Online Appendix to "Screening Under A Fixed Cost of Misrepresentation"

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8 Appendix. Multi-valued targeted type

This appendix characterize the optimal mechanism when the targeted type $\tau(.)$ may be multi-valued. Two additional issues needs to be addressed in this case. First, with multi-valued targeted types, $\tau(.)$ is a correspondence which can be equivalently represented as a discontinuous function with upwards jumps. Such jumps cannot be characterized by the following differential equation derived in Theorem 7 in the main paper:

$$\dot{\tau}^k(\theta) = \frac{f(\theta)[u_q(Q^k, \tau^{k-1}) - u_q(Q^k, \tau^k)]}{f(\tau^k)u_q(Q^k, \tau^k)} \prod_{s=1}^{k-1} \frac{u_q(Q^s, \tau^{s-1})}{u_q(Q^s, \tau^s)}(\theta), \quad k \in \{1, ..., M(\theta)\}, \tag{72}$$

which applies only where $\tau(.)$ is continuous. Second, the image $\tau(\theta)$ need not be convex for all θ, and so one would have to determine the boundaries of subintervals in $[\min \tau(\hat{\theta}), \max \tau(1)]$ where the law of motion is:

$$[u_q(q(\tau(\theta)), \theta) - u_q(q(\tau(\theta)), \tau(\theta))]\dot{q}(\tau(\theta)) = u_\theta(q(\tau(\theta)), \tau(\theta)) - 1(\tau(\theta) \geq \hat{\theta})u_\theta(q(\tau(\tau(\theta))), \tau(\theta)). \tag{73}$$

In order to tackle these issues, we introduce and work with a concept of an "attracted type," a generalized inverse of τ. Specifically, let $\underline{\tau} = \min \tau(\hat{\theta})$ and $\overline{\tau} = \max \tau(1)$. The attracted type function $\beta : [\underline{\tau}, \overline{\tau}] \to [\hat{\theta}, 1]$ is defined as follows:

$$\beta(\theta) = \theta' \text{ if } \theta \in [\min \tau(\theta'), \max \tau(\theta')].$$

This definition implies that $\beta(\theta) = \tau^{-1}(\theta)$ if $\tau^{-1}(\theta)$ is non-empty. If $\tau^{-1}(\theta)$ is empty, then

$\beta(\theta)$ a unique θ' s.t. $\min \tau(\theta') < \theta < \max \tau(\theta')$. Since $\tau(\theta)$ is strictly increasing and upper hemicontinuous by Theorem 4, $\beta(\theta)$ is well-defined, weakly increasing and continuous.[12]

To describe the chains of attracted types connected by binding incentive constraints, we use the concept of higher-order attracted types in a similar fashion to higher-order targeted types. Specifically, for $\theta \in [\underline{\tau}, \hat{\theta}]$ let $\beta^0(\theta) = \theta$ and $\beta^k(\theta) = \beta(\beta^{k-1}(\theta))$ for $k \geq 1$. Let $R(\theta)$ be the number of elements in the chain of attracted types, so that $\beta^k(\theta)$ exists for $k = 1, ..., R(\theta) - 1$. The maximal length of the chain of attracted types is $R = R(\underline{\tau})$. Since $\beta(.)$ is continuous and increasing, it maps the interval $[\beta^{k-1}(\theta), \beta^k(\theta)]$ onto the adjacent interval $[\beta^k(\theta), \beta^{k+1}(\theta)]$.

Then the following condition must hold for $\theta \in [\underline{\tau}, \overline{\tau}]$ in the optimal mechanism:

$$u_q(q(\theta), \theta) f(\theta) = [u_q(q(\theta), \beta(\theta)) - u_q(q(\theta), \theta)] \sum_{k=1}^{s} f(\beta^k(\theta)) \dot{\beta}^k(\theta), \tag{74}$$

where s is such that $\beta^s(\theta) \in (\max \tau(1), 1]$.

A formal proof of this claim is provided in the proof of Theorem 11. Condition (74) is the same as the optimality condition (13) in Theorem 6 in the paper, but restated using attracted type function $\beta(.)$. Intuitively, this condition reflects the optimal tradeoff between the marginal efficiency gain from raising $q(\theta)$ and the marginal cost of information rent that the principal has to provide to the types in every predecessor of θ in the chain of attracted types $\beta^k(\theta)$ for $k = 1, ..., s$.

Our next step is to generalize the optimal "law of motion" of $q(\theta)$ to the current case. Note that $\beta^{-1}(\theta)$ is well-defined as the convex hull of $\tau(\theta)$. If $\theta \in \tau(\beta(\theta))$ i.e., the incentive constraint $IC(\beta(\theta), \theta)$ is binding, for all θ in some open interval, then the corresponding law of motion, which we denote by $\dot{q}^{IC}(.)$, is obtained by rewriting (73) which yields:

$$\dot{q}^{IC}(\theta) \equiv \frac{u_\theta(q(\theta), \theta) - 1(\theta \geq \hat{\theta}) u_\theta(q(\min \beta^{-1}(\theta)), \theta)}{u_q(q(\theta), \beta(\theta)) - u_q(q(\theta), \theta)}. \tag{75}$$

On the other hand, by part 4 of Theorem 4 in the main paper, $q(\theta) = q^{fb}(\theta)$ for all $\theta \in [\theta_1, \theta_2]$ where θ_1 and θ_2 are the boundaries of the maximal interval on which $\beta(.)$ is constant (put otherwise, $\tau(\beta(\theta))$ is multi-valued. So, $\dot{q}(\theta) = \dot{q}^{fb}(\theta)$.

[12] To illustrate the relationship between τ and β, consider the following example: $\hat{\theta} = 0.6$, $\tau(\theta) = \theta - 0.3$ if $\theta \in [0.6, 0.8)$, $\tau(\theta) = \{0.5, 0.6\}$ if $\theta = 0.8$, $\tau(\theta) = \theta - 0.2$ if $\theta \in (0.8, 1]$. The corresponding β function is: $\beta(\theta) = \theta + 0.3$ if $\theta \in [0.3, 0.5)$, $\beta(\theta) = 0.8$ if $\theta \in [0.5, 0.6]$, $\beta(\theta) = \theta + 0.2$ if $\theta \in (0.6, 0.8]$. Particularly, note that a type in $(0.5, 0.6)$ is not in the image of $\tau(.)$, but $\beta(\theta) = 0.8$ for all $\theta \in [0.5, 0.6]$.

Thus, $\dot{q}(\theta) = \dot{q}^{IC}(\theta)$ when $IC(\beta(\theta), \theta)$ is binding, and $\dot{q}(\theta) = \dot{q}^{fb}(\theta)$ when it is not binding. To identify which of these two cases applies, consider the payoff of type θ when she imitates type θ', $U(\theta', \theta) = u(q(\theta), \theta') - u(q(\theta), \theta) - C + \int_{\hat{\theta}}^{\max\{\theta, \hat{\theta}\}} u_\theta(q(\min \beta^{-1}(s)), s)ds$. Then for $\theta \in [\underline{\tau}, \max \tau(1)]$, let $I(\theta) = \int_{\underline{\tau}}^{\theta} U_2(\beta(x), x)dx =$

$$= \int_{\underline{\tau}}^{\theta} [u_q(q(x), \beta(x)) - u_q(q(x), x)]\dot{q}(x) - u_\theta(q(x), x) + 1(x \geq \hat{\theta})u_\theta(q(\min \beta^{-1}(x)), x)dx. \tag{76}$$

As shown in the proof of Theorem 11 stated below, $I(\theta)$ tracks the slackness of $IC(\beta(\theta), \theta)$. Specifically, if $I(\theta) = 0$, then $IC(\beta(\theta), \theta)$ is binding; if $I(\theta) < 0$, $IC(\beta(\theta), \theta)$ is slack. Therefore, the optimal law of motion of $q(\theta)$ can be stated as follows:

$$\dot{q}(\theta) = \begin{cases} \dot{q}^{IC}(\theta) & \text{if } q(\theta) < q^{fb}(\theta), \\ \dot{q}^{fb}(\theta) & \text{if } q(\theta) = q^{fb}(\theta) \text{ and } I(\theta) < 0, \\ \min\{\dot{q}^{IC}(\theta), \dot{q}^{fb}(\theta)\} & \text{if } q(\theta) = q^{fb}(\theta) \text{ and } I(\theta) = 0. \end{cases} \tag{77}$$

The logic behind (77) is that, when $q(\theta)$ is below the first-best, the incentive constraint $IC(\beta(\theta), \theta)$ must be binding, and so $\dot{q}(\theta) = \dot{q}^{IC}(\theta)$. On the other hand, if $I(\theta) < 0$, then $IC(\beta(\theta), \theta)$ is slack and the optimal quantity must stay at the first-best level in a neighborhood of θ. This case arises when $\tau(.)$ is non-convex.

When $I(\theta) = 0$ and $q(\theta) = q^{fb}(\theta)$, we are in a boundary situation with binding $IC(\beta(\theta), \theta)$. In this case, if $\dot{q}^{IC}(\theta) > \dot{q}^{fb}(\theta)$, the types in a neighborhood of θ do not have IC constraints binding towards them and the quantities remains at the first-best level. On the other hand, if $\dot{q}^{IC}(\theta) < \dot{q}^{fb}(\theta)$, the types in a neighborhood of θ do have IC constraints binding towards , and the law of motion of q is given by (75).

Further, the boundary conditions for $\beta(.)$ and $q(.)$ on the interval $[\underline{\tau}, \overline{\tau}]$ where $\underline{\tau} = \min\{\theta : \beta(\theta) \neq \emptyset\}$, $\overline{\tau} = \max\{\theta : \beta(\theta) = 1\}$, are as follows:

$$\beta^k(\underline{\tau}) = \beta^{k-1}(\beta(\underline{\tau})), \tag{78}$$

$$q(\underline{\tau}) = q^{fb}(\underline{\tau}), \tag{79}$$

$$q(\overline{\tau}) = q^{fb}(\overline{\tau}), \tag{80}$$

$$u(q^{fb}(\underline{\tau}), \beta(\underline{\tau})) - u(q^{fb}(\underline{\tau}), \underline{\tau}) - C = 0. \tag{81}$$

The necessary conditions for optimality are presented in the following Theorem:

Theorem 11 *The following conditions must hold in an optimal mechanism* $(q(.), t(.))$*:*
(i) The optimality condition (74);
(ii) The law of motion (77);
(iii) The boundary conditions (78) - (81).

