c})}{2}.$$

By incentive compatibility,

$$V(\bar{c}) > \Pr(\bar{c})(\widehat{z}(\bar{c}) - \bar{c})$$

because if type $\bar{c}$ sells then his expected revenue is at least $\widehat{z}(\bar{c})$.[32] Therefore,

$$\frac{\widehat{z}(\bar{c}) - \bar{c}}{\bar{c}} < \frac{1-\Pr(\bar{c})}{2\Pr(\bar{c})}.$$

Then we obtain the following chain

$$1 - \beta_s(\bar{c}) = \frac{1-\Pr(\bar{c})}{\Pr(\bar{c})}\widehat{z}'(\bar{c}) > 2\widehat{z}'(\bar{c})\frac{\widehat{z}(\bar{c}) - \bar{c}}{\bar{c}} = \frac{2k(\bar{c})}{\bar{c}},$$

which implies

$$k(\bar{c})/\bar{c} < 1/2. \tag{93}$$

On the other hand, we know that $\Pr(\bar{c}) > 1/2$. This probability can be calculated by letting Γ be the queue length generated by all types less than $\bar{c}$, and noting that $\Pr(\bar{c}) = e^{-\Gamma}$. All these types visit all buyer types and generate then a queue length of $\Gamma = \frac{\bar{c}-k(\bar{c})}{k(\bar{c})}$ where the numerator is the mass of all such visiting types and the denominator is the mass of all posting buyer types (using symmetry between buyers and sellers). Therefore, $e^{-\frac{\bar{c}-k(\bar{c})}{k(\bar{c})}} > 1/2$ or

$$k(\bar{c}) > \bar{c}\frac{1}{1+\log 2} > \bar{c}/2. \tag{94}$$

However, (93) and (94) contradict each other, which completes the proof. *Q.E.D.*

[31] Formula (2) implies that Pr is convex if $\left(\frac{1-\Pr}{\widetilde{v}-c}\right)' > 0$. Taking this derivative, and using (2) to simplify imply that $\left(\frac{1-\Pr}{\widetilde{v}-c}\right)' = (1-\Pr)(2-\widetilde{v}')/(\widetilde{v}-c)^2 > 0$.

[32] Since $\widetilde{v}(c^+) = 1 - \bar{c}$, type $1 - \bar{c}$ buyer when optimally visiting must pay an expected amount of c^+ when winning, since he would be the lowest buyer type visiting type c^+. Therefore, by symmetry, type $\bar{c}$ seller obtains an expected revenue (conditional on selling) of $1 - c^+$. Then we just need to prove $1 - c^+ > \widetilde{v}(\bar{c})$. Note, that $\widetilde{v}' > 1$ implies $\widetilde{v}(c^+) - c^+ > \widetilde{v}(\bar{c}) - \bar{c}$ or $1 - \bar{c} - c^+ > \widetilde{v}(\bar{c}) - \bar{c}$, which implies that $1 - c^+ > \widetilde{v}(\bar{c})$.

Proof of Lemma 12

When $\overline{\beta}_s = 1$, and thus $\overline{c} = 0$, it holds that a zero measure of the buyers and sellers post in equilibrium, so posting is more profitable than visiting. Therefore, using continuity of the solutions in $\overline{\beta}_s$, it is sufficient for us to prove that when $\overline{\beta}_s = 0$, it holds that posting is less profitable than visiting. We show this below.

Let $R_B(\overline{c})$ and $R_S(\overline{c})$ denote the expected revenues in the two markets for a seller with type $\overline{c}$. Since $\Pr_S(\overline{c}) = \Pr_B(\overline{c})$, posting being less profitable than visiting is equivalent to

$$R_B(\overline{c}) > R_S(\overline{c}).$$

Note that for seller type $\overline{c}$ it is optimal to visit type $\underline{v} = 1 - \overline{c}$ in which case the seller's expected revenue is

$$R_B = (1 - \Pr(\overline{c}))(1 - \overline{c}) + (2\Pr(\overline{c}) - 1)Z_B \geq (1 - \Pr(\overline{c}))(1 - \overline{c}) + (2\Pr(\overline{c}) - 1)\overline{c}. \quad (95)$$

where Z_B is the expected revenue when a competing seller is present at a buyer with type $\underline{v}$. Note that there is no competing seller present with the complementary of the probability that a buyer of type $1 - \overline{c}$ trades, which is the same probability (by symmetry) that a seller of type $\overline{c}$ trades, i.e. $\Pr(\overline{c})$. Hence there is no competing seller with probability $1 - \Pr(\overline{c})$, and in this case the revenue is just the reservation price of the buyer which is $1 - \overline{c}$.

