Screening Under A Fixed Cost of Misrepresentation*

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Abstract

This paper studies an optimal screening problem in which an agent incurs a fixed cost of lying when she misrepresents her private information. In the optimal mechanism for this environment, local incentive constraints are not binding, and standard techniques for solving screening problems are not applicable. Significantly, the problem can no longer be dichotomized into two parts solved sequentially: an implementability part which involves an envelope condition and the monotonicity of the allocation, and an optimization part. We develop a new methodology to tackle this problem, characterize the optimal mechanism, and compute it in special cases. Our method involves introducing and characterizing an endogenous "targeted type" correspondence that reflects binding non-local incentive constraints, and then jointly solving for the targeted type correspondence and the optimal allocation. The optimal mechanism has a number of novel qualitative properties, such as lack of exclusion and first-best efficient allocation at high- and low- ends of the spectrum of types. Also, bunching never occurs, as the optimal quantity allocation is increasing in type irrespectively of type distribution.

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1 Introduction

This paper studies a screening problem in which an uninformed principal interacts with a privately informed agent who incurs a fixed cost of misrepresenting her private information. The analysis of the fixed cost of such misrepresentation, or lying, is novel and, as we argue below, well-motivated, and produces qualitatively new and interesting results.

Whereas most literature on contracts and mechanism design assumes that a privately informed party is unconstrained in her ability to misrepresent and manipulate her information, several strands in this literature have explored alternative frameworks in which misrepresentation is costly. A notable direction in this research, which originated in the work of Lacker and Weinberg (1989) and has been further developed by Maggi and Rodriguez-Clare (1995), Crocker and Morgan (1998) and Deneckere and Severinov (2022), considers settings in which an agent incurs a cost increasing in the size of her “lie” or type misrepresentation.

Another strand of literature on honesty in mechanisms, which includes Alger and Ma (2003), Alger and Renault (2006, 2007), and Severinov and Deneckere (2006) has explored situations in which a principal has to deal with a population of agents some of whom are “honest” and are not able to misrepresent their private information, whereas a complementary fraction consists of fully “strategic” agents who behave in a standard fashion. This paper differs from both of these literatures in studying a setting in which the cost of misrepresentation or lying is finite and does not depend on the magnitude of a “lie.”

Misrepresentation costs may exist for several reasons. First, misrepresenting the truth may require costly effort or actions either to manufacture evidence or, conversely, to hide evidence that reveals the true state of the world and to conceal one’s information. For example, a firm seeking a contract or an individual seeking a promotion may need to be perceived as productive, competent and/or creditworthy. This goal may be attained by manufacturing “evidence” exaggerating prior performance and concealing the risk of non-performance. It is plausible that the cost or the effort required to produce such favorable but false “evidence” is independent of the magnitude of misrepresentation. For instance, the cost of misrepresentation or concealment could involve the loss of future business or reputation which has a “once and for all” nature and is unrelated to the size of misrepresentation.

A fixed cost of lying associated with reputation loss is embodied in a common law legal principle “Falsus in uno, falsus in omnibus” (“false in one thing, false in everything”), which states that someone who lies about one matter is not credible regarding any other one.

From a game-theoretic perspective, a fixed cost of lying associated with a loss of reputation can be modeled as the payoff difference between a ‘’good equilibrium" and a "bad equilibrium" in a repeated interaction where any revealed lie triggers the bad equilibrium.[1]

Second, the cost of misrepresentation may have psychological or ethical nature, reflecting moral barriers, feelings of shame and of "betrayal" of one's identity, discomfort, or stress.[2] Since engaging in misrepresentation or not is often a binary decision, its size would not affect these negative psychological effects. These psychological costs have economic consequences. For instance, individuals may experience discomfort and awkwardness when selecting options that do not align with their identity and social perceptions, such as adults ordering kids' meals in a restaurant, or non-students claiming student discounts. The sellers can and apparently do exploit this "misrepresentation guilt" and concern of being negatively perceived by others in price discrimination, to influence purchase decisions via specific identity-based labelling of their products.

Third, studies in cognitive science and neuroscience indicate that lying is costly because it requires more cognitive resources (Van't Veer, Stel and van Beest (2014), Vrij et al. (2011), Christ et al. (2008)). Therefore, if the potential benefit of lying is small, people tend not to think about it and stay honest as a default choice. On the other hand, if the temptation to lie is high enough, individuals tend to take full advantage of it regardless of the extent of the lie. For example, for a consumer pretending to be mildly interested in a product may not be easier than pretending to be not interested at all. Thus, this literature provides support to the hypothesis of a fixed cost of lysing.

[1]For example, consider an (infinitely) repeated screening problem with two competing agents. In every period, each agent's private valuation is randomly drawn, and the principal chooses one of them to interact with in the current period. In each interaction, the agent reports her private information, the principal implements an allocation, and the agent's true information is revealed at the end of the period. Consider an equilibrium in which the principal switches to another agent if and only if misreporting by the currently employed agent is revealed. The principal's strategy is sequentially rational because the two agents have the same productivity distributions in every period. Then an active agent's fixed cost of lying in each period is the difference between the expected continuation payoffs following truth-telling and following lying.

[2]Behavioral psychologists have studied a number of physical symptoms associated with emotional discomfort and "feeling wrong" that people experience when lying, including blushing, gaze aversion, elevated eye-blink rate, etc. See, for example, Ekman(1988, 2003), Porter and Ten Brinke (2008).

There is substantial experimental evidence showing that individuals are averse to lying and incur a cost when doing so. In particular, Abeler, Becker and Falk (2014) measure the intrinsic cost of lying in a setup where other motives such as reputational and efficiency concerns, altruism and conditional cooperation can be ruled out, and find that lying costs are significant and widespread. Kajackaite and Gneezy (2017) report experimental data indicating that intrinsic costs of lying are positive and finite. They conclude that "the evidence suggests that lying is a "normal" good for which people compare the intrinsic cost and benefit of the lie, and when the benefit from lying is higher than the intrinsic cost of lying, they lie." Preuter, Jaeger and Stel (2024) experimentally confirm that lying is associated with negative psychological consequences, particularly, a decrease in self-esteem and negative emotions. Experimental studies have also found that the cost of lying exists in various settings such as public service (Hanna and Wang (2017)), banking industry(Cohn, Fehr and Marechal (2014)) and school misconduct (Cohn and Marechal (2018)).

While most experimental studies indicate that lying costs exist, the exact shape and nature of these costs remain unclear. Gneezy, Kajackaite and Sobel (2018) provide evidence that the size of a lie has a small effect on the cost of lying. On the other hand, Hilbig and Hessler (2013) find that willingness to lie decreases with the degree of misrepresentation, which suggests that the cost of lying is increasing in the size of lie. It is likely that in reality, the cost of lying includes both fixed and variable parts.

In this paper, we adopt the fixed cost of lying hypothesis and investigate its effect on the optimal mechanism and pricing. As we show, the presence of such cost reshapes the landscape of the optimal screening problem and leads to qualitatively new results.

Reflecting this approach, our formal model posits the existence of a set of messages M which are differentially costly for different agent types. A screening mechanism or a contract in this setting can be modelled as a mapping from M to the set of physical allocations (such as quantities and prices). Equivalently, one can consider M as a part of the allocation space, and represent a mechanism offered by the principal to the agent as a menu of allocations each of which contains submission of an element from M in addition to a standard allocation.

To provide some examples of message set M used in the real world, firms in many industries, such as car dealerships, insurance companies and airlines, try to elicit information related to customers' willingness to pay, including income, occupation, demographic status, as well as the customers' tastes and habits before making a sale to them. This information is often collected through voluntary questionnaires, although sometimes customers are requested to provide evidence supporting their claims. For instance, insurance companies price car insurance on the basis of self-reported consumer characteristics, such as the percentage of driving to and from work, which are typically unverifiable, and indeed appear to remain unverified. Car dealers are known to price discriminate on the basis of information, disclosures and signals obtained from customers through personal interviews.[3]

In the next section we formally describe our model and, to simplify the analysis, use an argument based on the Revelation Principle to map it into a direct mechanism in which the agent experiences a fixed cost when misrepresenting her type. Notably, we show that truthful reporting is a necessary part of behavior in the optimal mechanism, rather than an analytical short-cut, as in the standard setting.

The first significant difference between our problem and the standard screening environment is that local incentive constraints are no longer binding in our setting. Indeed, imitating a close-by type yields a lower payoff than truthtelling. Therefore, we can no longer use the standard Mirrlees' method to derive the agent's surplus from the first-order condition and omit incentive constraints. Instead, we need to identify non-local incentive constraints that are binding at the optimum. To this end, we introduce a concept of a "targeted type" $\tau(\theta)$ - to which type θ has a binding incentive constraint. Notably, $\tau(\theta)$ is endogenous, and its choice is one of the elements of the optimal design.