We can now use the optimality conditions of Theorem 11, in particular, (74), to obtain the differential equations characterizing the attracted type functions $\beta^k(.)$ and the corresponding quantities. To state them, let $\hat{\theta} = \beta(\underline{\tau})$, $G^k(\theta) = q(\beta^k(\theta))$ for $\theta \in [\underline{\tau}, \hat{\theta}]$, so that $G^k(.)$ is the quantity received by the k-th order attracted type $\beta^k(\theta)$. Also, with a slight abuse of notation, let $L(\theta, k) = \prod_{i=1}^{k-1} \frac{u_q(G^i(\theta)), \beta^i(\theta))}{u_q(G^i(\theta), \beta^{i+1}(\theta))}$, with $L(\theta, 1) = 1$ by convention. Then we have:

Corollary 2 *In an optimal mechanism, for* $\theta \in [\underline{\tau}, \hat{\theta}]$ *and* $s(\theta) \in \mathbf{N}$ *such that* $\beta^{s(\theta)}(\theta) \in [\min \tau(1), 1]$ *we have:*

$$\dot{\beta}^k(\theta) = \begin{cases} \frac{f(\theta)u_q(G^0(\theta),\theta))[u_q(G^k(\theta),\beta^{k+1}(\theta))-u_q(G^k(\theta),\beta^k(\theta))]}{f(\beta^k(\theta))u_q(G^k(\theta),\beta^{k+1}(\theta))[u_q(G^0(\theta),\beta^1(\theta))-u_q(G^0(\theta),\theta)]} L(\theta, k) & \text{if } k < s(\theta); \\ \frac{f(\theta)u_q(G^0(\theta),\theta)}{f(\beta^k(\theta))[u_q(G^0(\theta),\beta^1(\theta))-u_q(G^0(\theta),\theta)]} L(\theta, k) & \text{if } k = s(\theta); \end{cases} \tag{82}$$

Also, for $k = 0, ..., s(\theta) - 1$, $\dot{G}^k(\theta) =$

$$\begin{cases} \frac{u_\theta(G^0(\theta),\theta))}{u_q(G^0(\theta),\beta^1(\theta))-u_q(G^0(\theta),\theta)} & \text{if } G^0(\theta) < q^{fb}(\theta)),\ k = 0; \\ \frac{-u_{\theta q}(G^k(\theta),\beta^k(\theta))}{u_{qq}(G^k(\theta),\beta^k(\theta))} & G^0(\theta) = q^{fb}(\theta), I(\theta) < 0, k = 0; \\ \frac{f(\theta)u_q(G^0(\theta),\theta))[u_\theta(G^k(\theta),\beta^k(\theta))-u_\theta(G^{k-1}(\theta),\beta^k(\theta))]}{f(\beta^k(\theta))[u_q(G^0(\theta),\beta^1(\theta))-u_q(G^0(\theta),\theta)]u_q(G^k(\theta),\beta^{k+1}(\theta))} L(\theta, k) & G^k(\theta) < q^{fb}(\beta^k(\theta)),\ k \geq 1; \\ \frac{-f(\theta)u_q(G^0(\theta),\theta))[u_q(G^k(\theta),\beta^{k+1}(\theta))-u_q(G^k(\theta),\beta^k(\theta))]u_{\theta q}(G^k(\theta),\beta^k(\theta))}{f(\beta^k(\theta))[u_q(G^0(\theta),\beta^1(\theta))-u_q(G^0(\theta),\theta)]u_q(G^k(\theta),\beta^{k+1}(\theta))u_{qq}(G^k(\theta),\beta^k(\theta))} L(\theta, k) & G^k(\theta) = q^{fb}(\beta^k(\theta)), I(\beta^k(\theta)) < 0, k \geq 1; \\ \dot{\beta}^k(\theta) \min\{\dot{q}^{IC}(\beta^k(\theta)), \dot{q}^{fb}(\beta^k(\theta))\} & G^k(\theta) = q^{fb}(\beta^k(\theta)), I(\beta^k(\theta)) = 0. \end{cases} \tag{83}$$

Differential equations (82) and (83) describe the laws of motion of the high-order attracted types β^k and their corresponding quantities G^k. Together with the boundary conditions (78)- (81), these differential equations provide a characterization of the optimal mechanism when multi-valued targeted types exist. Particularly, consider the law of motion of quantities (83). Its first two cases specify the law of motion $\dot{q}^{IC}$ that applies when the quantities are below the first-best and is derived from the binding incentive constraint towards the respective types. The next two cases in (83) specify the law of motion for types who do not have "attracted types." In these cases, the law of motion is the rate that keeps the quantities at the first-best level. The last case of condition (83) specifies the law of motion for such θ where both the quantity is at the first-best and there is a type attracted to θ. The incentive constraints are binding, which is the smaller of $\dot{q}^{IC}$ and $\dot{q}^{fb}$, as the quantities cannot exceed the first-best.

Proof of Theorem 11 and Corollary 2:

First, the boundary conditions in part (iii) hold by the definitions of $\underline{\tau}$, $\overline{\tau}$, and $\hat{\theta}$.

Next, we derive the optimality condition (74) in the following Lemma. Note that this condition is equivalent to condition (13).

Lemma 18 *In an optimal mechanism, $\beta^k(.)$ is differentiable at θ for all $k \in \{1, ..., s\}$ and equation (74) holds for any $\theta \in [\min \tau(\hat{\theta}), \max \tau(1)]$ and s such that $\beta^s(\theta) \in [\min \tau(1), 1]$.*

Proof of Lemma 18: The proof is by contradiction. So, suppose that there exists $\tilde{\theta} \in [\min \tau(\hat{\theta}), \max \tau(1)]$ with differentiable $\beta^k(\tilde{\theta})$, $k = 1, ..., s$, such that

$$u_q(q(\tilde{\theta}), \tilde{\theta}) f(\tilde{\theta}) > [u_q(q(\tilde{\theta}), \beta(\tilde{\theta})) - u_q(q(\tilde{\theta}), \tilde{\theta})] \sum_{k=1}^{s} f(\beta^k(\tilde{\theta})) \dot{\beta}^k(\tilde{\theta}). \tag{84}$$

We will show that in this case the mechanism is not optimal, as the principal can get a higher profit by increasing the quantities assigned to the types around $\tilde{\theta}$ and collecting the additional revenue generated thereby, while providing increased information rents to types around $\beta^k(\tilde{\theta})$, $k = 1, ..., s$. The case when this inequality has the opposite sign is similar.

The proof proceeds through three steps. In Step 1, we construct an alternative mechanism $(\tilde{q}(.), \tilde{t}(.))$. In Steps 2 and 3 we show that this alternative mechanism is incentive compatible and more profitable, respectively, for the principal than the original one, when the quantity changes for the types near $\tilde{\theta}$ are sufficiently small.

Step 1. Constructing an Alternative Mechanism $(\tilde{q}(.), \tilde{t}(.))$.

Inequality (84) implies that there exists $\mu > 0$ such that

$$u_q(q(\tilde{\theta}), \tilde{\theta}) f(\tilde{\theta}) - [u_q(q(\tilde{\theta}), \beta(\tilde{\theta})) - u_q(q(\tilde{\theta}), \tilde{\theta})] \sum_{k=1}^{s} f(\beta^k(\tilde{\theta})) \dot{\beta}^k(\tilde{\theta}) - \mu > 0. \tag{85}$$

Note that the inequality (85) implies that $q(\tilde{\theta}) < q^{fb}(\tilde{\theta})$.

Now, for $\epsilon > 0$ small enough and $k = 0, ..., s$, let $\Theta_k(\epsilon) = [\beta^k(\tilde{\theta} - \epsilon) - (\frac{\delta_\tau}{2})^{s-k}\epsilon^2, \beta^k(\tilde{\theta} + \epsilon) + (\frac{\delta_\tau}{2})^{s-k}\epsilon^2]$. Since $\beta^k(\tilde{\theta}) < \beta^{k+1}(\tilde{\theta})$ for all $k \in \{0, ..., s-1\}$, Lemma 12 implies that $\Theta_k(\epsilon) \cup \Theta_k + 1(\epsilon)$ for all $k \in \{0, ..., s-1\}$, which we now assume.

The alternative mechanism $(\tilde{q}(.), \tilde{t}(.))$ differs from the original one, $(q(.), t(.))$, only as follows: (i) for $\theta \in \Theta_0(\epsilon)$, $\tilde{q}(\theta) = q(\theta) + \epsilon^5$ and $\tilde{t}(\theta) = t(\theta) + u(q(\theta) + \epsilon^5, \theta) - u(q(\theta), \theta)$; (ii) for $\theta \in \cup_{k=1}^{s} \Theta_k(\epsilon)$, $\tilde{q}(\theta) = q(\theta)$ and $\tilde{t}(\theta) = t(\theta) - \Delta(\epsilon)$, where $\Delta(\epsilon) \equiv \max_{\theta' \in \Theta_0(\epsilon)} u(q(\theta') + \epsilon^5, \overline{\theta}_1) - u(q(\theta'), \overline{\theta}_1) - u(q(\theta') + \epsilon^5, \theta') + u(q(\theta'), \theta')$ and $\overline{\theta}_1 = \max \Theta_1(\epsilon)$. So, $\Delta(\epsilon) > 0$ and $\lim_{\epsilon \to 0} \Delta(\epsilon) = 0$. Let $\tilde{V}(\theta)$ be the net payoff of type θ in $(\tilde{q}(.), \tilde{t}(.))$.

Step 2. Establishing individual rationality and incentive compatibility of the alternative mechanism for small $\epsilon > 0$.

IR constraints hold in $(\tilde{q}(.), \tilde{t}(.))$ because $\tilde{V}(\theta) > V(\theta)$ for $\theta \in \cup_{k=1}^{s} \Theta_k(\epsilon)$, and $\tilde{V}(\theta) = V(\theta)$ for all $\theta \in [0,1] \setminus \cup_{k=1}^{s} \Theta_k(\epsilon)$.

Now, let us focus on incentive constraints in the mechanism $(\tilde{q}(.), \tilde{t}(.))$, which we denote by $\tilde{IC}(\theta, \theta')$ for $(\theta, \theta') \in [0,1]^2$. First, if $\theta \in [0,1]$ and $\theta' \notin \cup_{k=0}^{s} \Theta_k(\epsilon)$, then $\tilde{IC}(\theta, \theta')$ holds because $\tilde{V}(\theta) \geq V(\theta)$, $\tilde{q}(\theta') = q(\theta')$, $\tilde{t}(\theta') = t(\theta')$ and $IC(\theta, \theta')$ holds.

Second, if $\theta \in [0,1]$ and $\theta' \in \Theta_s(\epsilon)$, then for small enough ϵ, $\tau^{-1}(\theta') = \emptyset$ since $\beta^{s+1}(\tilde{\theta}) = \emptyset$. Therefore, $IC(\theta, \theta')$ is slack in the original mechanism. Let $\delta > 0$ be the minimal slack over all $\theta \in [0,1]$ and all $\theta' \in \Theta_s(\epsilon)$. Note that $\tilde{V}(\theta) \geq V(\theta)$ for all $\theta \in [0,1]$, and $\tilde{V}(\theta') = V(\theta') + \Delta(\epsilon)$ for $\theta' \in \Theta_s(\epsilon)$. So, $\tilde{IC}(\theta, \theta')$ holds for sufficiently small ϵ s.t. $\Delta(\epsilon) \leq \delta$.