When seller-type $\overline{c}$ posts, the probability that exactly one buyer arrives is $-(1 - \Pr(\overline{c}))\log(1 - \Pr(\overline{c}))$ by the formula for the Poisson distribution. Letting Z_S denote the revenue conditional on at least two buyers arriving, we obtain

$$R_S = -(1 - \Pr(\overline{c}))\log(1 - \Pr(\overline{c})) * \overline{c} + (\Pr(\overline{c}) + (1 - \Pr(\overline{c}))\log(1 - \Pr(\overline{c})))Z_S. \quad (96)$$

By construction, if $\beta_s(\overline{c}) = 0$, it holds that $\frac{1-\Pr(\overline{c})}{\Pr(\overline{c})}\widehat{z}'(\overline{c}) = 1$. Since conditions (81) to (86) and $\beta_s(0) = \Pr(0) = 1$ together imply that $\widehat{z}'(0) = \frac{1+\sqrt{5}}{2}$ and $\widehat{z}'(c) \leq \widehat{z}'(0)$ for all $c > 0$[33] , it follows that $\Pr(\overline{c}) \in (0.5, \frac{\sqrt{5}-1}{2})$

To estimate Z_S, note that posting seller type $\overline{c}$ receives a bid above $1 - \overline{c}$ with probability $1 - \Pr(\overline{c})$. The chance of receiving at least two bids above $1 - \overline{c}$ is $1 - \Pr(\overline{c}) + \Pr(\overline{c})\log\Pr(\overline{c})$. When receiving just one bid above $1 - \overline{c}$ the revenue is less

[33]This is shown in the proof of Lemma 3.

than $1-\overline{c}$ by the rules of the second price auction. Therefore,

$$
\begin{aligned}
&(\Pr(\overline{c}) + (1-\Pr(\overline{c}))\log(1-\Pr(\overline{c})))Z_S \leq \\
&((\Pr(\overline{c}) + (1-\Pr(\overline{c}))\log(1-\Pr(\overline{c}))) - (1-\Pr(\overline{c}) + \Pr(\overline{c})\log\Pr(\overline{c})))\,(1-\overline{c}) \\
&+ (1-\Pr(\overline{c}) + \Pr(\overline{c})\log\Pr(\overline{c}))\,. \qquad (97)
\end{aligned}
$$

Combining (95)-(97), for $R_B > R_S$ it is sufficient to prove that for all $p \in (0.5, \frac{\sqrt{5}-1}{2})$,

$$\overline{c} < \frac{1-2p-(1-p)\log(1-p)}{3-5p-2(1-p)\log(1-p)+p\log p}. \qquad (98)$$

Note that by $\widehat{z}(\overline{c}) + \overline{c} \leq 1$ and concavity of $\widehat{z}$ we have:

$$\overline{c} < \frac{\overline{c}}{\widehat{z}(\overline{c}) + \overline{c}} < \frac{1}{1+\widehat{z}'(\overline{c})} = \frac{1}{1+\frac{p}{1-p}} = 1-p.$$

So, (98) holds if for all $p \in (0.5, \frac{\sqrt{5}-1}{2})$ that

$$1-p < \frac{1-2p-(1-p)\log(1-p)}{3-5p-2(1-p)\log(1-p)+p\log p}.$$

However, this fails by a small margin, so bounds need to be tightened further. When there are at least two bids above $1-\overline{c}$, which is with probability $\alpha(p) = 1-p+p\log p$, the revenue is not 1 as it was estimated above. Let the conditional expected revenue be denoted by $1-b$ with $b > 0$ the correcting term to be estimated. Then we modify the condition to

$$1-2p-(1-p)\log(1-p)+\alpha(p)b > \overline{c}\,(3-5p-2(1-p)\log(1-p)+p\log p)\,.$$

To estimate b we consider three events. If there are exactly two such visitors, which is with conditional probability $\frac{\frac{(\log p)^2}{2} * p}{\alpha(p)}$, then the revenue is the lower of two types above $1-\overline{c}$. Since higher types are less likely to visit (because of their higher posting probability), this expected value is less than what a uniform distribution would be, so less than $\frac{2(1-\overline{c})+1}{3}$, so the correction term is at least $1 - \frac{2(1-\overline{c})+1}{3} = 2\overline{c}/3$. Similarly, the correction term when there are exactly three visitors is at least $\overline{c}/2$; this happens with conditional probability $\frac{-\frac{(\log p)^3}{6} * p}{\alpha(p)}$. So, the correction term has a lower bound of

$\overline{c}[\frac{2}{3}\frac{(\log p)^2}{2}p - \frac{1}{2}\frac{(\log p)^3}{6}p] = \overline{c}p[\frac{(\log p)^2}{3} - \frac{(\log p)^3}{12}]$. Thus it is sufficient to have $1-2p-(1-p)\log(1-p) + \overline{c}p[\frac{(\log p)^2}{3} + \frac{(\log p)^3}{12}] > \overline{c}\,(3-5p-2(1-p)\log(1-p)+p\log p)$, or

$$\overline{c} < \frac{1-2p-(1-p)\log(1-p)}{3-5p-2(1-p)\log(1-p)+p\log p - p[\frac{(\log p)^2}{3} - \frac{(\log p)^3}{12}]}.$$

With this modification

$$1-p < \frac{1-2p-(1-p)\log(1-p)}{3-5p-2(1-p)\log(1-p)+p\log p - p[\frac{(\log p)^2}{3} - \frac{(\log p)^3}{12}]}.$$

Plotting this inequality for all $p \in (0.54, \frac{\sqrt{5}-1}{2})$ shows that it holds for all such values.