Further, targeted types form "chains." Specifically, if type θ targets some θ' i.e., $\tau(\theta) = \theta'$, and type θ' targets some θ'' i.e., $\tau(\theta') = \theta''$, then the types $\theta, \theta', \theta''$ are part of a single chain. Our methodology for characterizing the optimal mechanism involves optimizing over the chains of targeted types. This approach allows us to derive necessary and sufficient conditions for the optimal mechanism, which take the form of ordinary differential equations for the optimal quantity $q(\theta)$ and the targeted type $\tau(\theta)$. We are able to derive a closed-

[3]Car dealers employ various methods and techniques to elicit such information, such as the "Four Square Negotiating." Eskeldson (2000) describes it as follows: "A car salesperson will sit you down in front of a blank piece of paper divided into four quadrants. In each quadrant (s)he will fill in values for the price, the trade-in value, the down payment and the monthly lease rate." So this technique essentially boils down to querying a customer about their tastes and financial capabilities, which is then used to set the monthly payments. Figuring the optimal way to provide information in this negotiation is a cognitively demanding task for a customer. This technique is so common that apprentice salespersons are trained how to use it.

form solution and exhibit the optimal mechanism explicitly for quadratic utility function and uniform type distribution, under intermediate costs.

The overall structure of the optimal mechanism involves an endogenous partition of the type space into intervals such that any type in an interval targets some type in the adjacent lower interval.

Significantly, the monotonicity of the quantity allocation in type is no longer a necessary condition for implementation in our setup. In contrast to the standard setting, non-monotone allocations are implementable even when single-crossing property (SCP) holds. However, the optimal quantity allocation is strictly increasing in type, regardless of the type distribution. The reason behind this is two-fold. For one thing, an increasing quantity schedule is more profitable for the firm because it is more efficient. Further, under the fixed cost of misrepresentation, one can always make the quantity schedule at least slightly increasing locally, without affecting incentive constraints.

Full allocative efficiency is achieved in the optimal mechanism on intervals of low and high types: they are assigned the first-best quantities, while middle type experience downward quantity distortions. This result is in contrast to the standard "sacrifice efficiency of low types to extract more rent from the high types" logic. No quantity distortion is needed for low types because, with a positive fixed cost, it is not worth it for any type to imitate a low type even if the latter is assigned her first-best quantity. The intuition behind the efficiency of the allocation for the high-value types is different: because high types are assigned large quantities and pay large transfers, only nearby higher types can potentially want to imitate them. But since the imitator and the imitated types are close, the extra benefit from imitation is outweighed by the fixed cost of misrepresentation.

The efficiency of the allocation for the low types also means that there is no exclusion in the sense that every type with a positive valuation receives a positive quantity. Severinov and Deneckere (2006) established a no-exclusion property when there is a positive fraction of completely honest agents. This paper shows that this property also holds when there are intermediate barriers to the agents' opportunism in the form of a fixed cost.

The comparative statics of the optimal mechanism in the fixed cost of lying is as follows. As this cost decreases, the number of intervals in the partition generated by targeted types increases, the distance between a type and her targeted type gets smaller i.e., $\tau(\theta) \to \theta$, and the optimal quantity allocation and transfers converge to the standard second-best.

Conversely, the number of intervals decreases as the fixed cost becomes large. In the limit, binding incentive constraints disappear and the quantity allocation becomes the first-best. While not surprising, these limiting results provide an insight that the second-best and the first-best can be seen as two extreme cases as lying costs vary. Our model attains these limits and also allows us to understand what happens under intermediate costs of lying. Particularly, for a range of intermediate costs, the type partition contains two elements, two intervals of types. Put otherwise, every chain of types contains two types because a targeted type does not target another type herself. We characterize this case explicitly under quadratic utility and uniform type distribution.

Thus the contribution of this paper is two-fold. First, we characterize the optimal screening mechanism when the agent incurs a fixed cost of lying, and highlight important qualitative properties of this mechanism.

Our second contribution is methodological, as the paper develops new techniques to solve a class of screening problems in which local incentive constraints are not binding and which, in contrast to standard ones, cannot be dichotomized into two parts, an implementability part which involves an envelope condition and a monotonicity restriction on an allocation profile, and the second part that involves optimization under those constraints. The key elements of our approach, such as the characterization of binding non-local incentive constraints and the associated "targeted types," and solution techniques provide analytical instruments for different applications in which lying costs exist, and could also be useful for solving other problems with binding non-local incentive constraints.

Finally, it is instructive to compare our results with those in the literature studying the principal-agent problem with variable misrepresentation cost increasing in the magnitude of a 'lie" (e.g. Maggi and Rodriguez-Clare (1995)). First, while in our optimal mechanism the agent acts truthfully, lying occurs in their optimal mechanism: by inducing a type of an agent to lie, the principal can reduce the information rent paid to the types who may wish to mimic the former type because such mimicking will now require even larger lies from the latter. In contrast, under a fixed cost of lying in our model, lying by one type does not make it more costly for another type to mimic the former, so the principal does not benefit by inducing lying. Second, local incentive constraints are binding in the models without fixed costs of lying, while the only binding incentive constraints in our model are endogenously determined non-local ones. Third, allocative efficiency is improved in both cases compared with the standard screening model with no cost of lying. The main difference in this aspect is that in our model the first-best allocation is assigned on intervals of low types and high types. These qualitative differences between the optimal mechanisms in the two models may be useful in identifying the structure of lying costs in the subject population.

The remainder of the paper is organized as follows. Section 2 presents the formal model. Section 3 establishes important properties of the optimal screening mechanism. Section 4 formulates the screening problem as a dynamic optimization problem, derives the optimality conditions, and characterizes a closed-form solution in the uniform-quadratic case. Section 5 concludes. Appendix A includes the proof of Theorems 1- 4 and 8. Appendix B includes the proofs of Theorems 5, 6, 7, 9, 10, and Lemmas 1 and 2. The online Appendix contains the derivation of the optimal mechanism in the quadratic-uniform case and the analysis of the case where the set of types targeted by some type may be multi-valued.

2 Model and Preliminaries

Consider a principal and an agent engaged in some economic activity such as production, service provision or trade. The volume of this activity is represented by the variable $q \in Q$, which could be the output, the quantity or quality of the good transacted or service provided, etc. The agent's benefit from q is measured by the utility function $u(q, \theta)$, where θ is the agent's privately known preference parameter (type), while the principal's utility from q is $v(q)$. These assumptions are discussed in more detail below. The probability distribution of θ, $F(.)$, is assumed to be continuously differentiable over its support normalized to be $[0, 1]$ and to possess a density $f(.)$. In addition, both parties are assumed to be risk-neutral with respect to money. The principal controls the volume q and has the bargaining power allowing her to offer a take-it-or-leave-it contract to the agent.

To this standard screening model we add costly type misrepresentation, positing the existence of a set $\mathcal{M}$ of costly messages such that the agent experiences a cost $c(m, \theta)$ when sending a message $m \in \mathcal{M}$. The elements of $\mathcal{M}$ are broadly construed as messages and can represent statements, claims, pieces of evidence or documents, or actions serving as signals, such as passing a test or an inspection, demonstrating a skill or identity features, etc.

To reflect the fixed cost nature of misrepresentation in our setting, we assume that for for some $C > 0$ and each type θ there exists a non-empty set of messages $M(\theta) \subseteq \mathcal{M}$ such that $c(m', \theta) = 0$ and $c(m'', \theta) = C$ for all $m' \in M(\theta)$ and $m'' \in \mathcal{M} \setminus M(\theta)$, and the set $M(\theta) \cap (\mathcal{M} \setminus \cup_{\theta' \neq \theta} M(\theta'))$ is non-empty for any $\theta \in \Theta$. A message m in the latter set can then be interpreted as a truthful message by agent-type θ.

Exploiting the existence and nature of costly messages in this setting, a contract or mechanism can be represented as a mapping $\Theta \mapsto \mathcal{M} \times Q \times \mathbf{R}$ where $\mathbf{R}$ is the space of transfers. Alternatively, generalizing the idea of the Taxation Principle, we can model a mechanism as a mapping from $\mathcal{M}$ into the space of physical allocations and transfers $Q \times \mathbf{R}$.

Both of these mechanism design approaches can be simplified, as we do now for ease of exposition, by focussing on equivalent environments in which an agent-type θ incurs a fixed cost C of announcing a type $\theta' \neq \theta$, but has zero cost when announcing the true θ, thereby eliminating the need to refer to the costly message set $\mathcal{M}$.[4] With this simplification, a mechanism can be represented as a mapping $\Theta \mapsto Q \times \mathbf{R}$ from the set of type announcements into the set of quantity and transfer pairs, $(q(.), t(.))$. Because of misrepresentation costs, we cannot immediately invoke the Revelation principle to focus on truthtelling mechanisms. However, we will establish this property below in Theorem 1.[5]

We further simplify the exposition by casting our model as a relationship between a monopolistic seller firm, a principal, and a privately informed buyer, an agent. Our results obviously apply in other settings, such as a regulator and a firm, an employer and an employee. Thus, a consumer of type θ gets utility $u(q, \theta) - t$ from quantity/quality $q \in \mathbf{R}_+$ of the good provided in exchange for payment t. Without loss of generality, we assume that the firm has zero cost of production, since a model in which the firm incurs a cost of production $v(q)$ is equivalent to one in which this cost is identically zero, while the consumer's utility is $u(q, \theta) - v(q)$. The consumer's reservation utility is normalized to 0.