Third, $\tilde{IC}(\theta, \theta')$ holds for $\theta \in \Theta_1(\epsilon)$ and $\theta' \in \Theta_0(\epsilon)$ because we have:

$$\begin{aligned}
&\tilde{V}(\theta) = V(\theta) + \Delta(\epsilon) \geq u(q(\theta'), \theta) - t(\theta') - C + \Delta(\epsilon) \geq u(q(\theta'), \theta) - t(\theta') - C + \\
&[u(q(\theta') + \epsilon^5, \theta) - u(q(\theta') + \epsilon^5, \theta')] - [u(q(\theta'), \theta) - u(q(\theta'), \theta')] = u(\tilde{q}(\theta'), \theta) - \tilde{t}(\theta') - C,
\end{aligned}$$

where the first equality holds by construction; the first inequality holds by incentive compatibility of the original mechanism; the second inequality holds by definition of $\Delta(\epsilon)$, and because $\theta \leq \overline{\theta}_1$ and $u_{\theta q} > 0$; the last equality holds by definition of $\tilde{q}(\theta')$ and $\tilde{t}(\theta')$.

Fourth, if $\theta \in \cup_{k=1}^{s} \Theta_k(\epsilon)$ and $\theta' \in \cup_{k=1}^{s-1} \Theta_k(\epsilon)$, then $\tilde{V}(\theta) = V(\theta) + \Delta(\epsilon)$ and $\tilde{V}(\theta') = V(\theta') + \Delta(\epsilon)$ since $\tilde{q}(\theta) = q(\theta)$, $\tilde{t}(\theta) = t(\theta) - \Delta(\epsilon)$, $\tilde{q}(\theta') = q(\theta')$, and $\tilde{t}(\theta') = t(\theta') - \Delta(\epsilon)$. So, $\tilde{IC}(\theta, \theta')$ holds because $IC(\theta, \theta')$ holds.

Fifth, consider $\tilde{IC}(\theta, \theta')$ s.t. $\theta \notin \Theta_1(\epsilon)$, $\theta' \in \Theta_0(\epsilon)$. Now, suppose that $\frac{\theta + \beta(\theta')}{2} \geq \hat{\theta}$ in the original mechanism. Then applying Lemma 13 in the main paper, we get:

$$\tilde{V}(\theta) = V(\theta) > u(q(\theta'),\theta) - t(\theta') - C + \delta_V \left(\frac{\delta_\tau}{2}\right)^{2(s-1)} \frac{\epsilon^4}{16} = u(\tilde{q}(\theta'),\theta) - \tilde{t}(\theta') - C$$
$$+ \delta_V \left(\frac{\delta_\tau}{2}\right)^{2(s-1)} \frac{\epsilon^4}{16} - [u(q(\theta')+\epsilon^5,\theta) - u(q(\theta'),\theta) - u(q(\theta')+\epsilon^5,\theta') + u(q(\theta'),\theta')]$$
$$> u(\tilde{q}(\theta'),\theta) - \tilde{t}(\theta') - C,$$

where the first inequality holds because $\theta' \in \Theta_0(\epsilon) \equiv [\tilde{\theta} - \epsilon - (\frac{\delta_\tau}{2})^s \epsilon^2, \tilde{\theta} + \epsilon + (\frac{\delta_\tau}{2})^s \epsilon^2]$ and $\theta - \theta' \geq \delta_\tau[\beta(\theta) - \beta(\theta')]$ by Lemma 12 in the main paper. So, $\beta(\theta') \in [\beta(\tilde{\theta} - \epsilon) - \frac{1}{2}(\frac{\delta_\tau}{2})^{s-1}\epsilon^2, \beta(\tilde{\theta}+\epsilon) + \frac{1}{2}(\frac{\delta_\tau}{2})^{s-1}\epsilon^2]$. This and the fact that $\theta \notin \Theta_1(\epsilon)$ imply that $|\theta - \beta(\theta')| \geq \frac{1}{2}(\frac{\delta_\tau}{2})^{s-1}\epsilon^2$. Using the latter in

$$V(\theta_2) - U(\theta_1'|\theta_2) \geq \begin{cases} \delta_V \frac{(\theta_2-\theta_1)^2}{4} & \text{if } \frac{\theta_1+\theta_2}{2} \geq \hat{\theta} \\ \frac{\theta_1-\theta_2}{2} \min_\theta u_\theta(q(\theta_1'),\theta) & \text{if } \frac{\theta_1+\theta_2}{2} < \hat{\theta}. \end{cases} \tag{86}$$

of Lemma 13 in the main paper yields $V(\theta) - U(\theta'|\theta) \geq \delta_V (\frac{\delta_\tau}{2})^{2(s-1)} \frac{\epsilon^4}{16}$ for small enough ϵ. The second equality above holds by definitions of $\tilde{q}(\theta')$ and $\tilde{t}(\theta')$. The last inequality holds for small enough ϵ.

Now, suppose that $\frac{\theta+\beta(\theta')}{2} \leq \hat{\theta}$ in the original mechanism. Since $\beta(\theta') > \hat{\theta}$, it follows that $\theta < \hat{\theta}$ and so $\tau(\theta) = \emptyset$ and $IC(\theta,\theta')$ is slack in the original mechanism. Hence, when ϵ is sufficiently small, $\tilde{IC}(\theta,\theta')$ is slack in the modified mechanism as well.

Sixth, suppose that $\theta \notin \cup_{k=1}^s \Theta_k(\epsilon)$ and $\theta' \in \Theta_r(\epsilon)$, $r = 1, ..., s-1$, and ϵ is sufficiently small. Let us start with the case when $\frac{\theta+\beta(\theta')}{2} \geq \hat{\theta}$ in the original mechanism. We have:

$$\tilde{V}(\theta) = V(\theta) > u(q(\theta'),\theta) - t(\theta') - C + \delta_V \left(\frac{\delta_\tau}{2}\right)^{2(r-1)} \frac{\epsilon^4}{16}$$
$$= u(\tilde{q}(\theta'),\theta) - \tilde{t}(\theta') - C + \delta_V \delta_\tau (\frac{\delta_\tau}{2})^{2(r-1)} \frac{\epsilon^4}{16} - \Delta(\epsilon) > u(\tilde{q}(\theta'),\theta) - \tilde{t}(\theta') - C,$$

where the first inequality holds because $\theta' \in [\beta^r(\tilde{\theta} - \epsilon) - (\frac{\delta_\tau}{2})^{s-r}\epsilon^2, \beta^r(\tilde{\theta}+\epsilon) + (\frac{\delta_\tau}{2})^{s-r}\epsilon^2]$, and $\theta - \theta' \geq \delta_\tau[\beta(\theta) - \beta(\theta')]$ by Lemma 12. So, $\beta(\theta') \in [\beta^{r+1}(\tilde{\theta} - \epsilon) - \frac{1}{2}(\frac{\delta_\tau}{2})^{s-r-1}\epsilon^2, \beta^{r+1}(\tilde{\theta} + \epsilon) + \frac{1}{2}(\frac{\delta_\tau}{2})^{s-r-1}\epsilon^2]$. Therefore, $|\theta - \beta(\theta')| \geq \frac{1}{2}(\frac{\delta_\tau}{2})^{r-1}\epsilon^2$. Hence, inequality (86) in Lemma 13 implies that $V(\theta) - U(\theta'|\theta) \geq \delta_V(\frac{\delta_\tau}{2})^{2(r-1)}\frac{\epsilon^4}{16}$ for small enough ϵ. The second equality holds by definitions of $\tilde{q}(\theta')$ and $\tilde{t}(\theta')$. The last inequality holds for small enough ϵ.

Now, suppose that $\frac{\theta+\beta(\theta')}{2} \leq \hat{\theta}$ in the original mechanism. Since $\beta(\theta') > \hat{\theta}$, it follows that $\theta < \hat{\theta}$ and so $\tau(\theta) = \emptyset$ and $IC(\theta, \theta')$ is slack in the original mechanism. Hence, when ϵ is sufficiently small, $\tilde{IC}(\theta, \theta')$ is slack in the modified mechanism as well.

Seventh, suppose that $\{\theta, \theta'\} \subset \Theta_0(\epsilon)$. If $\theta > \theta'$. Since $V(.)$ and $q(.)$ are continuous, $\tilde{V}(.)$ and $\tilde{q}(.)$ are continuous on $\Theta_0(\epsilon)$, and so $\tilde{IC}(\theta, \theta')$ holds when ϵ is sufficiently small.

Step 3. Establishing that the mechanism $(\tilde{q}(.), \tilde{t}(.))$ is more profitable for the principal than the original mechanism.

The change in seller's profits from switching to the new mechanism is equal to

$$\Pi(\epsilon) = \int_{\Theta_0(\epsilon)} [u(q(\theta) + \epsilon^5, \theta) - u(q(\theta), \theta)] f(\theta) d\theta - \Delta(\epsilon) \sum_{k=1}^{s} \int_{\Theta_k(\epsilon)} f(\theta) d\theta.$$

Hence,

$$\lim_{\epsilon \to 0} \frac{\Pi(\epsilon)}{\epsilon^6} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\int_{\Theta_0(\epsilon)} u_q(q(\theta), \theta) f(\theta) d\theta - \max_{\theta' \in \Theta_0(\epsilon)} [u_q(q(\theta'), \overline{\theta}_1) - u_q(q(\theta'), \theta')] \sum_{k=1}^{s} \int_{\Theta_k(\epsilon)} f(\theta) d\theta \right)$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\int_{\tilde{\theta}-\epsilon}^{\tilde{\theta}+\epsilon} u_q(q(\theta), \theta) f(\theta) d\theta - \max_{\theta' \in \Theta_0(\epsilon)} [u_q(q(\theta'), \overline{\theta}_1) - u_q(q(\theta'), \theta')] \sum_{k=1}^{s} \int_{\beta^k(\tilde{\theta}-\epsilon)}^{\beta^k(\tilde{\theta}+\epsilon)} f(\theta) d\theta \right)$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\int_{\tilde{\theta}-\epsilon}^{\tilde{\theta}+\epsilon} u_q(q(\theta), \theta) f(\theta) - \max_{\theta' \in \Theta_0(\epsilon)} [u_q(q(\theta'), \overline{\theta}_1) - u_q(q(\theta'), \theta')] \sum_{k=1}^{s} \dot{\beta}^k(\theta) f(\beta^k(\theta)) d\theta \right)$$

$$= 2 \left(u_q(q(\tilde{\theta}), \tilde{\theta}) f(\tilde{\theta}) - [u_q(q(\tilde{\theta}), \beta(\tilde{\theta})) - u_q(q(\tilde{\theta}), \tilde{\theta})] \sum_{k=1}^{s} \dot{\beta}^k(\tilde{\theta}) f(\beta^k(\tilde{\theta})) \right) > 2\mu > 0,$$

where the first equality holds by definition of $\Delta(.)$. The second equality holds because $\Theta_k(\epsilon)$ converges to $[\beta^k(\tilde{\theta} - \epsilon), \beta^k(\tilde{\theta} + \epsilon)]$ at the same rate as ϵ^2. The third equality is obtained by a change of variables. The fourth equality holds since $\overline{\theta}_1 \to \beta(\tilde{\theta})$ and $\Theta_0(\epsilon) \to \tilde{\theta}$ as $\epsilon \to 0$; and the first inequality holds by (85). Therefore, $\Pi(\epsilon) > 0$ for small enough ϵ, which contradicts the optimality of the original mechanism. *Q.E.D.*

Next, to derive the law of motion of $q(.)$ in (77), let us prove the following Lemma.