However, to make it work for p close to 0.5 we need to improve on the bound $\overline{c} < 1-p$. If we were able to show that $\overline{c} < 0.46$, then it would be sufficient because numerical calculations show that for all $p \in [0.5, 0.54]$, it holds that

$$0.46 < \frac{1-2p-(1-p)\log(1-p)}{3-5p-2(1-p)\log(1-p)+p\log p - p[\frac{(\log p)^2}{3} - \frac{(\log p)^3}{12}]}.$$

So, the rest of the proof for point ii) is to show that $\overline{c} < 0.46$ for all $p \in [0.5, 0.54]$.

For $\overline{c} < 0.46$, it is sufficient to have $\widehat{z}(\overline{c}) - \overline{c} > 0.08$ because $\widehat{z}(\overline{c}) + \overline{c} < 1$. Take the solution and let c_γ be such that $-\Pr'(c_\gamma) = \gamma \geq 1$. If $\widehat{z}(\overline{c}) - \overline{c} > 0.08$, then we are done. So suppose that $\widehat{z}(\overline{c}) - \overline{c} \leq 0.08$. Then $-\Pr'(\overline{c}) = \frac{\Pr(\overline{c})}{\widehat{z}(\overline{c})-\overline{c}} \geq \frac{0.5}{0.08} = 6.25$. So, our construction works for all relevant p values as long as $\gamma < 6.25$. Let $p_\gamma = 1 - 0.08\gamma$. If $\Pr(c_\gamma) < p_\gamma$, then $\widehat{z}(\overline{c}) - \overline{c} > \widehat{z}(c_\gamma) - c_\gamma = (1-\Pr(c_\gamma))/(-\Pr'(c_\gamma)) > \frac{0.08\gamma}{\gamma} = 0.08$, and we are done.

So, suppose that $\Pr(c_\gamma) > p_\gamma$, and thus

$$1 - \beta_s(c_\gamma) = \frac{1-\Pr(c_\gamma)}{\Pr(c_\gamma)}\frac{k(\overline{c})}{\widehat{z}(c_\gamma) - c_\gamma} = \frac{\gamma k(\overline{c})}{\Pr(c_\gamma)} < \frac{\gamma k(\overline{c})}{1-0.08\gamma}.$$

We know by $\widehat{z}'' \leq 0$ that $\beta_s(c_\gamma) \leq \widehat{z}'(c_\gamma)(\widehat{z}'(c_\gamma) - 1)$. Therefore, using the previous display as well,

$$k(\overline{c}) > \frac{1-0.08\gamma}{\gamma}(1-\beta_s(c_\gamma)) > \frac{1-0.08\gamma}{\gamma}(1 - \widehat{z}'(c_\gamma)(\widehat{z}'(c_\gamma)-1)).$$

If $k(\overline{c}) = (\widehat{z}(\overline{c}) - \overline{c})\frac{\Pr(\overline{c})}{1-\Pr(\overline{c})} > 0.08\frac{\Pr(\overline{c})}{1-\Pr(\overline{c})}$, then we are done. Therefore, if $\frac{1-0.08\gamma}{\gamma}(1 - \widehat{z}'(c_\gamma)(\widehat{z}'(c_\gamma)-1)) > 0.08\frac{\Pr(\overline{c})}{1-\Pr(\overline{c})}$, then we are done. So, suppose that $\frac{1-0.08\gamma}{\gamma}(1 - \widehat{z}'(c_\gamma)(\widehat{z}'(c_\gamma)-1)) \leq 0.08\frac{\Pr(\overline{c})}{1-\Pr(\overline{c})} < 0.0935$, or

$$\widehat{z}'(c_\gamma)(\widehat{z}'(c_\gamma)-1) > 1 - \frac{0.0935\gamma}{1-0.08\gamma} = \frac{1-0.1735\gamma}{1-0.08\gamma}.$$

This is equivalent to

$$\widehat{z}'(c_\gamma) > 0.5(1+\sqrt{1+4*\frac{1-0.1735\gamma}{1-0.08\gamma}}) = 0.5(1+\sqrt{\frac{5-0.774\gamma}{1-0.08\gamma}}).$$

Let $c^- = \frac{0.08}{0.5(\sqrt{\frac{5-0.774\gamma}{1-0.08\gamma}}-1)}$. If $c_\gamma > c^-$, then $\widehat{z}(c_\gamma) - c_\gamma > (\widehat{z}'(c_\gamma)-1)c_\gamma > (\widehat{z}'(c_\gamma)-1)c^- > 0.5(\sqrt{\frac{5-0.774\gamma}{1-0.08\gamma}}-1) * \frac{0.08}{0.5(\sqrt{\frac{5-0.774\gamma}{1-0.08\gamma}}-1)} = 0.08$, and then $\widehat{z}(\overline{c}) - \overline{c} > \widehat{z}(c_\gamma) - c_\gamma > 0.08$, and we are done.

So, suppose that $c_\gamma < c^-$. Then $0.5 \geq 1 - \Pr(\overline{c}) > (\overline{c} - c^-)\gamma > (0.46 - c^-)\gamma$ because for all $c > c^-$ it holds that $-\Pr'(c) > \gamma$ by Pr being concave. So, if we show that there exists $\gamma \in (1, 6.25)$ such that

$$(0.46 - c^-)\gamma - 0.5 > 0,$$

then we obtain a contradiction and the proof is complete. One can easily plot this function to show that this indeed holds for all $\gamma \in (1.6, 2)$.