We adopt the following standard assumptions on $u(q, \theta)$:

Assumption 1 *(i) The function $u(q, \theta)$ is three times continuously differentiable on $\mathbf{R}_+ \times$*

[4]For clarity, an announcement of type θ in our mechanism represents sending a message $m \in M(\theta) \cap (\mathcal{M} \setminus \cup_{\theta' \neq \theta} M(\theta'))$.

[5]Technically, in this setting with costly type announcements, multiple types can make the same type announcement, but get different allocations. To allow for this possibility, we can construct our mechanism as a mapping from the set of type announcements into the set of quantity/transfer menus $\Theta \mapsto (Q \times \mathbf{R})^{\#\Theta}$, with typical menu $\{q(\theta, a), t(\theta), a) | a \in \Theta\}$ that an agent announcing type θ can choose from. However, this design is redundant by Theorem 1, the proof of which applies to this case verbatim. So we omit this for brevity.

$[0,1]$ *and strictly increasing in* θ *when* $q>0$*, with* $u(0,\theta)=u(q,0)=0$ *for all* θ*,* $q\geq 0$*;* *(ii)* $u_q(0,\theta)>0$ *if* $\theta>0$*;* $u_{qq}(q,\theta)<0$ *for all* (q,θ)*; there exists* $q^m>0$ *s.t.* $u_q(q^m,1)<0$*;* *(iii) there exist* $\underline{K}>0$ *and* $\overline{K}>0$ *such that* $\underline{K}<u_{q\theta}<\overline{K}$ *for all* $q>0$ *and* $\theta\in[0,1]$*.*

Assumption 1 implies that the first-best quantity $q^{fb}(\theta)\equiv\arg\max_q u(q,\theta)$ is well-defined, finite, strictly positive for $\theta>0$ and increasing in θ.

Our goal is to characterize the firm's optimal mechanism in this environment. To state the firm's problem, consider a mechanism $(q(\theta),t(\theta),A(\theta))\in\mathbf{R}_+\times\mathbf{R}\times[0,1]$, and let $1(A(\theta')\neq\theta)$ denote an indicator function equal to 1 when $A(\theta')\neq\theta$ and equal to zero otherwise. Then the firm's optimal mechanism solves the following problem:

$$\max_{q(\theta),t(\theta),A(\theta)}\int_0^1 t(\theta)f(\theta)d\theta$$

subject to the following incentive and individual rationality constraints:

$$u(q(\theta),\theta)-t(\theta)-C\times 1(A(\theta)\neq\theta)\geq u(q(\theta'),\theta)-t(\theta')-C\times 1(A(\theta')\neq\theta) \qquad \forall\theta,\theta'\in[0,1];$$
$$u(q(\theta),\theta)-t(\theta)-C\times 1(A(\theta)\neq\theta)\geq 0 \qquad \forall\theta\in[0,1].$$

As our first result demonstrates, an optimal mechanism involves no lying by the agent. Significantly, here truthtelling is not an analytical simplification as in the standard setting, but a necessary part of the optimal outcome.

Theorem 1 *Consider an incentive-compatible, individually rational mechanism* $(q(\theta),t(\theta),A(\theta))$ *such that* $A(\theta)\neq\theta$ *for a set of types* θ *of a positive measure. Then there exists an incentive-compatible, individually rational mechanism* $(\hat{q}(\theta),\hat{t}(\theta),\hat{A}(\theta))$ *with* $\hat{A}(\theta)=\theta$ *for almost all* θ*, which is strictly more profitable for the principal than the original mechanism.*

Thus, we can restrict consideration to incentive-compatible (IC) and individually rational (IR) mechanisms s.t. $A(\theta)=\theta$ for all θ. So, we can from now on denote a mechanism by a tuple $(q(.),t(.))$. An optimal mechanism $(q(.),t(.))$ solves the following problem:

$$\max_{q(\theta)\geq 0,t(\theta)}\int_0^1 t(\theta)f(\theta)d\theta \tag{1}$$

subject to:

$$u(q(\theta),\theta)-t(\theta)\geq u(q(\theta'),\theta)-t(\theta')-C \qquad \forall\theta,\theta'\in[0,1] \quad (IC), \tag{2}$$
$$u(q(\theta),\theta)-t(\theta)\geq 0 \qquad \forall\theta\in[0,1] \quad (IR). \tag{3}$$

The next result shows the existence of an optimal mechanism and provides a condition for its uniqueness:

Theorem 2 *An optimal mechanism exists. It is unique if* $u_{\theta qq}(q, \theta) \geq 0$ *for all* (q, θ).

Note that the condition $u_{\theta qq}(q, \theta) \geq 0$ is sufficient but not necessary for uniqueness; it guarantees the convexity of the constraint set defined by (2)-(3).

3 General Structure of An Optimal Mechanism

In this section, we will establish a number of important properties of an optimal mechanism. Let $V(\theta) = u(q(\theta), \theta) - t(\theta)$ be the net payoff of agent-type θ in an IC and IR mechanism $(q(.), t(.))$. Some useful properties of $V(.)$ and $q(.)$ are established in the next Theorem:

Theorem 3 *There exists an optimal mechanism* $(q(.), t(.))$ *such that for all* $\theta \in [0, 1]$:

1. $V(\theta)$ *is Lipschitz continuous,* $q(\theta)$ *and* $t(\theta)$ *are continuous in* θ, *with* $t(\theta) \geq 0$, *for all* $\theta \in [0, 1]$.
2. $V(\theta)$ *is non-decreasing;*
3. $q(\theta)$ *is strictly increasing;*
4. $0 < q(\theta) \leq q^{fb}(\theta)$ *for all* $\theta > 0$.

The proof of Theorem 3 shows that its claims *must hold a.e.* in an optimal mechanism, while a mechanism that fails any of these properties on a set of types of a positive measure (measure zero) is strictly (weakly) less profitable for the principal.

The continuity and monotonicity properties of Theorem 3 are standard in screening models without lying costs. In particular, monotonicity of q follows from the single-crossing assumption and is necessary for implementability, while $V(.)$ must be continuous and monotone. Yet, the nature and significance of monotonicity and continuity results in our model are different. Particularly, the presence of fixed costs creates a positive gap between the payoffs that the agent gets by reporting her true type and by imitating a close-by type, which makes it possible to implement non-monotone and discontinuous quantity schedules and payoff functions. To see this, suppose first that $q(.)$ and $V(.)$ are continuous and monotone. Then if type θ imitated type $\theta - \epsilon$ for some ϵ, her payoff would be $V(\theta - \epsilon) + u(q(\theta - \epsilon), \theta) - u(q(\theta - \epsilon), \theta - \epsilon) - C$ which is strictly less than her payoff $V(\theta)$ if ϵ is small. So, local incentive constraints are not binding for any type θ. Therefore, we can change $q(.)$ and $V(.)$ slightly and, in particular, choose them to be decreasing and/or discontinuous on some neighborhoods without violating any incentive constraints.

So, instead of implementability conditions, the proof of Theorem 3 relies on optimality arguments and shows that the principal can get a strictly higher payoff by modifying a mechanism in which $V(.)$ and $q(.)$ are non-monotone and/or discontinuous.

The no-exclusion property i.e., $q(\theta) > 0$ for all θ, is also due to the presence of fixed cost. Indeed, for every $\theta > 0$, there exists a sufficiently small $\underline{q}(\theta) > 0$ such that $u(\underline{q}(\theta), 1) - u(\underline{q}(\theta), \theta) < C$. Then assigning $\underline{q}(\theta)$ to an excluded type θ in exchange for transfer $u(\underline{q}(\theta), \theta)$ increases the seller's expected profit without inducing any other type to imitate θ.

Relying on Theorem 3, in the sequel we will assume without loss of generality that $q(.)$, $t(.)$ and $V(.)$ are increasing and continuous, and hence almost everywhere differentiable.

Although local incentive constraints will not binding when $V(.)$ and $q(.)$ are continuous, some incentive constraints must be binding when the fixed cost is not too large. Otherwise the optimal mechanism would, impossibly, involve the first-best quantities and full surplus extraction by the principal. Identifying and characterizing the set of binding incentive constraints is an important and challenging task, since such constraints are non-local.

To address this issue, we first establish general properties of the binding incentive constraint correspondence. To this end, let us define the targeted type correspondence $\tau(\theta)$ in an incentive compatible individually rational mechanism $(q(.), t(.))$ as follows:

$$\tau(\theta) = \begin{cases} \theta' \text{ s.t. } u(q(\theta), \theta) - t(\theta) = u(q(\theta'), \theta) - t(\theta') - C, & \text{if such } \theta' \text{ exists;} \\ \emptyset, & \text{otherwise.} \end{cases} \tag{4}$$

In words, $\tau(\theta)$ is the set of all such types θ' that incentive constraint $IC(\theta, \theta')$ of type θ is binding. We call the types in $\tau(\theta)$ "targeted types" of type θ. We will show below that $\tau(\theta)$ is non-empty for all sufficiently large θ. In contrast, $\tau(\theta)$ is empty when θ is sufficiently small: low types do not have binding incentive constraints and hence earn zero surplus. With a slight abuse of notation, for any set $\Theta \subseteq [0, 1]$, we let $\tau(\Theta) = \cup_{\theta \in \Theta} \tau(\theta)$. The following Theorem provides key properties of the correspondence $\tau(.)$.