Lemma 19 *If $IC(\beta(\theta), \theta)$, then $I(\theta) = 0$. If $IC(\beta(\theta), \theta)$ is slack, then $I(\theta) < 0$.*

Proof of Lemma 19: Let $S = \{\theta \in [\min \tau(\hat{\theta}), \max \tau(1)] : \theta \notin \tau(\beta(\theta))\}$. That is, S is the set of types such that $IC(\beta(\theta), \theta)$ is slack. Since τ is upper hemicontinuous and strictly increasing, $S = \cup_{i=1}^{\infty} (\underline{\theta}_i, \overline{\theta}_i)$ where $\underline{\theta}_i \leq \overline{\theta}_i \leq \underline{\theta}_{i+1}$, and for any $\theta \in [\min \tau(\hat{\theta}), \max \tau(1)] \setminus S_k$,

$IC(\beta(\theta),\theta)$ is binding and so $U_2(\beta(\theta),\theta)=0$. Also, there exists $\tilde{\theta}_i$ such that $\beta(\theta)=\tilde{\theta}_i$ for all $\theta\in[\underline{\theta}_i,\overline{\theta}_i]$, and $IC(\tilde{\theta}_i,\underline{\theta}_i)$ and $IC(\tilde{\theta}_i,\overline{\theta}_i)$ are binding.

Note that the number of intervals $(\underline{\theta}_i,\overline{\theta}_i)$ such that $IC(\tilde{\theta}_i,\theta)$ is non-binding for $\theta\in(\underline{\theta}_i,\overline{\theta}_i)$ is at most countable, because all such intervals are pairwise disjoint, their union is contained in $[0,1]$, and, being open, each such interval contains at least one rational number, while the number of rational numbers in an interval is countable.

Therefore, $U(\tilde{\theta}_i,\underline{\theta}_i)=U(\tilde{\theta}_i,\overline{\theta}_i)>U(\tilde{\theta}_i,\theta)$ for any $\theta\in(\underline{\theta}_i,\overline{\theta}_i)$, and hence

$$\int_{\underline{\theta}_i}^{\theta} U_2(\tilde{\theta}_i,s)ds \begin{cases} <0 & \text{if } \theta\in(\underline{\theta}_i,\overline{\theta}_i) \\ =0 & \text{if } \theta=\overline{\theta}_i \end{cases}. \tag{87}$$

Let $\overline{\theta}_0=\beta(\min\tau(\hat{\theta}))$. If $IC(\beta(\theta),\theta)$ is binding, then $\theta\in[\overline{\theta}_i,\underline{\theta}_{i+1}]$ for $i\geq 0$ and $U_2(\beta(\theta),\theta)=0$. So using (87), we obtain $I(\theta)=\int_{\min\tau(\hat{\theta})}^{\theta}U_2(\beta(x),x)dx=0$.

If $IC(\beta(\theta),\theta)$ is slack, then $\theta\in(\underline{\theta}_i,\overline{\theta}_i)$ for some $i\in\{1,..,N\}$, and so $I(\theta)=\int_{\min\tau(\hat{\theta})}^{\theta}U_2(\beta(x),x)dx=\int_{\underline{\theta}_i}^{\theta}U_2(\tilde{\theta}_i,s)ds<0$, where the first equality holds by the definition of $I(.)$, the second equality holds because $U_2(\beta(\theta),\theta)=0$ for all $\theta\in[\min\tau(\hat{\theta}),\max\tau(1)]\setminus S_k$, and the inequality holds by (87). It follows that $IC(\beta(\theta),\theta)$ is binding/non-binding if $I(\theta)=0/I(\theta)<0$. This completes the proof of Lemma 19. *Q.E.D.*

Now, we are in a position to complete the derivation of the law of motion for $q(.)$. We need to consider three cases.

(1) *Suppose that* $q(\theta)<q^{fb}(\theta)$. Since $q(.)$ is continuous by Theorem 3 in the main paper, it follows that there exists $\epsilon>0$ s.t. for all $\theta'\in(\theta-\epsilon,\theta+\epsilon)$, $q(\theta')<q^{fb}(\theta')$, and so by Lemma 9 in the main paper, $IC(\beta(\theta'),\theta')$ is binding. Hence $U_2(\beta(\theta),\theta)=0$, and (75) must hold i.e., $\dot{q}(\theta)=\dot{q}^{IC}((\theta)$.

(2) *Now suppose that* $I(\theta)<0$. Then $IC(\beta(\theta),\theta)$ is slack by Lemma 19. By continuity of $q(.)$ and $I(.)$, there exists $\epsilon>0$ s.t. for all $\theta'\in(\theta-\epsilon,\theta+\epsilon)$, we also have $I(\theta')<0$ and hence $IC(\beta(\theta'),\theta')$ and $q(\theta')=q^{fb}(\theta')$ by Lemma 9. So $\dot{q}(\theta)=\dot{q}^{fb}(\theta)$.

(3) *Now suppose that* $q(\theta)=q^{fb}(\theta)$ *and* $I(\theta)=0$. Then we must have $\dot{q}(\theta)\leq\dot{q}^{fb}(\theta)$, for otherwise $q(\theta')>q^{fb}(\theta')$ for $\theta'\in(\theta,\theta+\epsilon)$ for some $\epsilon>0$, which would contradict Lemma 9.

Suppose also that $\dot{q}^{fb}(\theta)<\dot{q}^{IC}(\theta)$. Then $\dot{q}(\theta)<\dot{q}^{IC}(\theta)$, and so $U_2(\beta(\theta),\theta)<0$. Hence, there exists $\epsilon>0$ s.t. for all $\theta'\in(\theta,\theta+\epsilon)$, $I(\theta')<0$ which implies that $IC(\beta(\theta'),\theta')$ is slack by Lemma 19, and so $q(\theta')=q^{fb}(\theta')$ by Lemma 9. Hence $\dot{q}(\theta)=\dot{q}^{fb}(\theta)$.

Now suppose that $\dot{q}^{fb}(\theta) \geq \dot{q}^{IC}(\theta)$. If $\dot{q}(\theta) > \dot{q}^{IC}(\theta)$, then $U_2(\beta(\theta), \theta) > 0$. Hence, there exists $\epsilon > 0$ s.t. for all $\theta' \in (\theta, \theta + \epsilon)$, $I(\theta') > 0$ which contradicts Lemma 19.

On the other hand, if $\dot{q}(\theta) < \dot{q}^{IC}(\theta)$, then $U_2(\beta(\theta), \theta) < 0$. Hence, there exists $\epsilon > 0$ s.t. for all $\theta' \in (\theta, \theta+\epsilon)$, $I(\theta') < 0$ and, by Lemma 19, $IC(\beta(\theta'), \theta')$ is slack, and so $q(\theta') = q^{fb}(\theta')$ by Lemma 9. But this contradicts $\dot{q}(\theta) < \dot{q}^{IC}(\theta) \leq \dot{q}^{fb}(\theta)$. Hence, $\dot{q}(\theta) = \dot{q}^{IC}(\theta)$. This completes the derivation of the law of motion of $q(.)$ in (77).

Finally, let us establish (83) and (82). For $\theta \in [\underline{\tau}, \overline{\tau}]$, let $A(\theta) = \sum_{k=1}^{s} f(\beta^k(\theta))\dot{\beta}^k(\theta)$. Then by recursion,

$$A(\theta) = \begin{cases} \dot{\beta}(\theta)[f(\beta(\theta)) + A(\beta(\theta))] & \text{if } \beta^2(\theta) \neq \emptyset \\ \dot{\beta}(\theta) f(\beta(\theta)) & \text{if } \beta^2(\theta) = \emptyset \end{cases} \tag{88}$$

Next, let

$$B(\theta) = \frac{u_q(q(\theta), \theta)}{u_q(q(\theta), \beta(\theta)) - u_q(q(\theta), \theta)}. \tag{89}$$

The optimality condition (74) in Theorem 11 implies that

$$A(\theta) = f(\theta)B(\theta) = \begin{cases} \dot{\beta}(\theta) f(\beta(\theta))[1 + B(\beta(\theta))] & \text{if } \beta^2(\theta) \neq \emptyset, \\ \dot{\beta}(\theta) f(\beta(\theta)) & \text{if } \beta^2(\theta) = \emptyset. \end{cases} \tag{90}$$

Therefore,

$$\dot{\beta}(\theta) = \begin{cases} \frac{f(\theta)B(\theta)}{f(\beta(\theta))[1+B(\beta(\theta))]} & \text{if } \beta^2(\theta) \neq \emptyset \\ \frac{f(\theta)B(\theta)}{f(\beta(\theta))} & \text{if } \beta^2(\theta) = \emptyset \end{cases} \tag{91}$$

Since $\beta^k(\theta) = \prod_{i=0}^{k-1} \dot{\beta}(\beta^i(\theta))$, we have

$$\dot{\beta}^k(\theta) = \begin{cases} \prod_{i=0}^{k-1} \frac{f(\beta^i(\theta))B(\beta^i(\theta))}{f(\beta^{i+1}(\theta))[1+B(\beta^{i+1}(\theta))]} & \text{if } \beta^{k+1}(\theta) \neq \emptyset, \\ \frac{f(\beta^{k-1}(\theta))B(\beta^{k-1}(\theta)}{f(\beta^k(\theta))} \prod_{i=0}^{k-2} \frac{f(\beta^i(\theta))B(\beta^i(\theta))}{f(\beta^{i+1}(\theta))[1+B(\beta^{i+1}(\theta))]} & \text{if } \beta^{k+1}(\theta) = \emptyset. \end{cases}$$

Using (89) in the above and setting $Q^k(\theta) = q(\beta^k(\theta))$ yields for $\theta \in [\underline{\tau}, \hat{\theta}]$:

$$\dot{\beta}^k(\theta) = \begin{cases} \frac{f(\theta)u_q(Q^0(\theta),\theta))[u_q(Q^k(\theta),\beta^{k+1}(\theta))-u_q(Q^k(\theta),\beta^k(\theta))]}{f(\beta^k(\theta))[u_q(Q^0(\theta),\beta^1(\theta))-u_q(Q^0(\theta),\theta)]u_q(Q^k(\theta),\beta^{k+1}(\theta))} \prod_{i=1}^{k-1} \frac{u_q(Q^i(\theta)),\beta^i(\theta))}{u_q(Q^i(\theta),\beta^{i+1}(\theta))} & \text{if } k < s(\theta) \\ \frac{f(\theta)u_q(Q^0(\theta),\theta)}{f(\beta^k(\theta))[u_q(Q^0(\theta),\beta^1(\theta))-u_q(Q^0(\theta),\theta)]} \prod_{i=1}^{k-1} \frac{u_q(Q^i(\theta)),\beta^i(\theta))}{u_q(Q^i(\theta),\beta^{i+1}(\theta))} & \text{if } k = s(\theta) \end{cases} \tag{92}$$