Proof of Lemma 13

We need to prove that an appropriate solution of our system exists, as it is stated in Lemma 13. We solve the system (81) to (83), using the initial conditions (85), (86). So, we start the solution at $c = \overline{c}$, and work our way back to $c = 0$. Note, that the system satisfies Lipshitz-continuity at point $\overline{c}$, and at any point where $\widehat{z}(c) > c$. So, a unique solution exists on $[c^*, \overline{c}]$ as long as $\widehat{z}(c) > c$ for any c on this interval. For the end conditions, note that (82), and (83) imply

$$1 - \beta_s(\overline{c}) = \frac{1-\Pr(\overline{c})}{\Pr(\overline{c})} \frac{k(\overline{c})}{\widehat{z}(\overline{c}) - \overline{c}}. \tag{99}$$

Therefore, for a given value of $\overline{c}, \beta_s(\overline{c})$ conditions (85), (86), and (99) determine the values of $\Pr(\overline{c}), k(\overline{c})$, and $\widehat{z}(\overline{c})$.

To prove the Lemma, we need to find a solution of our system for any possible value of $\beta_s(\overline{c})$. Given the above discussion, the only free parameter to choose at the end point is $\overline{c}$. By choosing $\overline{c}$ appropriately we need to ensure that $\Pr(0) = \beta_s(0) = 1$. We also need to make sure that $\beta_s, \Pr$ remain decreasing at the interval $[0, \overline{c}]$.

In what follows, we take $\beta_s(\overline{c})$ and $\overline{c}$ as given, and solve that system for $c \in (c^*, \overline{c})$ until, at some c^*, either we hit a singularity or other conditions are satisfied:

(i) $\widehat{z}(c^*) - c^* = 0$ (we use the equivalent condition $\beta_s(c^*) = \widehat{z}'(c^*)(\widehat{z}'(c^*) - 1)$ for this)

(ii) $\beta_s'(c^*) = 0$

(iii) $\widehat{z}''(c^*) = 0$

(iv) $\widehat{z}'(c^*) = 2$

(v) $\Pr(c^*) = 1$.

These are related to the necessary conditions that follow directly from the necessary equilibrium conditions (81) to (83). In particular, for $c \in (c^*, \overline{c})$, condition (i) would hold as $\widehat{z}(c) - c > 0$, and the conditions (ii)-(v) would hold as strict inequalities, with the left-hand side smaller than the right-hand side.

In the next section, we derive two necessary conditions for $k(\overline{c})/\overline{c}$ for a given value of $\beta_s(\overline{c})$. The first necessary condition is $\widehat{z}''(c) < 0$ for all $c > 0$. It implies that $\overline{\beta}_s = \beta_s(\overline{c}) < \widehat{z}'(\overline{c})(\widehat{z}'(\overline{c}) - 1)$, which then boils down to $k(\overline{c})/\overline{c} > \tau(\overline{\beta}_s)$ for a function τ which is characterized in the next section, and where it is also shown that $\tau(0) = 0.59, \tau(0.5) = 0.76$, and $\tau(1) = 1$. The second necessary condition, which follows from the concavity of $\widehat{z}$, is that $\widehat{z}' \leq \widehat{z}/c$. This condition boils down to $k(\overline{c})/\overline{c} \leq \omega(\overline{\beta}_s)$. The function ω is characterized in the next section, and it is also shown that $\omega(0) = 0.65, \tau(0.5) = 0.78$, and $\tau(1) = 1$.

Remark: Since τ and ω are fairly close for all values of $\overline{\beta}_s$, the ratio $k(\overline{c})/\overline{c}$ can be estimated quite precisely if $\overline{\beta}_s = \beta_s(\overline{c})$ is known.

The condition that $k(\overline{c})/\overline{c} \in (\tau(\beta_s), \omega(\beta_s))$ implies that $\widehat{z}(\overline{c}) - \overline{c} > 0$, $\beta_s'(\overline{c}) < 0, \widehat{z}''(\overline{c}) < 0$ by the Corollary below.

We start by providing some useful auxiliary results, including the ones that establish the above inequalities at $\overline{c}$. First, by using straightforward calculus we obtain:

$$\Pr(c)(\widehat{z}'(c) - 1) < 1 \Longleftrightarrow \beta_s'(c) < 0,$$

$$\widehat{z}'(c) < 2 \Longleftrightarrow Pr''(c) < 0$$

$$\beta_s(c) < \widehat{z}'(c)(\widehat{z}'(c) - 1) \Longleftrightarrow \widehat{z}''(c) < 0.$$

$\beta_s(c) < \widehat{z}'(c)(\widehat{z}'(c) - 1)$ is equivalent to $k(\overline{c})/\overline{c} > \tau(\beta_s)$ by construction (see below), and thus $\widehat{z}''(\overline{c}) < 0$ holds when $k(\overline{c})/\overline{c} > \tau(\beta_s)$. (Upon using (82) and (83), this condition boils down to an upper bound on $\overline{c}$.)

The condition that $k(\overline{c})/\overline{c} < \omega(\beta_s)$ (which is equivalent to $\widehat{z}'(c) < \widehat{z}(\overline{c})/\overline{c}$, and by (82) and (83), boils down to a lower bound on $\overline{c}$) implies that $\widehat{z}'(\overline{c}) < 1.61$ and thus $\beta_s'(\overline{c}) < 0$, $Pr''(\overline{c}) < 0$ and $\widehat{z}''(\overline{c}) < 0$.