Theorem 4 *In an optimal mechanism,*

1. *For any fixed cost* $C \in (0, \overline{C})$*, where* $\overline{C} \equiv \max_{\theta \in [0,1]} u(q^{fb}(\theta), 1) - u(q^{fb}(\theta), \theta)$*, there exists* $\hat{\theta} \in (0,1)$ *such that* $\tau(\theta) \neq \emptyset$ *iff* $\theta \in [\hat{\theta}, 1]$.

2. *The correspondence* $\tau(\theta)$ *is strictly increasing,*[6] *upper hemicontinuous and closed-valued on* $[\hat{\theta}, 1]$*, and satisfies* $\max \tau(\theta) < \theta$ *and* $\min \tau(\theta) > 0$.

3. *For all* $\theta \in [0, \max \tau(\hat{\theta})] \cup [\min \tau(1), 1]$*,* $q(\theta) = q^{fb}(\theta)$.

4. *If* $\theta_1, \theta_2 \in \tau(\theta)$ *for some* θ *and* $\theta_1 < \theta_2$*, then* $q(\theta') = q^{fb}(\theta')$ *for all* $\theta' \in [\theta_1, \theta_2]$.

5. $V(\theta) = 0$ *for all* $\theta \in [0, \hat{\theta}]$*,* $V(\theta) > 0$ *for all* $\theta \in (\hat{\theta}, 1]$.

6. *For any* θ *s.t.* $\tau(\theta) \neq \emptyset$*,* $\theta - \max \tau(\theta) \geq \frac{C}{\overline{K} \times q^{fb}(1)}$.

Theorem 4 implies that our screening problem is non-trivial iff $C < \overline{C} = \max_{\theta \in [0,1]} u(q^{fb}(\theta), 1) - u(q^{fb}(\theta), \theta)$, and then only sufficiently high types have binding incentive constraints and earn positive surpluses, while only intermediate types are "targeted." This is intuitive since no type can earn a sufficient surplus to cover the fixed cost C by imitating a low type. Likewise, imitating a high type does not give enough surplus for the imitator to cover the cost C, because high types pay large transfers for high quantities. So, the types below $\tau(\hat{\theta})$ and above $\tau(1)$ are not targeted and are assigned their first-best quantities. Figure 1 illustrates the targeted type correspondence τ and quantity q in the optimal mechanism.

By Theorem 4, the correspondence $\tau(.)$ is strictly increasing on its domain, $[\hat{\theta}, 1]$. Therefore, it is a.e. single-valued, differentiable and satisfies the following first-order condition:

$$u_q(q(\tau(\theta)), \theta)\dot{q}(\tau(\theta)) - \dot{t}(\tau(\theta)) = 0. \tag{5}$$

Differentiating $V(\theta) = u(q(\tau(\theta)), \theta) - t(\tau(\theta)) - C$ and using (5) yields for almost all $\theta \in [\hat{\theta}, 1]$:

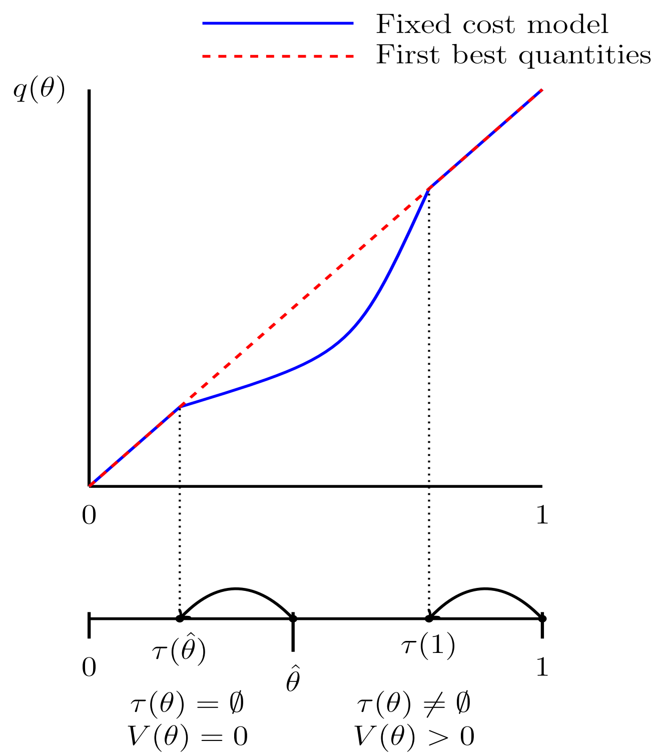
$$V'(\theta) = u_\theta(q(\tau(\theta)), \theta). \tag{6}$$

Since $V(.)$ is Lipschitz continuous and non-decreasing, it is an integral of its derivative $\dot{V}(\theta)$ a.e. So, since $V(\theta) = 0$ for $\theta \in [0, \hat{\theta}]$, we have for $\theta \in [0, 1]$:

$$V(\theta) = \int_{\hat{\theta}}^{\max\{\theta, \hat{\theta}\}} u_\theta(q(\max \tau(s)), s) ds, \tag{7}$$

[6] τ is strictly increasing when the following is true: If $\theta > \theta'$, $t \in \tau(\theta)$ and $t' \in \tau(\theta')$, then $t > t'$.

Figure 1: Structure of targeted types and quantities in the optimal mechanism.



where the max operator in the argument of $\tau(.)$ of the integrand is chosen without loss of generality because $\tau(\theta)$ is singleton almost everywhere. Equation (7) is the analog of the well-known envelope condition, yet with the argument $q(\max\tau(\theta))$, rather than $q(\theta)$ in the integrand. Using (7) yields:

$$t(\theta) = u(q(\theta),\theta) - \int_{\hat{\theta}}^{\max\{\theta,\hat{\theta}\}} u_\theta(q(\max\tau(s)),s)ds. \tag{8}$$

Using (8) and integrating by parts yields the following expression for the seller's profits:

$$\begin{aligned}&\int_0^1 \left(u(q(\theta),\theta) - \int_{\hat{\theta}}^{\max\{\theta,\hat{\theta}\}} u_\theta(q(\max\tau(s)),s)ds \right) f(\theta)d\theta = \\ &\int_0^1 u(q(\theta),\theta)f(\theta) - \int_{\hat{\theta}}^1 (1-F(\theta))u_\theta(q(\max\tau(\theta)),\theta)d\theta \end{aligned}\tag{9}$$

Since $q(\theta) = q^{fb}(\theta)$ on $T(\tau(\hat{\theta}),\tau(1)) \equiv [0,\min\tau(\hat{\theta})] \cup [\max\tau(1),1]$ by Theorem 4, we can rewrite (9) as follows:

$$\begin{aligned}&\int_{\theta\in T(\tau(\hat{\theta}),\tau(1))} u(q^{fb}(\theta),\theta)f(\theta)d\theta + \int_{\min\tau(\hat{\theta})}^{\max\tau(1)} u(q(\theta),\theta)f(\theta)d\theta - \int_{\hat{\theta}}^1 (1-F(\theta))u_\theta(q(\max\tau(\theta)),\theta)d\theta = \\ &\int_{\theta\in T(\tau(\hat{\theta}),\tau(1))} u(q^{fb}(\theta),\theta)f(\theta)d\theta + \int_{\hat{\theta}}^1 u(q(\max\tau(\theta)),\tau(\theta))f(\tau(\theta))\dot{\tau}(\theta) - (1-F(\theta))u_\theta(q(\max\tau(\theta)),\theta)d\theta,\end{aligned}\tag{10}$$

where the equality holds by a change of variables from θ to $\tau(\theta)$ in the second term on the first line.

The objective (10) differs from the objective in the standard adverse selection problem, because here type θ has binding incentive constraint to $\tau(\theta)$, not a local one. Reflecting this, the argument of $u_\theta(.,\theta)$ is $q(\max\tau(\theta))$, not $q(\theta)$, under the second integral in (10). Therefore, we cannot simply maximize the integrand of (10) pointwise to solve for the optimal $q(.)$ as in the standard case. So, below we develop a new solution method for our problem which, in particular, operates with "chains" of targeted types.