Equations (83) are derived from (74), (77), and the definition of Q^k. *Q.E.D.*

The optimal mechanism in the quadratic-uniform case under an intermediate cost C

Consider the following system of ordinary differential equation system presented in section 4.4 in the main paper:

$$\dot{\tau} = \frac{\theta - \tau}{\tau - Q}, \tag{93}$$

$$\dot{Q} = \frac{Q}{\tau - Q}, \tag{94}$$

With boundary conditions:

$$Q(1) = \tau(1) \tag{95}$$

$$Q(\hat{\theta}) = \tau(\hat{\theta}) \tag{96}$$

$$Q(\hat{\theta})(\hat{\theta} - \tau(\hat{\theta})) = C \tag{97}$$

First, let us make a change of variables:

$$y = \tau - Q, \qquad z = \tau + Q \tag{98}$$

Then the system (93)-(94) is equivalent to the following system:

$$\dot{y}y = \theta - z \tag{99}$$

$$\dot{z} = \frac{\theta}{y} - 1 \tag{100}$$

Differentiating (99) yields:

$$\ddot{y}y + (\dot{y})^2 = 1 - \dot{z} = 2 - \frac{\theta}{y} \tag{101}$$

Let us make another change of variables: $w = \frac{y^2}{4}$. Then (101) becomes:

$$\ddot{w}(\theta) = 1 - \frac{\theta}{4\sqrt{w(\theta)}} \tag{102}$$

The general solution to the differential equation (102) is parametric. Specifically, let b_1, b_2 and b_3 be some constants and $t \in [0, \infty)$ be a parameter. Then:

$$\theta(t) = b_1 t + b_2 t^{\frac{\sqrt{5}-1}{2}} + b_3 t^{-\frac{\sqrt{5}+1}{2}} \tag{103}$$

$$\frac{y^2(t)}{4} \equiv w(t) = \left(\frac{1}{2} b_1 t + \frac{\sqrt{5}-1}{4} b_2 t^{\frac{\sqrt{5}-1}{2}} - \frac{\sqrt{5}+1}{4} b_3 t^{-\frac{\sqrt{5}+1}{2}} \right)^2 \tag{104}$$

Indeed, note that we have:

$$\frac{d\frac{y^2(t)}{4}}{dt} \equiv \frac{dw(t)}{dt} = \left(b_1 + \frac{3-\sqrt{5}}{2}b_2 t^{\frac{\sqrt{5}-3}{2}} + \frac{3+\sqrt{5}}{2}b_3 t^{-\frac{\sqrt{5}+3}{2}}\right)\left(\frac{1}{2}b_1 t + \frac{\sqrt{5}-1}{4}b_2 t^{\frac{\sqrt{5}-1}{2}} - \frac{\sqrt{5}+1}{4}b_3 t^{-\frac{\sqrt{5}+1}{2}}\right) \tag{105}$$

$$\frac{d^2\frac{y^2(t)}{4}}{dt^2} \equiv \frac{d^2w(t)}{dt^2} = \frac{1}{2}\left(b_1 + \frac{3-\sqrt{5}}{2}b_2 t^{\frac{\sqrt{5}-3}{2}} + \frac{3+\sqrt{5}}{2}b_3 t^{-\frac{\sqrt{5}+3}{2}}\right)^2$$
$$+ \left(-\frac{7-3\sqrt{5}}{2}b_2 t^{\frac{\sqrt{5}-5}{2}} - \frac{7+3\sqrt{5}}{2}b_3 t^{-\frac{\sqrt{5}+5}{2}}\right)\left(\frac{1}{2}b_1 t + \frac{\sqrt{5}-1}{4}b_2 t^{\frac{\sqrt{5}-1}{2}} - \frac{\sqrt{5}+1}{4}b_3 t^{-\frac{\sqrt{5}+1}{2}}\right) \tag{106}$$

$$\theta'(t) = b_1 + \frac{\sqrt{5}-1}{2}b_2 t^{\frac{\sqrt{5}-3}{2}} - \frac{\sqrt{5}+1}{2}b_3 t^{-\frac{\sqrt{5}+3}{2}} \tag{107}$$

$$\theta''(t) = \frac{\sqrt{5}-1}{2}\frac{\sqrt{5}-3}{2}b_2 t^{\frac{\sqrt{5}-5}{2}} + \frac{\sqrt{5}+1}{2}\frac{\sqrt{5}+3}{2}b_3 t^{-\frac{\sqrt{5}+5}{2}} \tag{108}$$

Note that $\frac{d^2w}{d\theta^2} = \frac{\ddot{w}(t)}{(\theta'(t))^2} - \dot{w}(t)\frac{\theta''}{\theta'(t)^3}$. Therefore, the ODE (102) can be rewritten as follows:

$$\frac{\ddot{w}(t)}{(\theta'(t))^2} - \dot{w}(t)\frac{\theta''(t)}{\theta'(t)^3} = 1 - \frac{\theta(t)}{4\sqrt{w(t)}} \tag{109}$$

Note that we must have $0 \leq y < \theta$, since $y = \tau - Q$, $\tau < \theta$, and the optimal quantity Q cannot be greater than its first-best level, which in this case is equal to τ. So,

$$y(t) = \left| b_1 t + \frac{\sqrt{5}-1}{2}b_2 t^{\frac{\sqrt{5}-1}{2}} - \frac{\sqrt{5}+1}{2}b_3 t^{-\frac{\sqrt{5}+1}{2}} \right| \tag{110}$$

We can without loss of generality take that $\theta(1) = 1$. Indeed, if $\theta(t_1) = 1$ for some $t_1 \in (0, \infty)$, $t_1 \neq 1$, then we can replace the parameter t with the parameter $s = \frac{t}{t_1}$, and replace the constants b_1, b_2, b_3 with constants b_1', b_2', b_3' such that $b_1' = b_1 t_1$, $b_2' = b_2 t_1^{\frac{\sqrt{5}-1}{2}}$ and $b_3' = b_3 t_1^{-\frac{\sqrt{5}+1}{2}}$. Then we would have $\theta(s) = \theta(t)$ and $y(s) = y(t)$ for all $t \in [0, \infty)$, with $\theta(s)_{s=1} = 1$.

Using $\theta(1) = 1$ in (103) yields $b_1 + b_2 + b_3 = 1$. Also, $\theta(1) = 1$ and the boundary condition $\tau(1) = Q(1)$ imply that $y(1) = 0$. In turn, the latter implies that $b_1 + \frac{\sqrt{5}-1}{2}b_2 - \frac{\sqrt{5}+1}{2}b_3 = 0$. Now, we can solve for b_2 and b_3 in terms of b_1 to obtain:

$$b_2 = -b_1\frac{5+3\sqrt{5}}{10} + \frac{\sqrt{5}+1}{2\sqrt{5}}.$$

$$b_3 = b_1 \frac{3\sqrt{5}-5}{10} + \frac{\sqrt{5}-1}{2\sqrt{5}}.$$

Then (103) and (110) become:

$$\theta(t) = b_1 \left(t - \frac{1+3\sqrt{\frac{1}{5}}}{2} t^{\frac{\sqrt{5}-1}{2}} + \frac{3\sqrt{\frac{1}{5}}-1}{2} t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{\sqrt{5}+1}{2\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} + \frac{\sqrt{5}-1}{2\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} \tag{111}$$

$$y(t) = \left| b_1 \left(t - \frac{1+\sqrt{\frac{1}{5}}}{2} t^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2} t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{1}{\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} \right| \tag{112}$$

At first, let us suppose that the expression under the absolute value sign on the right-hand side of (112) is positive i.e:[13]

$$y(t) = b_1 \left(t - \frac{1+\sqrt{\frac{1}{5}}}{2} t^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2} t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{1}{\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} \tag{113}$$

Next, we solve the differential equation (100) for z, which we will also parameterize by t. So, we have $z'(t) \equiv \frac{dz}{dt} = z'(\theta)\theta'(t)$. By (111) and (113), $y(t) = \theta'(t)t$. Then (100) can be rewritten as:

$$z'(t) = \left(\frac{\theta}{y} - 1 \right) \theta'(t) = \frac{\theta}{\theta'(t)t}\theta'(t) - \theta'(t) = \frac{\theta}{t} - \theta'(t). \tag{114}$$

Substituting (111) for $\theta(t)$ we obtain:

$$z'(t) = b_1 \left(-\frac{1}{\sqrt{5}} t^{\frac{\sqrt{5}-3}{2}} + \frac{1}{\sqrt{5}} t^{-\frac{\sqrt{5}+3}{2}} \right) + \frac{\sqrt{5}-1}{2\sqrt{5}} t^{\frac{\sqrt{5}-3}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}} t^{-\frac{\sqrt{5}+3}{2}} \tag{115}$$

Integrating (115) yields:

$$z(t) = b_1 \left(-\frac{1+\sqrt{\frac{1}{5}}}{2} t^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2} t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{1}{\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} + k \tag{116}$$

where k is a constant of integration. Now, let us show that equation (99), $\dot{y(\theta)}y = \theta - z$, implies that the constant of integration k is equal to zero. Note that $y'(t) = \dot{y(\theta)}\theta'(t)$. So we can rewrite (99) as $y'(t)y = (\theta - z)\theta'(t)$. Since $y = \theta'(t)(t)$, the previous equation can be rewritten as follows: $y'(t)t = (\theta - z)$

[13]Later we will show that this is, indeed, the case since the opposite case when this expression is negative leads to a contradiction.

Next, from (113) we obtain:

$$y'(t)t = b_1\left(t - \frac{1}{\sqrt{5}}t^{\frac{\sqrt{5}-1}{2}} + \frac{1}{\sqrt{5}}t^{-\frac{\sqrt{5}+1}{2}}\right) + \frac{1-\sqrt{\frac{1}{5}}}{2}t^{\frac{\sqrt{5}-1}{2}} + \frac{1+\sqrt{\frac{1}{5}}}{2}t^{-\frac{\sqrt{5}+1}{2}} \tag{117}$$

Also, (111) and (116) yield:

$$\theta(t) - z(t) = b_1\left(t - \frac{1}{\sqrt{5}}t^{\frac{\sqrt{5}-1}{2}} + \frac{1}{\sqrt{5}}t^{-\frac{\sqrt{5}+1}{2}}\right) + \frac{\sqrt{5}-1}{2\sqrt{5}}t^{\frac{\sqrt{5}-1}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}}t^{-\frac{\sqrt{5}+1}{2}} - k \tag{118}$$

Equating (117) and (118) yields $k = 0$.

Furthermore, observe that $z(t) - y(t) = -b_1 t$. Since $z(t) - y(t) = 2Q(t)$, it follows that $Q(t) = -\frac{b_1}{2}t$ and so $b_1 < 0$.