The condition that $k(\overline{c})/\overline{c} \in (\tau(\beta_s), \omega(\beta_s))$ also implies that (using straightforward calculations) that $\widehat{z}(\overline{c}) - \overline{c} > 0$, $\beta_s'(\overline{c}) < 0$, $\widehat{z}'(\overline{c}) < 2$, and $\Pr(\overline{c}) < 1$.

Corollary 2 *If $k(\overline{c})/\overline{c} \in (\tau(\beta_s), \omega(\beta_s))$, then there is c^*, $c^* < \overline{c}$, such that for all $c \in (c^*, \overline{c})$ it holds that $\widehat{z}''(c) < 0$, $\widehat{z}(\overline{c}) - \overline{c} > 0$, $\beta_s'(\overline{c}) < 0$, $\widehat{z}'(\overline{c}) < 2$, and $\Pr(\overline{c}) < 1$.*

The following result shows the significance of some of these conditions:

Lemma 14 *Suppose that $\beta_s(c) = \widehat{z}'(c)(\widehat{z}'(c) - 1)$, $\widehat{z}'(c) < 2$ and $\widehat{z}''(c) < 0$. Then $\beta_s(c) = 1$ and $\int_0^c \beta_s(x)dx = 0$.*

Proof. By construction, $\widehat{z}(c) - c = \frac{\beta_s(c) - \widehat{z}'(c)(\widehat{z}'(c)-1)}{\widehat{z}''(c)} = 0$. Then

$$\widehat{z}'(c)(\widehat{z}(c) - c) = \int_0^c \beta_s(x)dx = 0.$$

Then $\Pr'(x) = -\frac{1-\Pr(x)}{\widehat{z}(x)-x}$ for all x implies that $\Pr(c) = 1$, for otherwise $\Pr'(c) = -\infty$ but Pr is concave at c because $\widehat{z}'(c) < 2$, so $\Pr'(c) = -\infty$ is impossible because then $\Pr'$ could not increase as we decreased the cost type.

Then $\lim_{x\to c} \frac{1-\Pr(x)}{\widehat{z}(x)-x} = \lim_{x\to c} \frac{-\Pr'(x)}{\widehat{z}'(x)-1}$ and thus $\lim_{x\to c}(1 - \beta_s(x)) = \lim_{x\to c} \frac{-\Pr'(x)}{\widehat{z}'(x)-1}$. It is then sufficient to prove $\Pr'(c) = 0$ because by concavity of $\widehat{z}(c)$ we have $\widehat{z}'(c) > \widehat{z}'(\overline{c}) > 1$ and thus the denominator $\widehat{z}'(x) - 1$ is bounded away from zero. Applying l'Hospital's rule to $\Pr'(x) = -\frac{1-\Pr(x)}{\widetilde{v}(x)-x}$ and using $\widetilde{v}'(c) \in (1, 2)$ we obtain that $\Pr'(c) = 0$ must indeed hold, and thus $\beta_s(c) = 1$. Q.E.D.

If we inspect the critical point c^* above, there is one favorable outcome where $\widehat{z}(c^*) - c^* = 0$. Then, if $c^* \neq 0$, the solution would be a full fledged solution of the system (81) to (86) with all the right conditions at c^* including $\beta_s(c^*) = \Pr(c^*) = 1$. Among the other four cases, $\beta_s'(c^*) = 0$ would imply $\Pr(c^*)(\widehat{z}'(c^*) - 1) = 1$, which means that either $\Pr(c) = 1$ or $\widehat{z}'(c) = 2$ must have occurred for some $c \in (c^*, \overline{c})$. Therefore, we can concentrate on

iii) $\widehat{z}''(c^*) = 0$,

iv) $\widehat{z}'(c^*) = 2$, and

v) $\Pr(c^*) = 1$.

In what follows, we argue that if both $\beta_s < \widehat{z}'(\widehat{z}' - 1)$ and $\widehat{z}' < \widehat{z}/c$ holds on an interval $(c_i, \overline{c}]$, then on that interval $\widehat{z}''(c) < 0$, $\widehat{z}'(c) < 2$, and $\Pr(c) < 1$. The inequality $\widehat{z}''(c) < 0$ follows because $\beta_s = \widehat{z}'(\widehat{z}' - 1) + \widehat{z}''(\widehat{z} - c)$, and $\widehat{z} - c > 0$. Also, since $\widehat{z}'(\overline{c}) \leq (1+\sqrt{5})/2 = 1.61$ by virtue of $k(\overline{c})/\overline{c} < \omega(\overline{\beta}_s)$, it follows that $\widehat{z}'(c) < 1.61$ for all c. Finally, since $\Pr' = -(1-\Pr)/(\widehat{z}-c)$, Pr stays below 1 as long as $\widehat{z}-c > 0$.[34]

Therefore, if both $\beta_s \leq \widehat{z}'(\widehat{z}' - 1)$ and $\widehat{z}' \leq \widehat{z}/c$ holds on an interval $[c^*, \overline{c}]$, then all the other necessary conditions numbered i) to v) hold as well, and thus we can work with the two conditions $\beta_s < \widehat{z}'(\widehat{z}' - 1)$ and $\widehat{z}' < \widehat{z}/c$**,** and let the critical point c^* be defined as the point where one of the two conditions (or both) are violated first, that is, where at least one of $\beta_s(c^*) = \widehat{z}'(c^*)(\widehat{z}'(c^*) - 1)$, $\widehat{z}'(c^*) = \widehat{z}(c^*)/c^*$ hold but for all $c \in (c^*, \overline{c}]$, $\beta_s < \widehat{z}'(\widehat{z}' - 1)$ and $\widehat{z}' < \widehat{z}/c$.