To begin, we can differentiate (8) a.e. since $t(\theta)$ and $\tau(\theta)$ are increasing to obtain:

$$\dot{t}(\theta) = u_q(q(\theta),\theta)\dot{q}(\theta) + u_\theta(q(\theta),\theta) - 1(\theta \geq \hat{\theta})u_\theta(q(\tau(\theta)),\theta). \tag{11}$$

Combining (5) and (11) then yields the following "law of motion" of $q(.)$ for almost all $\theta \in [\hat{\theta},1]$ which must be satisfied in any incentive compatible mechanism:

$$[u_q(q(\tau(\theta)),\theta) - u_q(q(\tau(\theta)),\tau(\theta))]\dot{q}(\tau(\theta)) = u_\theta(q(\tau(\theta)),\tau(\theta)) - 1(\tau(\theta) \geq \hat{\theta})u_\theta(q(\tau(\tau(\theta))),\tau(\theta)). \tag{12}$$

(73) implies that $q(.)$ is increasing on $[\tau(\hat{\theta}), \tau(1)]$. Indeed, if $\tau(\theta) \in [\tau(\hat{\theta}), \hat{\theta})$, then $q(.)$ is increasing because $u_q(q(\tau(\theta)), \theta) - u_q(q(\tau(\theta)), \tau(\theta)) > 0$ and $u_\theta(q(\tau(\theta)), \tau(\theta)) > 0$. The rest of the argument is by contradiction, so suppose that $q(.)$ is decreasing at some $\theta \in [\hat{\theta}, \tau(1)]$. Let $\theta^d = \inf\{\theta \in [\hat{\theta}, \tau(1)] | \dot{q}(\theta) \leq 0\}$. Then $\theta \in \tau(\theta')$ for some $\theta' \in [\hat{\theta}, 1]$. Hence, (73) and the assumption that $q'(\theta^d) \leq 0$ imply that $q(\tau(\theta^d)) \geq q(\theta^d)$ which, by continuity of $q(.)$, implies that $\dot{q}(\theta) \leq 0$ for some $\theta \in [\tau(\theta^d), \theta^d]$. A contradiction.

Additionally, by Theorem 4, $q(\theta) = q^{fb}(\theta)$ for $\theta \in [0, \tau(\hat{\theta})] \cup [\tau(1), 1]$ and $q^{fb}(.)$ is increasing by assumption. So we can drop the requirement that $q(.)$ is increasing below.

Significantly, the expected seller's profits (10) and the transfers in (8) are completely determined by the triple $(q(.), \tau(.), \hat{\theta})$. So, we can reformulate our optimal mechanism design problem in terms of finding such optimal triple. To proceed, let us introduce the following definition reflecting the properties of an optimal mechanism established above.

Definition 1 *A triple $(q(.), \tau(.), \hat{\theta})$, where $\hat{\theta} \in [0,1]$, $q(.) : [0,1] \mapsto [0,1]$, and $\tau(.) : [\hat{\theta}, 1] \rightrightarrows [0,1]$ is admissible if:*

(i) $\tau(.)$ is strictly increasing, upperhemicontinuous, closed-valued, satisfying $\max \tau(\theta) < \theta$ for all $\theta \in [\hat{\theta}, 1]$;

(ii) $q(\theta)$ is continuous on $[0,1]$ and satisfies $q(\theta) = q^{fb}(\theta)$ for $\theta \in [0, \tau(\hat{\theta})] \cup [\tau(1), 1]$;

(iii) $(\tau(.), q(.))$ satisfy (73) on $[\hat{\theta}, 1]$;

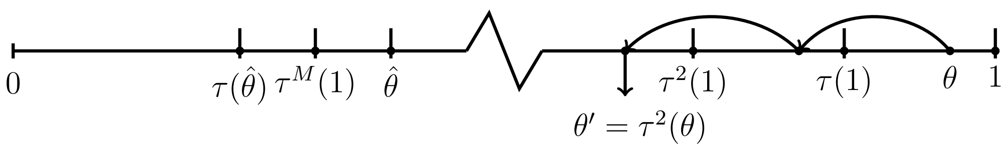
(iv) $\tau(\hat{\theta}) = \min\{\theta | u(q^{fb}(\theta), \hat{\theta}) - u(q^{fb}(\theta), \theta) = C\}$.

Note that part (i) of Definition 1 reflects properties 2 and 6 in Theorem 4, while part (ii) reflects property 1 in Theorem 3 and property 3 in Theorem 4. The equation in part (iv) is the boundary condition at $\hat{\theta}$. Since in an optimal mechanism $q(\theta) = q^{fb}(\theta)$ for all $\theta \in [0, \tau(\hat{\theta})]$, and $u(q^{fb}(\theta'), \theta) - u(q^{fb}(\theta'), \theta')$ is quasiconcave in θ' by assumption, the min operator in part (iv) is necessary for incentive compatibility. The following Theorem shows that we can reformulate our problem in terms of finding an optimal admissible triple.

Theorem 5 *If $(q(.), \tau(.), \hat{\theta})$ is an admissible triple, then its corresponding mechanism $(q(.), t(.))$, with $t(.)$ given by (8), is incentive compatible.*

If an admissible triple $(q(.), \tau(.), \hat{\theta})$ maximizes (10), then its corresponding mechanism $(q(.), t(.))$ is optimal.

Figure 2: Chains of Targeted Types.



4 Deriving the Optimal Mechanism

4.1 Chains of Targeted Types

In this section we characterize an optimal triple. To this end, we introduce a new construct, "chains of targeted types" connected by binding incentive constraints between them. Specifically, define a higher-order targeted type recursively as follows. For any $\theta \in [0, 1]$ and any integer $k \geq 1$, let $\tau^0(\theta) = \theta$ and $\tau^k(\theta) = \tau(\tau^{k-1}(\theta))$.

Then $(\theta, t^1 ..., t^k)$ is a chain of targeted types originating from θ if $t^1 \in \tau(\theta)$, $t^i \in \tau(t^{i-1})$ for all $i \in \{2, .., k\}$ and $\tau(t^k) = \emptyset$. The last condition can be equivalently stated as $t^k < \hat{\theta}$. Since $\tau(\theta)$ is strictly increasing, the maximal length of any chain of targeted types starting at θ is equal to M where $(1, \bar{t}^1, ..., \bar{t}^M)$ is the chain of targeted types s.t. $\bar{t}^1 = \max \tau(1)$ and $\bar{t}^i = \max \tau(\bar{t}^{i-1})$ for all $i \in \{1, ..., M\}$. By Theorem 4, $M < \infty$ and any chain of targeted types $(\theta, t^1 ..., t^k)$ for $\theta \in (\max \tau(1), 1)$ is such that $\bar{t}^{i+1} < t^i < \bar{t}^i$. So, there exists $\theta^M \in [\max \tau(1), 1]$ such that the length of any chain of targeted types originating from θ, $M(\theta)$, is equal to $M - 1$ iff $\theta \in [\max \tau(1), \theta^M)$, and is equal to M iff $\theta \in [\theta^M, 1]$.

Our next result derives the central optimality condition for the optimal mechanism:

Theorem 6 *In an optimal mechanism, for almost all $\theta \in [\hat{\theta}, 1]$, $\tau^s(\theta)$ is single-valued and differentiable for all s s.t. $\tau^s(\theta) \neq \emptyset$ and the following optimality condition holds:*

$$u_q(q(\tau^s(\theta)), \tau^s(\theta))f(\tau^s(\theta))\dot{\tau}^s(\theta) =$$
$$[u_q(q(\tau^s(\theta)), \tau^{s-1}(\theta)) - u_q(q(\tau^s(\theta)), \tau^s(\theta))] \sum_{k=1}^{s} f(\tau^{s-k}(\theta))\dot{\tau}^{s-k}(\theta). \tag{13}$$

Condition (13), derived be a perturbation method, is analogous to the well-known optimality condition in the standard adverse selection problem, $u_q(q(\theta), \theta)f(\theta) = (1-F(\theta))u_{q\theta}(q(\theta), \theta)$.

The difference between the two conditions reflects the fact that here binding incentive constraints are non-local. For this reason, standard methods for deriving an optimal mechanism are not applicable here, and we have to develop an alternative one.

Intuitively, the left-hand side of (13) is the marginal efficiency gain from raising the quantity of type $\tau^s(\theta)$, while its right-hand side is the associated marginal increase in the information rents of the predecessors of $\tau^s(\theta)$ in the chain of targeted types, $(\theta, \tau(\theta), ..., \tau^{s-1}(\theta))$, which ensures that the incentive constraints in this chain continue to hold. The multiplier term $f(\tau^{s-k}(\theta))\dot{\tau}^{s-k}(\theta)$ for $k = 1, ..., s$, reflects the relative probability weight of the types around $\tau^{s-k}(\theta)$ whose information rents need to be increased by the amount $[u_q(q(\tau^s(\theta)), \tau^{s-1}(\theta)) - u_q(q(\tau^s(\theta)), \tau^s(\theta))]$.

4.2 Optimal Mechanism when $\tau(.)$ is single-valued.

In this section, we characterize the optimal mechanism in the case when the function $\tau(.)$ is single-valued.[7] First, we provide sufficient conditions for this in the following Lemma.

Lemma 1 *(i) $\tau(\theta)$ is single-valued for all θ s.t.* $\max \tau(\theta) \leq \hat{\theta}$ *if $G(\theta, \theta') \equiv u(q^{fb}(\theta'), \theta) - u(q^{fb}(\theta'), \theta')$ is strictly quasiconcave in θ', on the interval $[0, \theta - \frac{C}{K}]$.*[8]

(ii) $\tau(\theta)$ is single-valued for all $\theta \in [\hat{\theta}, 1]$ if $G(\theta, \theta')$ is strictly quasi-concave in θ', $f'(\theta) \geq 0$, $u_{qqq}(q, \theta) \leq 0$ and $u_{\theta qq}(q, \theta) \leq 0$ for all θ and $q \leq q^{fb}(\theta)$.