Now, let us confirm that, as claimed, the expression under the absolute value sign on the right-hand side of (112) is positive. The proof is by contradiction, so suppose otherwise i.e.,

$$y(t) = -b_1\left(t - \frac{1+\sqrt{\frac{1}{5}}}{2}t^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}t^{-\frac{\sqrt{5}+1}{2}}\right) - \frac{1}{\sqrt{5}}t^{\frac{\sqrt{5}-1}{2}} + \frac{1}{\sqrt{5}}t^{-\frac{\sqrt{5}+1}{2}} \tag{119}$$

Then (111) and (119) yield $y(t) = -\theta'(t)t$ and so, instead of (114), we now have:

$$z'(t) = \left(\frac{\theta}{y} - 1\right)\theta'(t) = \frac{\theta}{-\theta'(t)t}\theta'(t) - \theta'(t) = -\frac{\theta}{t} - \theta'(t) = \frac{\theta}{t} - \theta'(t) - 2\frac{\theta}{t}. \tag{120}$$

Substituting (111) for $\theta(t)$ in (120) and integrating yields:

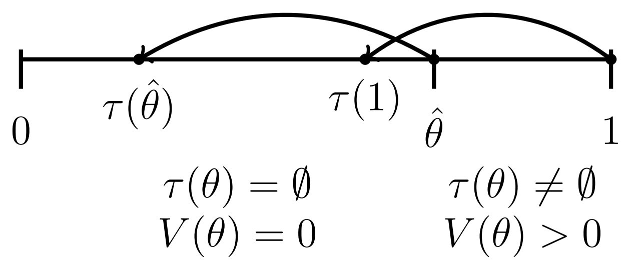
$$\begin{aligned} z(t) = {} & b_1\left(-\frac{1+\sqrt{\frac{1}{5}}}{2}t^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}t^{-\frac{\sqrt{5}+1}{2}}\right) + \frac{1}{\sqrt{5}}t^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}}t^{-\frac{\sqrt{5}+1}{2}} \\ & - 2b_1\left(t - \frac{1+3\sqrt{\frac{1}{5}}}{\sqrt{5}-1}t^{\frac{\sqrt{5}-1}{2}} - \frac{3\sqrt{\frac{1}{5}}-1}{\sqrt{5}+1}t^{-\frac{\sqrt{5}+1}{2}}\right) + \frac{\sqrt{5}+1}{\sqrt{5}(\sqrt{5}-1)}t^{\frac{\sqrt{5}-1}{2}} - \frac{\sqrt{5}-1}{\sqrt{5}(\sqrt{5}-1)}t^{-\frac{\sqrt{5}+1}{2}} + k_2 \end{aligned} \tag{121}$$

where k_2 is a constant of integration.

Since in this case $y = -\theta'(t)t$, the equation $y'(t)y = (\theta - z)\theta'(t)$ (i.e., equation (99) parameterized by t) can be rewritten as $-y'(t)t = (\theta - z)$. This equation can be rewritten as follows using (111) and (121) and differentiating (119):

$$-2b_1\left(t - \frac{1+3\sqrt{\frac{1}{5}}}{\sqrt{5}-1}t^{\frac{\sqrt{5}-1}{2}} - \frac{3\sqrt{\frac{1}{5}}-1}{\sqrt{5}+1}t^{-\frac{\sqrt{5}+1}{2}}\right) + \frac{\sqrt{5}+1}{\sqrt{5}(\sqrt{5}-1)}t^{\frac{\sqrt{5}-1}{2}} - \frac{\sqrt{5}-1}{\sqrt{5}(\sqrt{5}-1)}t^{-\frac{\sqrt{5}+1}{2}} + k_2 = 0$$

Figure 4: Structure of targeted types $\tau(.)$ and informational rents $V(.)$ under intermediate costs of lying



which cannot hold on any neighborhood of t.

Thus, we have confirmed that $y(t)$ is given by (113), and hence $y(t) = \theta'(t)t$. Since $y(t) \geq 0$, it follows that $\theta'(t) > 0$.

So, to complete the solution, it remains to characterize b_1 and $\hat{t}$ such that $\hat{t} < 1$ and $y(\hat{t}) = 0$ and $y(t) \geq 0$ for all $t \in [\hat{t}, 1]$. We will then have $\hat{\theta} = \theta(\hat{t}) < 1$. For this, we need to compute $y'(t)$ and $y''(t)$. We have:

$$y'(t) = b_1 + \frac{(\sqrt{5}-1) - 2b_1}{2\sqrt{5}} \hat{t}^{\frac{\sqrt{5}-3}{2}} + \frac{(\sqrt{5}+1) + 2b_1}{2\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}+3}{2}} \tag{122}$$

$$y''(t) = -\frac{3-\sqrt{5}}{2} \frac{(\sqrt{5}-1) - 2b_1}{2\sqrt{5}} \hat{t}^{\frac{\sqrt{5}-5}{2}} - \frac{\sqrt{5}+3}{2} \frac{(\sqrt{5}+1) + 2b_1}{2\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}+5}{2}} \tag{123}$$

Using (113) and (117) we obtain:

$$\begin{aligned} y(t) - ty'(t) = b_1 &\left(t - \frac{1+\sqrt{\frac{1}{5}}}{2} t^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2} t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{1}{\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} \\ &- b_1 \left(t - \frac{1}{\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} + \frac{1}{\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} \right) - \frac{1-\sqrt{\frac{1}{5}}}{2} t^{\frac{\sqrt{5}-1}{2}} - \frac{1+\sqrt{\frac{1}{5}}}{2} t^{-\frac{\sqrt{5}+1}{2}} = \\ &b_1 \left(-\frac{\sqrt{5}-1}{2\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} - \frac{\sqrt{5}+1}{2\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{3-\sqrt{5}}{2\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} - \frac{3+\sqrt{5}}{2\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} \end{aligned} \tag{124}$$

As established above, $b_1 < 0$. In fact, let us show that $b_1 \in [-\frac{\sqrt{5}+1}{2}, -1)$.

First, let us rule out $b_1 < -\frac{\sqrt{5}+1}{2}$. Observe that if $b_1 < -\frac{\sqrt{5}+1}{2}$, then by (122) $y'(t) < 0$ for all $t \geq 1$. Since $y(1) = 0$, it follows that $y(t) < 0$ for all $t > 1$ and $y(1-\epsilon) > 0$ for sufficiently small $\epsilon > 0$. Further, observe from (113) that $y(t) > 0$ when t is sufficiently small, with $\lim_{t\to 0+} y(t) = \infty$. Finally, (124) implies that $y'(t) < 0$ if $y(t) = 0$. So, if $b_1 < -\frac{\sqrt{5}+1}{2}$ then there does not exist $\hat{t} \neq 1$ such that $y(\hat{t}) = 0$.

Consider now $b_1 \in [-\frac{\sqrt{5}+1}{2}, 0]$. Note that in this case: (i) by (123), $y''(t) < 0$ for all t; (ii) $y(t) < 0$ when t is sufficiently small, with $\lim_{t\to 0+} y(t) = -\infty$, (iii) $y(t) < 0$ when t is sufficiently large, with $\lim_{t\to\infty} y(t) = -\infty$. (iv) By (122) $y'(1) = b_1 + 1$.

So, if $b_1 \in (-1, 0]$, then $y'(1) > 0$. This, in combination with (i)-(iii) above, implies that if $b_1 \in (-1, 0]$, then there exists a unique $\hat{t}$, $\hat{t} \neq 1$ such that $y(\hat{t}) = 0$ and, moreover, $\hat{t} > 1$ and $y(t) > 0$ for all $t \in (1, \hat{t})$. But we also have $y(t) = \theta'(t)t$ and $\theta(1) = 1$. So $\theta(t) > 1$ for all $t \in (1, \hat{t})$. This contradicts the fact that $\theta(t) \in [0, 1]$. Hence, we can rule out $b_1 \in (-1, 0]$.

Similarly, we can rule out $b_1 = -1$ because in this case $y(t) = 1$ only if $t = 1$.

Finally, if $b_1 \in [-\frac{\sqrt{5}+1}{2}, -1)$, then (i)-(iv) above imply that there exists $\hat{t}$, $\hat{t} < 1$ such that $y(\hat{t}) = 0$, and $y(t) > 0$ for all $t \in (\hat{t}, 1)$. Also, since $y(t) = \theta'(t)t$ and $\theta(1) = 1$, it follows that $\theta(t) \in [0, 1)$ for all $t \in (\hat{t}, 1)$. Moreover,

$$\begin{aligned}\theta(t) - y(t) &= b_1 \left(-\frac{1}{\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} + \frac{1}{\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{\sqrt{5}-1}{2\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} = \\ &= \frac{\sqrt{5}-1-2b_1}{2\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} + \frac{\sqrt{5}+1+2b_1}{2\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} \end{aligned} \tag{125}$$

To summarize, $\theta(t) - y(t) > 0$ and $\theta(t) \leq 1$ for all $t \in [\hat{t}, 1]$ when $b_1 \in [-\frac{\sqrt{5}+1}{2}, -1]$, as required for the solution. We conclude that $b_1 \in [-\frac{\sqrt{5}+1}{2}, -1)$.

Thus, the two remaining parameters completing the solution are $\hat{t} \in (0, 1)$ and $b_1 \in [-\frac{\sqrt{5}+1}{2}, -1)$. They are jointly determined as the solutions to the two equations: $y(\hat{t}) = 0$ where $y(\hat{t})$ is given by (119) and the boundary condition $Q(\hat{t})(\theta(\hat{t}) - \tau(\hat{t})) = C$.

Setting (119) to zero at $\hat{t}$ yields:

$$b_1 = -\frac{\frac{1}{\sqrt{5}} \hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}+1}{2}}}{\hat{t} - \frac{1+\sqrt{\frac{1}{5}}}{2} \hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2} \hat{t}^{-\frac{\sqrt{5}+1}{2}}} \tag{126}$$

Differentiating (126) we obtain for $\hat{t} \in (0,1)$:

$$\frac{\partial b_1}{\partial \hat{t}} = -\frac{\frac{\sqrt{5}-1}{2\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-3}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+3}{2}}}{\hat{t} - \frac{1+\sqrt{\frac{1}{5}}}{2}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}\hat{t}^{-\frac{\sqrt{5}+1}{2}}} + \frac{\left(\frac{1}{\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+1}{2}}\right)\left(1 - \frac{1}{\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-3}{2}} + \frac{1}{\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+3}{2}}\right)}{\left(\hat{t} - \frac{1+\sqrt{\frac{1}{5}}}{2}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}\hat{t}^{-\frac{\sqrt{5}+1}{2}}\right)^2}$$

$$= \frac{\frac{3-\sqrt{5}}{2\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{3+\sqrt{5}}{2\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+1}{2}} + \hat{t}^{-2}}{\left(\hat{t} - \frac{1+\sqrt{\frac{1}{5}}}{2}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}\hat{t}^{-\frac{\sqrt{5}+1}{2}}\right)^2} > 0 \tag{127}$$

where the last inequality follows from the fact that $\frac{3-\sqrt{5}}{2\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{3+\sqrt{5}}{2\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+1}{2}} + \hat{t}^{-2} = 0$ for $\hat{t} = 1$ and $\frac{\partial\left(\frac{3-\sqrt{5}}{2\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{3+\sqrt{5}}{2\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+1}{2}} + \hat{t}^{-2}\right)}{\partial \hat{t}} = \frac{(3-\sqrt{5})(\sqrt{5}-1)}{4\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-3}{2}} + \frac{(\sqrt{5}+1)(3+\sqrt{5})}{4\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+3}{2}} - 2\hat{t}^{-3} < 0$ for $\hat{t} \in (0,1)$.