To operationalize the solution, let c^* be such that $\beta_s(c^*) = \widehat{z}'(c^*)(\widehat{z}'(c^*) - 1)$ or $\widehat{z}'(c^*) = \widehat{z}(c^*)/c^*$, whichever occurs closer to $\overline{c}$ as we decrease c. Define the condition $\overline{c} = c_h(\overline{\beta}_s)$ as equivalent to $k(\overline{c})/\overline{c} = \tau(\overline{\beta}_s)$ as defined above (see Corollary 5). Then at $\overline{c} = c_h(\beta_s)$ it holds that $c^* = \overline{c}$ and $\widehat{z}''(c^*) = 0$ because $\beta_s(c^*) = \widehat{z}'(c^*)(\widehat{z}'(c^*) - 1)$ and $\widehat{z}(c^*) - c^* > 0$. The condition $\overline{c} = c_h(\overline{\beta}_s)$ is equivalent to $k(\overline{c})/\overline{c} = \tau(\overline{\beta}_s)$ as defined in the next section. On the other hand, at $\overline{c} = c_l(\overline{\beta}_s)$ it holds that $c^* = \overline{c}$ and $\widehat{z}'(c^*) = \widehat{z}(c^*)/c^*$, $\widehat{z}''(c^*) < 0$. The condition $\overline{c} = c_l(\overline{\beta}_s)$ is equivalent to $k(\overline{c})/\overline{c} = \omega(\overline{\beta}_s)$ as defined in the next section. Increasing $\overline{c}$ slightly above $c_l(\overline{\beta}_s)$, it holds that $\widehat{z}'(\overline{c}) < \widehat{z}(\overline{c})/\overline{c}$, and thus $c^* < \overline{c}$. When $\overline{c}$ is close to c_l, it still holds that $\beta_s(c^*) < \widehat{z}'(c^*)(\widehat{z}'(c^*) - 1)$.

By continuity, and the fact that at $\overline{c} = c_h$ the corresponding condition has $\beta_s(c^*) = \widehat{z}'(c^*)(\widehat{z}'(c^*) - 1)$, so there must be a $c_m \in (c_l, c_h)$ such that if $\overline{c} = c_m$, then $\beta_s(c^*) = \widehat{z}'(c^*)(\widehat{z}'(c^*)-1)$ and $\widehat{z}'(c^*) = \widehat{z}(c^*)/c^*$, and for any $\overline{c} > c_m$, $\widehat{z}'(c^*) < \widehat{z}(c^*)/c^*$. At $\overline{c} = c_m$ it holds that $\widehat{z}''(c^*) < 0$, because (by the monotonicity of β_s) $\int_0^c \beta_s(x)dx/c$ is decreasing in c as we increase c from c^*, and $(\widehat{z}(c)/c)$ has a zero derivative at c^* by $\widehat{z}'(c^*) = \widehat{z}(c^*)/c^*$. Therefore, $\widehat{z}' = \frac{\int_0^c \beta_s(x)dx/c}{\frac{\widehat{z}(c)-c}{c}}$ is decreasing at c^*. Since $\beta_s(c^*) = \widehat{z}'(c^*)(\widehat{z}'(c^*) - 1)$ and $\widehat{z}''(c^*) < 0$, it must hold that $\widehat{z}(c^*) - c^* = 0$. Then using Lemma 14 and the way c_m was defined, if $\overline{c} = c_m$ then $\int_0^{c^*} \beta_s(x)dx = 0$, and thus $c^* = 0$,[35] which concludes the proof. Q.E.D.

Bounds on $k(\overline{c})/\overline{c}$

We can provide tight bounds on $k(\overline{c})/\overline{c}$ inspecting equations (81) and (82) and observing the following:

[34] Alternatively, $\beta_s(c) < \widetilde{v}'(c)(\widetilde{v}'(c) - 1) < 1$, and then (82), and (83) imply that $\Pr(c) < 1$ as well.