Next, will proceed to characterize the optimal triple $(q(.), \tau(.), \hat{\theta})$ assuming that $\tau(.)$ is single-valued, and optimizing over the chains of targeted types starting from $[\tau(1), 1]$.

To this end, we introduce the following reformulation of quantity assignments: $Q^k(\theta) = q(\tau^k(\theta))$ for all $\theta \in [\tau(1), 1]$ and $k = 1, ..., M$. The significance of this reformulation lies in the fact that the domain of $Q^k(.)$ and $\tau^k(.)$, for all $k \in \{1, ..., M\}$, is the same and given by $[\tau(1), 1]$. Moreover, since $\tau(.)$ is continuous, it is surjective and so for every $\theta' \in [\hat{\theta}, 1]$ there exists $k \in \{1, ..., M\}$ and $\theta \in [\tau(1), 1]$ such that $\theta' = \tau^k(\theta)$, and hence $q(\theta') = Q^k(\theta)$.

[7] The optimal mechanism when the targeted type $\tau(.)$ is multi-valued is characterized in an online Appendix.

[8] It is straightforward to show that the following conditions, in combination with Assumption 1 parts (ii) and (iii), are sufficient to ensure the concavity of $G(\theta, .)$: $u_{\theta\theta}(q^{fb}(\theta'), \theta) \geq 0$, $u_{qqq}(q^{fb}(\theta), \theta) \leq 0$, $u_{q\theta\theta}(q^{fb}(\theta), \theta) \leq 0$, $u_{qq\theta}(q^{fb}(\theta), \theta) \leq 0$ for any $\theta \in [0, 1]$.

Thus, we can derive the optimal triple by solving for the optimal profiles $Q^k(\theta)$ and $\tau^k(\theta)$ for $\theta \in [\tau(1), 1]$ and $k \in \{1, ..., M\}$.

Specifically, we can rewrite the objective (10) as follows:

$$\int_{\theta \in \in [0, \min \tau(\hat{\theta})] \cup [\max \tau(1), 1]} u(q^{fb}(\theta), \theta) f(\theta) d\theta +$$
$$\sum_{k=1}^{M-1} \int_{\tau(1)}^{1} u(Q^k(\theta), \tau^k(\theta)) f(\tau^k(\theta)) \dot{\tau}^k(\theta) - (1 - F(\tau^{k-1}(\theta))) u_\theta(Q^k(\theta), \tau^{k-1}(\theta)) \dot{\tau}^{k-1}(\theta) d\theta$$
$$+ \int_{\theta^M}^{1} u(Q^M(\theta), \tau^M(\theta)) f(\tau^M(\theta)) \dot{\tau}^M(\theta) - (1 - F(\tau^{M-1}(\theta))) u_\theta(Q^M(\theta), \tau^{M-1}(\theta)) \dot{\tau}^{M-1}(\theta) d\theta \tag{14}$$

In this new notation, the law of motion (73) can be rewritten as follows:

$$\dot{Q}^k(\theta) = \frac{u_\theta(Q^k(\theta)), \tau^k(\theta)) - 1(\tau^k(\theta) \geq \hat{\theta}) u_\theta(Q^{k+1}(\theta), \tau^k(\theta))}{u_q(Q^k(\theta), \tau^{k-1}(\theta)) - u_q(Q^k(\theta), \tau^k(\theta))} \dot{\tau}^k(\theta). \tag{15}$$

We also have the following boundary conditions:

$$\tau^{k+1}(1) = \tau^k(\tau(1)) \qquad \text{for } k = 1, ..., M-1; \tag{16}$$
$$Q^{k+1}(1) = Q^k(\tau(1)) \qquad \text{for } k = 1, ..., M-1; \tag{17}$$
$$Q^1(1) = q^{fb}(\tau(1)); \tag{18}$$
$$Q^M(\theta^M) = q^{fb}(\tau^M(\theta^M)); \tag{19}$$
$$u(q^{fb}(\tau^M(\theta^M)), \tau^{M-1}(\theta^M)) - u(q^{fb}(\tau^M(\theta^M)), \tau^M(\theta^M)) - C = 0, \tag{20}$$

where $\tau^M(\theta^M)$ is the smallest root of (20).

The boundary conditions (16)-(17) ensure the continuity of $\tau(.)$ and $Q(.)$ at all juncture points $\tau^k(1)$. (18) and (19) need to hold by part (ii) in the Definition 1 of an admissible triple. Condition (20) imposes zero surplus for type $\hat{\theta}$ and reflects part (iv) of this Definition.

We can now use the optimality condition (13) and the law of motion (15) to derive the first-order differential equations for $\dot{Q}^k(.)$ and $\dot{\tau}^k(.)$ that characterize the optimal mechanism. As we will see below, these equations are simple because they are linear in $\dot{Q}^k(.)$ and $\dot{\tau}^k(.)$ and do not involve any other derivatives either of Q^s or τ^s, $s \neq k$.

To reminder a reader of the notation, recall that $M(\theta)$ is the number of elements in the chain of targeted types starting from $\theta \in [\tau(1), 1]$ and $\hat{\theta} = \tau^{M-1}(\theta^M)$, so that $M(\theta) = M$ if

$\theta \in [\theta^M, 1]$, and $M(\theta) = M - 1$ if $\theta \in [\tau(1), \theta^M)$.[9]

Theorem 7 *The optimal profile $(Q^k(\theta), \tau^k(\theta))$, $k \in \{1, ..., M\}$, is a unique solution to the following system of differential equations with boundary conditions (16)-(20):*

$$\dot{\tau}^k(\theta) = \frac{f(\theta)[u_q(Q^k, \tau^{k-1}) - u_q(Q^k, \tau^k)]}{f(\tau^k)u_q(Q^k, \tau^k)} \prod_{s=1}^{k-1} \frac{u_q(Q^s, \tau^{s-1})}{u_q(Q^s, \tau^s)}(\theta), \quad k \in \{1, ..., M(\theta)\}; \tag{21}$$

$$\dot{Q}^k(\theta) = \begin{cases} \frac{f(\theta)[u_\theta(Q^k,\tau^k) - u_\theta(Q^{k+1},\tau^k)]}{f(\tau^k)u_q(Q^k,\tau^k)} \prod_{s=1}^{k-1} \frac{u_q(Q^s,\tau^{s-1})}{u_q(Q^s,\tau^s)}(\theta), & k \in \{1, ..., M(\theta) - 1\}, \\ \frac{f(\theta)u_\theta(Q^k,\tau^k)}{f(\tau^k)u_q(Q^k,\tau^k)} \prod_{s=1}^{k-1} \frac{u_q(Q^s,\tau^{s-1})}{u_q(Q^s,\tau^s)}(\theta), & k = M(\theta). \end{cases} \tag{22}$$

The solution to the system (21)-(22) is unique by standard arguments, and is continuously differentiable because $\dot{Q}^k(\theta) = \dot{Q}^{k-1}(\tau(\theta))\dot{\tau}(\theta)$ and $\dot{\tau}^k(\theta) = \dot{\tau}^{k-1}(\tau(\theta))\dot{\tau}(\theta)$.

The optimal admissible triple $(q(.), \tau(.), \hat{\theta})$ is then defined as follows: $\hat{\theta} = \tau^{M-1}(\theta^M)$, $\tau(\theta) = \tau^{k+1}(\theta')$ and $q(\theta) = Q^k(\theta')$, where $\theta' \in [\tau(1), 1]$ is s.t. $\theta = \tau^k(\theta')$ for some $k \in \{1, ..., M-1\}$.

We conclude this section with a Lemma providing a limiting result in the fixed cost C. For the purposes of this Lemma we slightly modify the notation and let $q(\theta|C)$ and $V(\theta|C)$ be the quantity and the net payoff of the type θ, respectively, and let $M(C)$ be the maximal length of a chain of targeted types in the unique optimal mechanism under fixed cost C. Also, let $q^{sb}(\theta)$ and $V^{sb}(\theta)$ be the optimal quantity and the net payoff of type θ, respectively, in the solution to the standard screening problem with zero cost of misrepresentation.