Recall that $Q(\hat{t}) = \tau(\hat{t}) = -\frac{b_1}{2}\hat{t}$. Also, since $y(\hat{t}) = 0$, $\theta(\hat{t})$ is given by the right-hand side of (125). Using this, we can rewrite the boundary condition $Q(\hat{t})(\theta(\hat{t}) - \tau(\hat{t})) = C$ as follows:

$$F(b_1, \hat{t}, C) \equiv -\frac{b_1}{2}\left(b_1\left(\frac{\hat{t}^2}{2} - \frac{1}{\sqrt{5}}\hat{t}^{\frac{\sqrt{5}+1}{2}} + \frac{1}{\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}-1}{2}}\right) + \frac{\sqrt{5}-1}{2\sqrt{5}}\hat{t}^{\frac{\sqrt{5}+1}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}-1}{2}}\right) - C = 0 \tag{128}$$

Next, from (127) and (128) we get $\frac{dF}{dC} = -1 < 0$ and

$$\frac{dF(b_1(\hat{t}), \hat{t}, C)}{d\hat{t}} = -\frac{b_1}{2}y(\hat{t}) - \frac{\partial b_1}{\partial \hat{t}}\left(\frac{b_1}{2}\hat{t}^2 + \frac{\sqrt{5}-1-4b_1}{4\sqrt{5}}\hat{t}^{\frac{\sqrt{5}+1}{2}} + \frac{\sqrt{5}+1+4b_1}{4\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}-1}{2}}\right) > 0.$$

The last inequality holds since: (i) $y(\hat{t}) = 0$; (ii) $\frac{\partial b_1}{\partial \hat{t}} > 0$ as shown in (127); (iii) the multiplier of $\frac{\partial b_1}{\partial \hat{t}}$, $\frac{b_1}{2}\hat{t}^2 + \frac{\sqrt{5}-1-4b_1}{4\sqrt{5}}\hat{t}^{\frac{\sqrt{5}+1}{2}} + \frac{\sqrt{5}+1+4b_1}{4\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}-1}{2}}$, is negative when $\hat{t} = 1$ and $b_1 < -1$ and is increasing in $\hat{t}$ at any $\hat{t} \in (0,1)$ and $b_1 < -1$.

Next, applying l'Hospital's rule to (126) we obtain:

$$\lim_{\hat{t}\to 1} b_1(\hat{t}) = -\frac{\lim_{\hat{t}\to 1}\left(\frac{\sqrt{5}-1}{2\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-3}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+3}{2}}\right)}{\lim_{\hat{t}\to 1}\left(1 - \frac{1}{\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-3}{2}} + \frac{1}{\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+3}{2}}\right)} = -1.$$

So, $\lim_{\hat{t}\to 1} F(b_1(\hat{t}), \hat{t}, C) = \frac{1}{4} - C$.

On the other hand, $\lim_{\hat{t}\to 0} b_1(\hat{t}) = -\frac{\sqrt{5}+1}{2}$, and so $\lim_{\hat{t}\to 0} F(b_1(\hat{t}), \hat{t}, C) = -C$, From the above we conclude that for $C \in (0, \frac{1}{4})$ there exist a unique solution $\hat{t} \in (0,1)$ to the equation $F(b_1(\hat{t}), \hat{t}, C) = 0$ and that $\frac{d\hat{t}}{dC} > 0$.

Now let us establish the interval of C on which our solution applies. The upper bound of C is equal to $\frac{1}{4}$, since for $C > \frac{1}{4}$ no incentive constraints are binding. To establish the lower bound of C, $\underline{C}_1$, note that our solution applies when $\hat{\theta} \geq \tau(1)$. At $\underline{C}_1$ we then have $\hat{\theta} = \tau(1) = Q(1)$. Let $\hat{t}_m$, and $b_{1,m}$ denote the parameter values where the latter condition holds. Then we can rewrite the boundary condition $Q(\hat{\theta})(\hat{\theta} - \tau(\hat{\theta})) = C$ as follows:

$$\begin{aligned} Q(\hat{t}_m)(Q(1) - Q(\hat{t}_m)) &= \underline{C}_1 \\ \frac{(b_{1,m})^2}{4}\hat{t}_m(1-\hat{t}_m) &= \underline{C}_1 \end{aligned} \tag{129}$$

So, $\underline{C}_1$, $\hat{t}_m$, and $b_{1,m}$ are determined by (126), (129) and condition $\theta(\hat{t}_m) = \tau(1) = Q(1)$ Since $\tau(1) = Q(1) = -\frac{b_{1,m}}{2}$, we can equate the latter to $\theta(\hat{t}_m)$ as given by (125), since $y(\hat{t}_m) = 0$, to obtain:

$$b_{1,m}\left(-\frac{1}{2} + \frac{1}{\sqrt{5}}\hat{t}_m^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}}\hat{t}_m^{-\frac{\sqrt{5}+1}{2}}\right) = \frac{\sqrt{5}-1}{2\sqrt{5}}\hat{t}_m^{\frac{\sqrt{5}-1}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}}\hat{t}_m^{-\frac{\sqrt{5}+1}{2}} \tag{130}$$

Using (126) in (130) and simplifying yields:

$$\begin{aligned} &-\left(\frac{1}{\sqrt{5}}\hat{t}_m^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}}\hat{t}_m^{-\frac{\sqrt{5}+1}{2}}\right)\left(-\frac{1}{2} + \frac{1}{\sqrt{5}}\hat{t}_m^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}}\hat{t}_m^{-\frac{\sqrt{5}+1}{2}}\right) = \\ &\left(\frac{\sqrt{5}-1}{2\sqrt{5}}\hat{t}_m^{\frac{\sqrt{5}-1}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}}\hat{t}_m^{-\frac{\sqrt{5}+1}{2}}\right)\left(\hat{t}_m - \frac{1+\sqrt{\frac{1}{5}}}{2}\hat{t}_m^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}\hat{t}_m^{-\frac{\sqrt{5}+1}{2}}\right) \end{aligned} \tag{131}$$

The last equation simplifies to:

$$\hat{t}_m^{\sqrt{5}+1}(1-\sqrt{5}) + \hat{t}_m^{\sqrt{5}} - \hat{t}_m(1+\sqrt{5}) + 2\sqrt{5}\hat{t}_m^{\frac{\sqrt{5}-1}{2}} - 1 = 0 \tag{132}$$

The approximate root of the last equation in $(0,1)$ is $\hat{t}_m = 0.187169$. Then from (126) we obtain $b_{1,m} \approx -1.554$ and from (129), $\underline{C}_1 \approx 0.0918$.

Let us now establish some useful comparative statics results. First, we have:

$$\frac{d\hat{\theta}}{dC} = \frac{\partial\hat{\theta}}{\partial b_1}\frac{\partial b_1}{\partial \hat{t}}\frac{d\hat{t}}{dC} + \theta'(\hat{t})\frac{d\hat{t}}{dC} = \left(\hat{t} - \frac{1+3\sqrt{\frac{1}{5}}}{2}\hat{t}^{\frac{\sqrt{5}-1}{2}} + \frac{3\sqrt{\frac{1}{5}}-1}{2}\hat{t}^{-\frac{\sqrt{5}+1}{2}}\right)\frac{\partial b_1}{\partial \hat{t}}\frac{d\hat{t}}{dC} > 0 \tag{133}$$

The second equality follows from the fact that $\theta'(\hat{t}) = \frac{y(\hat{t})}{\hat{t}} = 0$ and (111), while the last inequality holds because, as established above, $\frac{\partial b_1}{\partial \hat{t}} > 0$, $\frac{d\hat{t}}{dC} > 0$, and $\hat{t} - \frac{1+3\sqrt{\frac{1}{5}}}{2}\hat{t}^{\frac{\sqrt{5}-1}{2}} + \frac{3\sqrt{\frac{1}{5}}-1}{2}\hat{t}^{-\frac{\sqrt{5}+1}{2}} = 0$ if $\hat{t} = 1$ and $\frac{\partial\left(\hat{t} - \frac{1+3\sqrt{\frac{1}{5}}}{2}\hat{t}^{\frac{\sqrt{5}-1}{2}} + \frac{3\sqrt{\frac{1}{5}}-1}{2}\hat{t}^{-\frac{\sqrt{5}+1}{2}}\right)}{\partial \hat{t}} < 0$ for any $\hat{t} \in (0,1)$.

We can now confirm that $\hat{\theta} > \tau(1)$ for $C \in (\underline{C}_1, \frac{1}{4})$. We have shown above that $\frac{d\hat{\theta}}{dC} > 0$. Next, since $\tau(1) = Q(1) = -\frac{b_1}{2}$, we have $\frac{d\tau(1)}{dC} = -\frac{1}{2}\frac{db_1}{d\hat{t}} < 0$ where b_1 is given by (126). So, since $\hat{\theta} = \tau(1)$ at $C = \underline{C}_1$, it follows that $\hat{\theta} > \tau(1)$ when $C \in (\underline{C}_1, \frac{1}{4})$, as required.

To obtain the comparative statics for $\tau(\hat{\theta})$, recall that $\tau(\hat{\theta}) = Q(\hat{\theta}) = -\frac{b_1}{2}\hat{t}$. Therefore, $\frac{d\tau(\hat{\theta})}{dC} = \frac{d\tau(\hat{\theta})}{d\hat{t}}\frac{d\hat{t}}{dC} = \left(-\frac{b_1}{2} - \frac{1}{2}\hat{t}\frac{\partial b_1}{\partial \hat{t}}\right)\frac{d\hat{t}}{dC}$. Using (126) and (127) we obtain:

$$
\begin{aligned}
&-\frac{b_1}{2} - \frac{1}{2}\hat{t}\frac{\partial b_1}{\partial \hat{t}} = \frac{1}{2}\frac{\frac{1}{\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+1}{2}}}{\hat{t} - \frac{1+\sqrt{\frac{1}{5}}}{2}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}\hat{t}^{-\frac{\sqrt{5}+1}{2}}} - \frac{1}{2}\hat{t}\frac{\frac{3-\sqrt{5}}{2\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{3+\sqrt{5}}{2\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+1}{2}} + \hat{t}^{-2}}{(\hat{t} - \frac{1+\sqrt{\frac{1}{5}}}{2}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}\hat{t}^{-\frac{\sqrt{5}+1}{2}})^2} \\
=&\frac{1}{2}\frac{(\frac{1}{\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+1}{2}})(\hat{t} - \frac{1+\sqrt{\frac{1}{5}}}{2}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}\hat{t}^{-\frac{\sqrt{5}+1}{2}}) - \hat{t}\left(\frac{3-\sqrt{5}}{2\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{3+\sqrt{5}}{2\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+1}{2}} + \hat{t}^{-2}\right)}{\left(\hat{t} - \frac{1+\sqrt{\frac{1}{5}}}{2}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}\hat{t}^{-\frac{\sqrt{5}+1}{2}}\right)^2} \\
=&\frac{\frac{\sqrt{5}-1}{2\sqrt{5}}\hat{t}^{\frac{\sqrt{5}+1}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}-1}{2}} - \frac{\sqrt{5}+1}{10}\hat{t}^{\sqrt{5}-1} + \frac{\sqrt{5}-1}{10}\hat{t}^{-(\sqrt{5}+1)} - \frac{4}{5}\hat{t}^{-1}}{(\hat{t} - \frac{1+\sqrt{\frac{1}{5}}}{2}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}\hat{t}^{-\frac{\sqrt{5}+1}{2}})^2} \qquad (134)
\end{aligned}
$$