[35] In particular, $\widetilde{v}(c^*) - c^* = 0$, and $\widetilde{v}'(c^*) = \widetilde{v}(c^*)/c^*$ with $\widetilde{v}'(c^*) > \widetilde{v}'(\overline{c}) > 1$, implies $c^* = \widetilde{v}(c^*) = 0$ must hold.

$$\begin{aligned}\beta_s(\overline{c}) &< \widehat{z}'(\overline{c})(\widehat{z}'(\overline{c})-1) \Longrightarrow \\ \frac{e^{-\frac{\overline{c}-k(\overline{c})}{k(\overline{c})}}}{1-e^{-\frac{\overline{c}-k(\overline{c})}{k(\overline{c})}}} &> \frac{1+\sqrt{1+4\beta_s}}{2(1-\beta_s)},\end{aligned}$$

and

$$\widehat{z}'(\overline{c}) = \frac{k(\overline{c})}{\widehat{z}(\overline{c})-\overline{c}} < \frac{\widehat{z}(\overline{c})}{\overline{c}} \Longrightarrow \frac{e^{-\frac{\overline{c}-k(\overline{c})}{k(\overline{c})}}}{1-e^{-\frac{\overline{c}-k(\overline{c})}{k(\overline{c})}}} > \frac{1+\sqrt{1+4k(\overline{c})/\overline{c}}}{2(1-\beta_s)}$$

The first condition boils down to $k(\overline{c})/\overline{c} > \tau = \frac{1}{1-\log(\frac{\frac{1+\sqrt{1+4\beta_s}}{2(1-\beta_s)}}{\frac{1+\sqrt{1+4\beta_s}}{2(1-\beta_s)}+1})}$ with τ increasing and $\tau(0) = 0.59, \tau(0.5) = 0.76$ and $\tau(1) = 1$.

The second condition boils down to $k(\overline{c})/\overline{c} < \omega$ with ω increasing and $\omega(0) = 0.65, \omega(0.5) = 0.78$ and $\omega(1) = 1$.

Analysis of Example 1 and Proof of Proposition 8

6.1.2 Analysis of Example 1

Let us start with the one-type case where all sellers have types $c = 0$ and all buyers have types $v = 1$. When there are equal numbers of buyers and sellers, both sides randomize between posting and visiting. The equilibrium queue length λ^*, that is, the ratio of visitors to posters, is the same in both submarkets. The posting side obtains a payoff of 1 when at least two visitors show up, which occurs with probability $1-e^{-\lambda^*}(1+\lambda^*)$. Therefore, the payoff of a posting trader is $U^P = 1-e^{-\lambda^*}(1+\lambda^*)$. A visitor makes a surplus of 1 if and only if he is the only visitor at that poster, which occurs with probability $e^{-\lambda^*}$, and zero otherwise. So her payoff is: $U^V = e^{-\lambda^*}$. Setting $U^P = U^V$, we obtain

$$e^{-\lambda^*} = 1-e^{-\lambda^*}(1+\lambda^*),$$

which has a unique solution $\lambda^* \approx 1.146$.

To summarize: the equilibrium of the one-type model is such that both buyers and sellers mix, and the queue length in each submarket is $\lambda^* \approx 1.146$.

Now, suppose let us introduce a zero measure of buyers of types $v = 1-\alpha$ and sellers of types $c = \alpha$, for some $\alpha \in [0, 1)$. By visiting, they obtain a payoff of

$$U_1^V = e^{-\lambda^*}(1-\alpha),$$

because the queue length is λ^* in the one-type model, and the surplus is $1-\alpha$ if no other buyer visits. The utility from posting is

$$U_1^P = (1-\alpha)(1-(1+\lambda_1)e^{-\lambda_1}),$$

because a posting buyer earns a profit if at least two sellers visit, which occurs with probability $1-(1+\lambda_1)e^{-\lambda_1}$ given that the queue length is λ_1.

Importantly, the queue length when posting a reservation price of $1-\alpha$ satisfies $\lambda_1 < \lambda^*$ because the queue length is λ^* when a reservation price of 1 is posted (recall that we are dealing with a reverse auction here where the sellers bid). Therefore,

$$1-(1+\lambda_1)e^{-\lambda_1} < 1-(1+\lambda^*)e^{-\lambda^*} = e^{-\lambda^*}$$

where the equality comes from how λ^* was defined. Given this,

$$U_1^P < U_1^V,$$

and a small measure of low types enter by visiting. The above argument works for any $\alpha > 0$, and shows that any type $x < 1$ prefers visiting over posting.

Now, let us introduce a zero (small) set of buyers with value $v_1 = 1+\alpha$. By visiting, such types obtain the payoff

$$U_3^V = e^{-\lambda^*}(1+\alpha) + (1-e^{-\lambda^*})\alpha = (1+\alpha) - (1-e^{-\lambda^*}),$$

because the queue length is λ^* in the one-type model, and the surplus is $1+\alpha$ if no other buyer visits and α otherwise. Their payoff from posting is

$$U_3^P = (1+\alpha)(1-e^{-\lambda_3}) - \lambda_3 e^{-\lambda^*},$$

because the poster has to provide an expected utility of $e^{-\lambda^*}$ to each of his visitor and the total surplus generated is $(1+\alpha)(1-e^{-\lambda_3})$.

Importantly, the queue length when posting a reservation price of $1+\alpha$ satisfies $\lambda_3 > \lambda^*$ because the queue length is λ^* when the less favorable reservation price of 1 is posted (recall again that we are dealing here with a reverse auction where the sellers bid). Also, the utility from the visitors must satisfy

$$U_2 = e^{-\lambda_3}(1+\alpha) = e^{-\lambda^*}.$$

Therefore,

$$U_3^P = (1+\alpha) - e^{-\lambda^*} - \lambda_3 e^{-\lambda^*}$$
$$U_3^P - U_3^V = 1 - e^{-\lambda^*} - e^{-\lambda^*}(1+\lambda_3).$$

By construction, $1-(1+\lambda^*)e^{-\lambda^*} = e^{-\lambda^*}$, and thus

$$U_3^P - U_3^V = 1 - e^{-\lambda^*} - e^{-\lambda^*}(1+\lambda_3) = e^{-\lambda^*}(\lambda^* - \lambda_3) < 0.$$

Therefore, a small measure of high entering types will choose to visit. The above argument works for any $\alpha > 0$, and shows that any type $x > 1$ prefers visiting over posting as well.