Lemma 2 *For all $\theta \in [0, 1]$, $\lim_{C\downarrow 0} q(\theta|C) = q^{sb}(\theta)$, $\lim_{C\downarrow 0} V(\theta|C) = V^{sb}(\theta)$, $\lim_{C\downarrow 0} M(C) = \infty$.*

4.3 The Optimal Mechanisms under Intermediate Costs.

In this section we characterize the optimal mechanism for an intermediate range of fixed costs C when every chain of targeted types includes no more than two elements i.e., $\tau^2(\theta) = \emptyset$ for all θ. In this case, the optimal mechanism takes a particularly simple form exhibited

[9] Note that the number of elements in the partition, M, is bounded from above because by Theorem 4, $\theta - \max \tau(\theta) \geq \frac{C}{\overline{K} \times q^{fb}(1)}$.

below. We will also derive a closed form solution in this case under the linear-quadratic utility and uniform type distribution. We need the following result to begin with:

Theorem 8 *There exists $\underline{C} \in (0, \overline{C})$ such that in the optimal mechanism $\tau(\tau(\theta)) = \emptyset$ for all θ if $C \in (\underline{C}, \overline{C})$.*[10]

When $C \in (\underline{C}, \overline{C})$, a typical chain of targeted types has at most two elements, as illustrated in Figure 1. So, boundary conditions (16) and (17) do not apply, and all types in $[0, \tau(1)]$ get zero net payoffs. Moreover, the last term in the law of motion (73) is zero for all θ. Therefore, the differential equations (21) and (22) can be rewritten as follows:

$$\dot{\tau} = \frac{f(\theta)(u_q(Q,\theta) - u_q(Q,\tau))}{f(\tau)u_q(Q,\tau)}, \tag{23}$$

$$\dot{Q} = \frac{f(\theta)u_\theta(Q,\tau)}{f(\tau)u_q(Q,\tau)}. \tag{24}$$

The next Theorem shows that equations (23) and (24) with boundary conditions (18)-(20) uniquely characterize the optimal admissible triple and hence the optimal mechanism.

Theorem 9 *Suppose that $C \in (\underline{C}, \overline{C})$ and $u_{\theta qq}(q,\theta) \geq 0$ for all $(q,\theta) \in \mathbf{R}_+ \times [0,1]$. Then there exists a unique admissible triple $(\tau(\theta), q(\theta), \hat{\theta})$ such that its corresponding $(\tau(\theta), Q(\theta))$ is a solution to the system (23) - (24) on $[\tau(\hat{\theta}), \tau(1)]$ with boundary conditions (18)-(20).*

The next Theorem provides comparative statics results for the optimal mechanism. Additional comparative statics results are given in the next subsection.

Theorem 10 *Suppose that $u_{\theta qq}(q,\theta) \geq 0$ for all $(q,\theta) \in \mathbf{R}_+ \times [0,1]$ and $C_i \in (\underline{C}, \overline{C})$, $i \in 1, 2$. Let $(q_i(\theta), \tau_i(\theta), \hat{\theta}_i)$ be the optimal triple for C_i. If $C_2 > C_1$, then:*
(1) $\hat{\theta}_2 > \hat{\theta}_1$; (2) $\tau_2(\hat{\theta}_2) > \tau_1(\hat{\theta}_1)$;
(3) $\tau_2(\theta) < \tau_1(\theta)$ for $\theta \in [\hat{\theta}_2, 1]$; (4) $q_2(\theta) > q_1(\theta)$ for $\theta \in [\tau_2(\hat{\theta}_2), \tau_2(1)]$.

4.4 Quadratic-Uniform Example

In this section we derive a solution when $u(q,\theta) = \theta q - \frac{q^2}{2}$, θ is uniformly distributed on $[0,1]$, and C lies in an intermediate range, so that the maximal length of the chain

[10]This condition is equivalent to $\tau(1) < \hat{\theta}$.

of targeted types is either one or two i.e., $M(C) = 1$ or $M(C) = 2$. We derive the solution analytically in the case $M(C) = 1$ and numerically in the case $M(C) = 2$. The derivations for the case $M(C) = 1$ are provided in the online Appendix available at http://severinov.com/working_papers/Online_Appendix_screening_fixed_cost_2025.pdf. The code for numerical computations in $M = 2$ case is available at:
https://github.com/sseverinov/fixed_cost_screen_code.

First, the cost range $[\underline{C}_1, \overline{C}_1]$ such that $M(C) = 1$ (equivalently, $\tau(1) \leq \hat{\theta}$) satisfies $\underline{C}_1 = 0.09$, and $\overline{C}_1 = 0.25$. The cost range for $M(C) = 2$ is $[\underline{C}_2, \underline{C}_1]$, where $\underline{C}_2 \approx 0.04$.

The differential equations (23)-(24) in the case $M(C) = 1$ are:

$$\dot{\tau} = \frac{\theta - \tau}{\tau - Q}, \tag{25}$$

$$\dot{Q} = \frac{Q}{\tau - Q}, \tag{26}$$

while the boundary conditions (18)-(20) become:

$$Q(1) = \tau(1), \tag{27}$$

$$Q(\hat{\theta}) = \tau(\hat{\theta}), \tag{28}$$

$$Q(\hat{\theta})(\hat{\theta} - \tau(\hat{\theta})) = C. \tag{29}$$

The unique solution to the system (25)-(29) is parameterized by $t \in [\hat{t}, 1]$ as follows:

$$\theta(t) = b_1 \left(t - \frac{1 + 3\sqrt{\frac{1}{5}}}{2} t^{\frac{\sqrt{5}-1}{2}} + \frac{3\sqrt{\frac{1}{5}} - 1}{2} t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{\sqrt{5}+1}{2\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} + \frac{\sqrt{5}-1}{2\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}}, \tag{30}$$

$$Q(t) = -\frac{b_1}{2} t, \tag{31}$$

$$\tau(t) = b_1 \left(\frac{t}{2} - \frac{1 + \sqrt{\frac{1}{5}}}{2} t^{\frac{\sqrt{5}-1}{2}} - \frac{1 - \sqrt{\frac{1}{5}}}{2} t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{1}{\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}}, \tag{32}$$

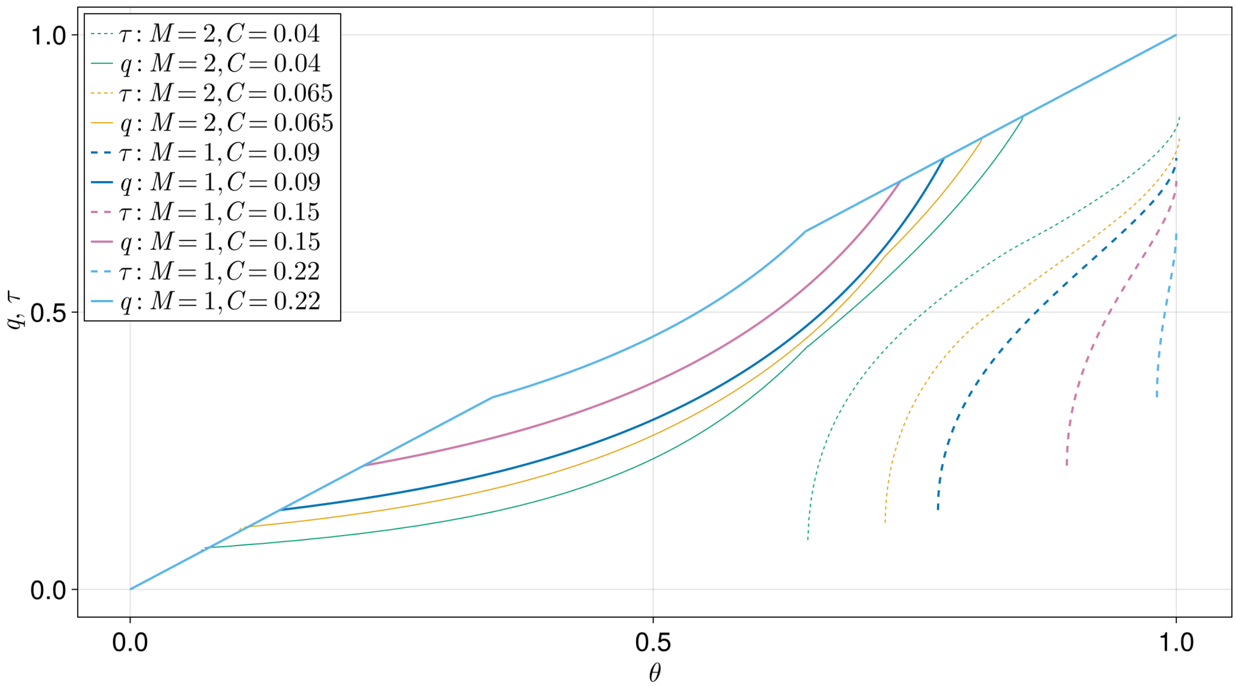
$$b_1 = -\frac{\frac{1}{\sqrt{5}} \hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}+1}{2}}}{\hat{t} - \frac{1+\sqrt{\frac{1}{5}}}{2} \hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2} \hat{t}^{-\frac{\sqrt{5}+1}{2}}}, \tag{33}$$

$$C = -\frac{b_1}{2} \left(b_1 \left(\frac{\hat{t}^2}{2} - \frac{1}{\sqrt{5}} \hat{t}^{\frac{\sqrt{5}+1}{2}} + \frac{1}{\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}-1}{2}} \right) + \frac{\sqrt{5}-1}{2\sqrt{5}} \hat{t}^{\frac{\sqrt{5}+1}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}-1}{2}} \right). \tag{34}$$

For $C \in [\underline{C}_1, \overline{C}_1]$, $(\theta(t), Q(t), \tau(t))$ define the optimal triple $(q(\theta), \tau(\theta), \hat{\theta})$ uniquely via (30)-(34) and the boundary conditions (27)-(29). Particularly, $\hat{t}$ and b are determined by

(33) and (34); $\theta(\hat{t}) = \hat{\theta}$ and $\theta(1) = 1$; $\tau(\theta) = \tau(t)$ where $\theta = \theta(t)$; and the optimal quantity $q(\theta)$ is defined by $q(\theta) = Q(t)$ where $\theta = \tau(t)$ for $\theta \in [\tau(\hat{t}), \tau(1)]$.