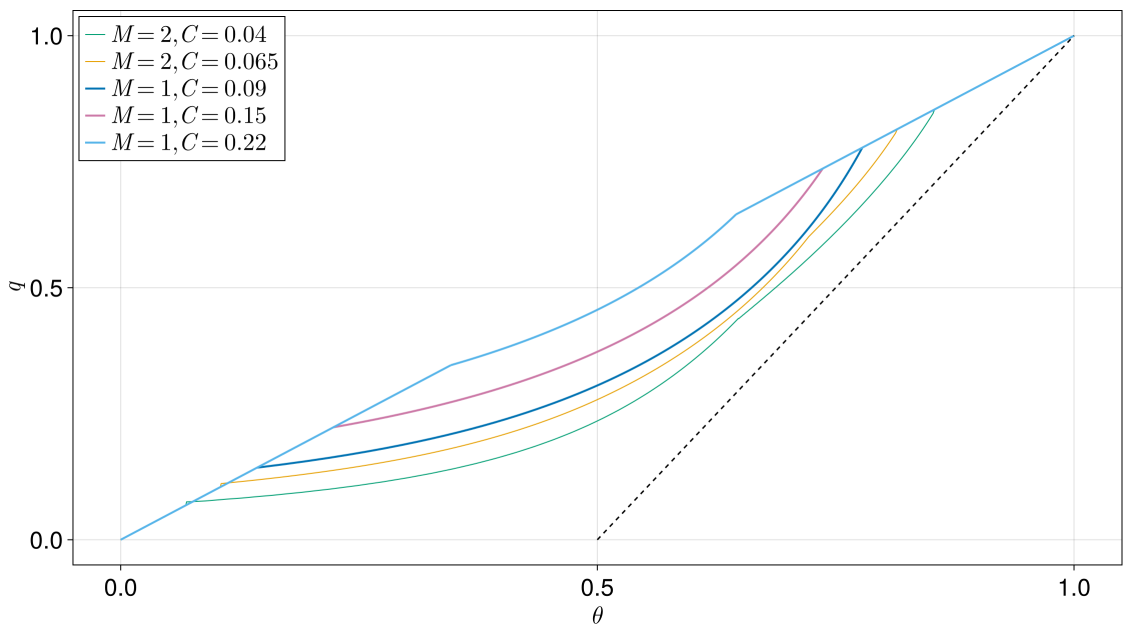
Let $G(\hat{t})$ be the numerator of the last equation in (134). Note that $G(1) = 0$, and $\frac{\partial G}{\partial \hat{t}} = \frac{1}{\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+1}{2}} - \frac{4}{10}\hat{t}^{\sqrt{5}-2} - \frac{4}{10}\hat{t}^{-(\sqrt{5}+2)} + \frac{4}{5}\hat{t}^{-2} = \frac{1}{\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+1}{2}}$ $- \frac{4}{10}\hat{t}^{-2}(\hat{t}^{-\sqrt{5}} - 1)(1 - \hat{t}^{\sqrt{5}})$. So, $\frac{\partial G}{\partial \hat{t}} < 0$ for all $\hat{t} \in (0,1)$. Hence, $G(\hat{t}) > 0$ for all $\hat{t} \in (0,1)$, which by (134) means that $-\frac{b_1}{2} - \frac{1}{2}\hat{t}\frac{\partial b_1}{\partial \hat{t}} > 0$ for $\hat{t} \in (0,1)$. Since $\frac{d\hat{t}}{dC} > 0$, we conclude that $\frac{d\tau(\hat{\theta})}{dC} > 0$.

Finally, let us show that Q is convex in τ. Note that $\tau(t) = y(t) + Q(t) = y(t) - \frac{b_1}{2}t$. Therefore, $\frac{dy}{d\tau} = \frac{y'(t)}{\tau'(t)} = \frac{y'(t)}{y'(t) - \frac{b_1}{2}}$ and $\frac{d^2y}{d\tau^2} = \frac{\frac{d\frac{dy}{d\tau}}{dt}}{\tau'(t)} = \frac{-\frac{b_1}{2}y''(t)}{(y'(t) - \frac{b_1}{2})^3}$. From (122), $\tau'(t) = y'(t) - \frac{b_1}{2} = \frac{1}{2}b_1 + \frac{(\sqrt{5}-1)-2b_1}{2\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-3}{2}} + \frac{(\sqrt{5}+1)+2b_1}{2\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+3}{2}}$ which is equal to $\frac{b_1}{2} + 1 > 0$ when $t = 1$. Since $y''(t) < 0$ by (123), it follows that $y'(t) - \frac{b_1}{2} > 0$ for $t \in (0,1)$. So, $\frac{d^2y}{d\tau^2} < 0$. Since $Q = \tau - y$, we have $\frac{dQ}{d\tau} = 1 - \frac{dy}{d\tau} = \frac{-\frac{b_1}{2}}{y'(t) - \frac{b_1}{2}} > 0$ and $\frac{d^2Q}{d\tau^2} = -\frac{d^2y}{d\tau^2} > 0$.

Also, since $Q(t) = q(\tau(t))$, we have $Q'(t) = q'(\tau(t))\tau'(t)$. So, since $Q'(t) > 0$ and $\tau'(t) > 0$, it follows that $q'(\theta) \equiv q'(\tau(t)) > 0$ for $\theta = \tau(t)$. Finally, differentiating $Q'(t) = q'(\tau(t))\tau'(t)$ we get: $0 = Q''(t) = q''(\tau(t))(\tau'(t))^2 + q'(\tau(t))\tau''(t)$. Since $\tau''(t) = y''(t) < 0$,

Figure 5: Optimal mechanism in quadratic-uniform case

(a) Optimal quantities



(b) Optimal targeted types

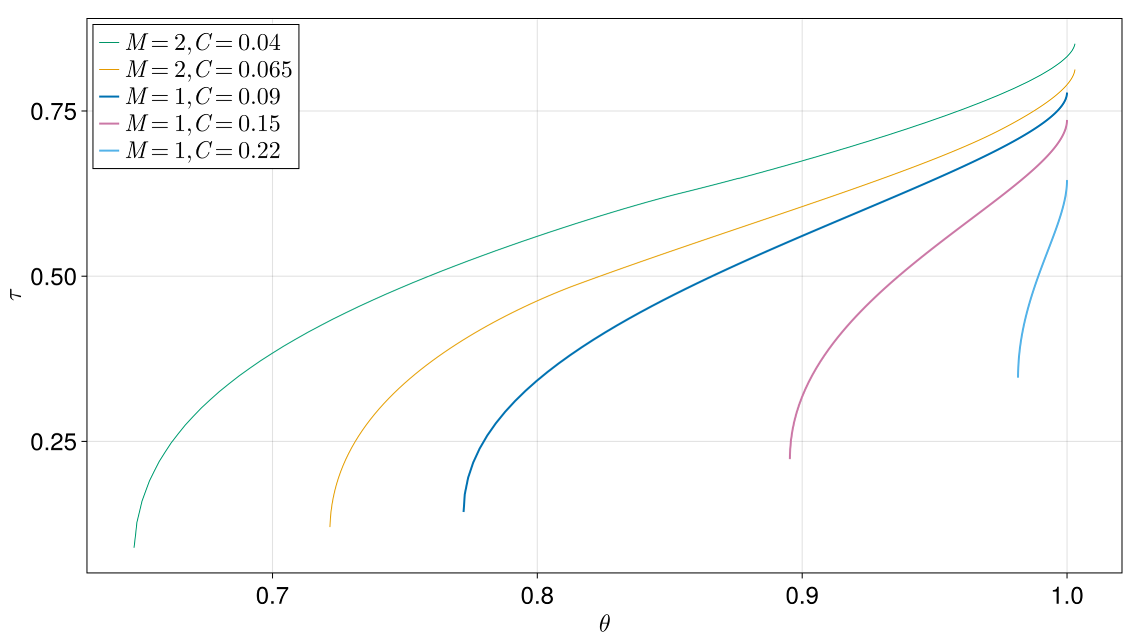
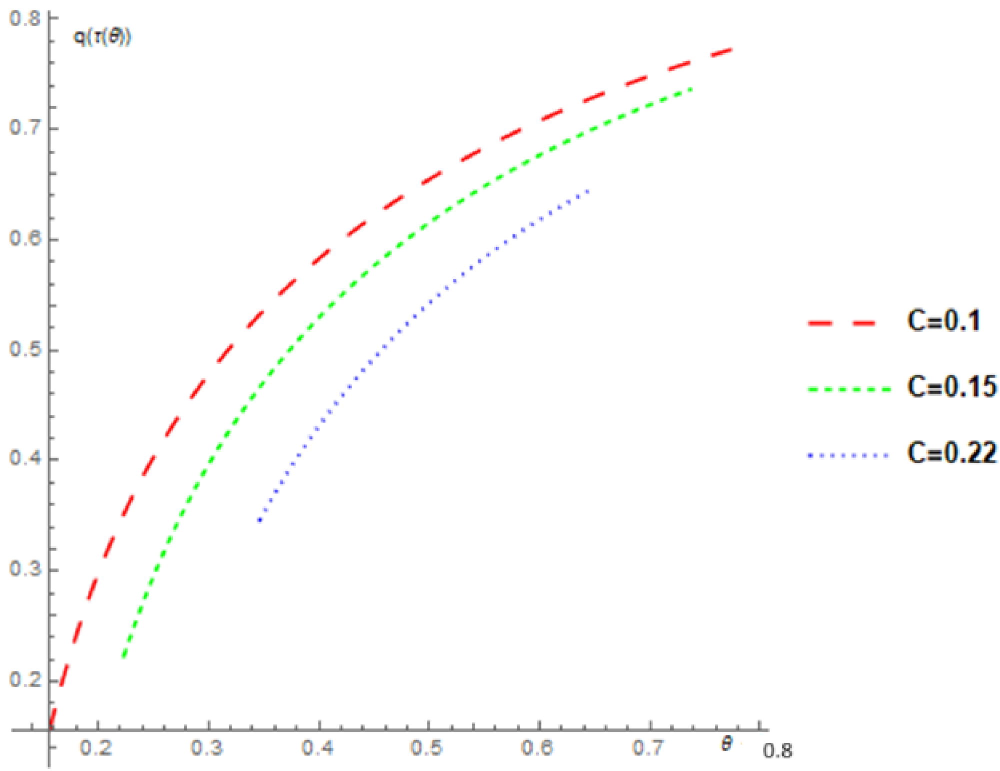


Figure 6: Quantity $q(\tau(\theta))$ of the targeted type $\tau(\theta)$



we conclude that $q''(\theta) \equiv q''(\tau(t)) > 0$ for $\theta \in (\tau(\hat{\theta}), \tau(1))$. So $q(\theta)$ is strictly increasing and convex for $\theta \in (\tau(\hat{\theta}), \tau(1))$. This completes the analysis of the quadratic-uniform case.