6.1.3 Proof of Proposition 8

Let $e^* = (1-\alpha)e^{-\lambda^*}$ where λ^* solves $e^x = 2 + x$, and thus $\lambda^* \approx 1.146$. The proof is in two Steps where Step 1 recalls the Analysis of Example 1, and Step 2 uses a simple continuity argument.

Step 1: Low types enter with probability zero, and they are indifferent between entering or not.

We show that in the (unique) equilibrium all low types enter with zero probability but gain a utility of e^* upon entering when they visit. Also, we show that, using calculations from Example 1, that they make a utility of lower than e^* when they post, which would establish the statement of Step 1.

Suppose that only high types enter, then as it is shown in Example 1 both buyers and sellers post and visit, and the equilibrium queue length is λ^*. Therefore, the utility of a low type from visiting is indeed $(1-\alpha)e^{-\lambda^*} = e^*$ because they make a surplus of $1-\alpha$ if and only if no other visitor is present. In the Analysis of Example 1 above it was shown that any type $v < 1$ makes a lower profit from posting than from visiting. Therefore, Step 1 is complete.

Step 2: For some $\widetilde{e} < e^*$, and $e \in (\widetilde{e}, e^*)$ low types enter with a positive probability and they visit with probability one.

First, it is clear that for any $e < e^*$ the equilibrium features a positive entry by the low types because the outcome where only high types enter would provide a profitable deviation for low types by Step 1. Moreover, the utility from visiting is strictly larger than the utility from posting when $e = e^*$. Therefore, by continuity of the equilibrium in entry costs, this still holds in a neighborhood of e^*, which completes Step 2.

7 Online Appendix 2: Multiple Submarkets

First, let us consider homogenous sellers. Let $c = 0$ and $v \in [0,1]$ with cdf F. Let $s = 1$ be the number of sellers without loss of generality, and let $b > 0$ be the mass of buyers with $\mu = b/s$. Since sellers are homogenous it is easy to see that the welfare maximizing (constrained efficient) allocation assigns the same queue length to each seller. In what follows, we calculate this queue length. Let $T(v) = \mu(1 - F(v))$ be the mass of buyers above type v. Then the probability that the highest type is less than v (or no buyer visits at all) at a given seller with queue length T is $G(v) = e^{-T(v)}$, and total welfare is

$$W = \int_0^1 G'(v)v dv = 1 - \int_0^1 G(v)dv = 1 - \int_0^1 e^{-T(v)}dv$$

after integration by parts.

To calculate the welfare from having two submarkets, let s_i, b_i denote masses of sellers and buyers in market $i = 1, 2$, and let $\mu_i = b_i/s_i$ denote the aggregate market tightness. Let F_i be the distribution of buyer types on market i, and $T_i = \mu_i(1 - F_i)$. Then by a similar argument as above, the welfare becomes

$$W^{(2)} = 1 - s_1 \int_0^1 e^{-T_1(v)}dv - s_2 \int_0^1 e^{-T_2(v)}dv.$$

We would like to show that $W \geq W^{(2)}$, which is then equivalent to

$$\int_0^1 e^{-T(v)}dv \leq s_1 \int_0^1 e^{-T_1(v)}dv + s_2 \int_0^1 e^{-T_2(v)}dv.$$

By feasibility conditions, and $s_1 + s_2 = 1$, we have that $s_1T_1 + s_2T_2 = T$, so we need to show that

$$\int_0^1 e^{-(s_1T_1(v)+s_2T_2(v))}dv \leq s_1 \int_0^1 e^{-T_1(v)}dv + s_2 \int_0^1 e^{-T_2(v)}dv.$$

We show that this holds point-wise for the integrands or

$$e^{-(s_1T_1+s_2T_2)} \leq s_1e^{-T_1} + s_2e^{-T_2}.$$

This last display directly follows from convexity of function e^{-x} and $s_1 + s_2 = 1$.

Let us now assume that the sellers' costs are distributed according to a cdf H. Then we can repeat the analysis in the previous subsection to show that merging two markets improves welfare.

Formally, fix a cost level c and let $W(c)$ and $W^{(2)}(c)$ be defined as above taking the allocations of buyers as given. The above analysis implies that $W(c) \geq W^{(2)}(c)$ for all c, and thus

$$TW = \int_0^1 W(c)dc \geq \int_0^1 W^{(2)}(c)dc = TW^{(2)},$$

which completes our proof. Note, that $W(c)$ includes already the mass of sellers on the market with type c (denoted as $f_s(c)$), while the calculations of $W^{(2)}(c)$ assume that $f_s^1(c), f_s^2(c)$ are present in the two submarkets with $f_s^1(c) + f_s^2(c) = f_s(c)$.

Thus, having a single market for posting maximizes constrained welfare.