Figure 3: Optimal mechanism in quadratic-uniform case: quantities and targeted types



Next, let us consider $M(C) = 2$. In this case, the differential equations (21)-(22) can be rewritten as follows:

$$\dot{\tau}_1 = \frac{\theta - \tau_1}{\tau_1 - Q_1}, \tag{35}$$

$$\dot{Q}_1 = \begin{cases} \frac{Q_1 - Q_2}{\tau_1 - Q_1} & \text{if } \theta \in [\theta^M, 1] \\ \frac{Q_1}{\tau_1 - Q_1} & \text{if } \theta \in [\tilde{\theta}, \theta^M) \end{cases}, \tag{36}$$

$$\dot{\tau}_2 = \frac{\tau_1 - \tau_2}{\tau_2 - Q_2} \frac{\theta - Q_1}{\tau_1 - Q_1}, \tag{37}$$

$$\dot{Q}_2 = \frac{Q_2}{\tau_2 - Q_2} \frac{\theta - Q_1}{\tau_1 - Q_1}. \tag{38}$$

The differential equations (35) and (36) are defined on $[\tau(1), 1]$, while the differential equations (37) and (38) are defined on an interval $[\theta^M, 1]$, where $0 < \tau(1) < \theta^M < 1$. The

Table 1: θ^M, $\hat{\theta}, \tau(1)$ and $\tau(\hat{\theta})$ in the optimal mechanism

C	θ^M	$\hat{\theta}$	$\tau(1)$	$\tau(\hat{\theta})$
0.04	0.87	0.65	0.85	0.07
0.05	0.93	0.68	0.83	0.09
0.07	0.98	0.73	0.81	0.11
0.08	0.99	0.75	0.79	0.12
0.09	0.78	0.78	0.78	0.15
0.1	0.80	0.80	0.77	0.16
0.15	0.90	0.90	0.74	0.23
0.22	0.98	0.98	0.65	0.35
0.25	1	1	0.5	0.5

corresponding boundary conditions are:

$$\tau_1(\tau_1(1)) = \tau_2(1), \tag{39}$$

$$Q_1(1) = \tau_1(1), \tag{40}$$

$$Q_1(\tilde{\theta}) = Q_2(1), \tag{41}$$

$$Q_2(\theta^M) = \tau_2(\theta^M), \tag{42}$$

$$\tau_2(\theta^M)(\tau_1(\theta^M) - \tau_2(\theta^M)) = C. \tag{43}$$

The analytical solution for the case $M(C) = 1$ and the numerical solution for the case $M(C) = 2$ are represented in Figure 3 and Table 1 for several values of C.

The solution exhibits several notable properties. First, an increase in the cost of lying C leads to a higher efficiency of the quantity allocation. As illustrated in Figure 3, the optimal quantities increase towards the first-best, and the interval of targeted types $[\tau(\hat{\theta}), \tau(1)]$, who are assigned inefficiently low quantities, shrinks from both ends as the cost C increases, disappearing entirely at $\overline{C}_1 = 0.25$.

Furthermore, the targeted type function $\tau(.)$ is lower for the same θ at a higher C. But the shrinking of targeted types interval $[\tau(\hat{\theta}), \tau(1)]$ means that the lowest targeted type, $\tau(\hat{\theta})$, and the lowest type who targets anyone, $\hat{\theta}$, increase in C, while $\tau(1)$ decreases in C.

Note that θ^M is the first element in the lowest chain of targeted types with the maximal number of elements. When $M = 2$, this chain is $(\theta^M, \hat{\theta}, \tau(\hat{\theta}))$ and has three elements. At C around 0.04, θ^M is close to $\tau(1)$, so for lower costs C the maximal chain grows to four elements. In the opposite direction, as C grows from 0.04, θ^M increases, reaching 1 at $C = 0.09$. So, $C = 0.09$ is the threshold cost value such that the maximal chain length is 2 $(M = 1)$ for $C > 0.09$.

Finally, the principal extracts more surplus from the agent as C increases, first, because the assigned quantity converges to the first-best and, second, because an agent's surplus $V(\theta) = \int_{\hat{\theta}}^{\theta} u_\theta(q(\tau(\theta')), \theta')d\theta'$ decreases in C. The latter occurs because the cutoff type $\hat{\theta}$ increases in C and the targeted type $\tau(\theta)$ decreases in C for all θ. The latter effect dominates an increase in $q(.)$ to lower $q(\tau(\theta))$. This can be most easily seen from equations (30)-(34) and is illustrated graphically in the online Appendix. Since $\hat{\theta}$ converges to 1 as $C \to 0.25$, the principal extracts all surplus in the limit.

5 Conclusions

In this paper, we have explored screening under a fixed cost of lying. The introduction of this cost significantly reshapes the nature of the solution to this problem, in large part due to binding non-local incentive constraints. We develop a solution method to this problem by introducing a notion of an endogenous "targeted types" and chains of targeted types.

Our analysis delivers several qualitatively novel results. Particularly, in contrast to the environments where the cost of misrepresentation is increasing in the magnitude of a lie, there is no lying in our optimal mechanism. The overall allocative efficiency is higher compared with the standard screening model with no lying cost. Moreover, the standard exclusion property is not robust to a small fixed cost of lying. On the contrary, low types, as well as high types, get an efficient allocation. These results distinguish our model from the models with a variable cost of lying and no cost of lying and provide empirical predictions useful for identifying the structure of lying costs in the population.

While this paper focusses on type-independent fixed cost of lying, we believe that the key elements of our methodological approach, such as the characterization of binding non-local incentive constraints and the chains of targeted types, are applicable in other settings where the relevant incentive constraints are non-local and under more general types of lying costs with a fixed component. We plan to explore these directions in our future research.

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6 Appendix A

Part 1

Proof of Theorem 1: Suppose that the mechanism $(q(\theta), t(\theta), A(\theta))$ is such that $A(\theta) \neq \theta$ for all $\theta \in \Theta^l$, and the set Θ^l has a positive measure. Now consider an alternative mechanism $(\hat{q}(\theta), \hat{t}(\theta), \hat{A}(\theta))$ such that $(\hat{q}(\theta), \hat{t}(\theta), \hat{A}(\theta)) = (q(\theta), t(\theta), A(\theta))$ for all θ such that $A(\theta) = \theta$ and $(\hat{q}(\theta), \hat{t}(\theta), \hat{A}(\theta)) = (q(\theta), t(\theta)+C, \theta)$ for θ such that $A(\theta) \neq \theta$. Clearly, $(\hat{q}(\theta), \hat{t}(\theta), \hat{A}(\theta))$ is strictly more profitable for the firm, provided that it is incentive compatible and individually rational. The individual rationality of the new mechanism follows immediately from the individual rationality of the original mechanism. So we only need to show that the new mechanism is incentive compatible. Indeed, for all $\theta, \theta' \in [0,1]$ we have:

$$u(\hat{q}(\theta), \theta) - \hat{t}(\theta) - C \times 1(\hat{A}(\theta) \neq \theta) = u(\hat{q}(\theta), \theta) - \hat{t}(\theta) = u(q(\theta), \theta) - t(\theta) - C \times 1(A(\theta) \neq \theta)$$
$$\geq u(q(\theta'), \theta) - t(\theta') - C \times 1(A(\theta') \neq \theta) \geq u(\hat{q}(\theta'), \theta) - \hat{t}(\theta') - C \times 1(\hat{A}(\theta') \neq \theta),$$

where the first equality holds because $\hat{A}(\theta) = \theta$ for all $\theta \in [0,1]$, the second equality holds by definition of $(\hat{q}(\theta), \hat{t}(\theta), \hat{A}(\theta))$, the first inequality holds because $(q(\theta), t(\theta), A(\theta))$ is incentive compatible, and the second inequality holds because $(q(\theta), t(\theta), A(\theta))$ is incentive compatible, $\hat{q}(\theta') = q(\theta')$, $\hat{t}(\theta') \geq t(\theta')$ and $\hat{A}(\theta') = \theta' \neq \theta$ when $\theta' \neq \theta$. *Q.E.D.*

Part 2

In this part we provide proof of Theorems 2, 3, 4 and 8 via a series of Lemmas.

Theorem 2 follows from Lemmas 3, 4 and 10.

Theorem 3 follows from Lemmas 3, 5, 9 and 11.

Theorem 4 follows from Lemmas 6, 7, 9, 14, 15, 16 and Corollary 1.

Theorem 8 follows from Lemmas 16 and 17.

The first Lemma shows that the payment t is non-negative for almost every type.

Lemma 3 *In any optimal mechanism $(q(.), t(.))$, $t(\theta) \geq 0$ for almost all $\theta \in [0,1]$. Furthermore, there exists an optimal mechanisms such that $t(\theta) \geq 0$ for all $\theta \in [0,1]$.*

Proof of lemma 3: Suppose that the mechanism $(q(.), t(.))$ is such that $t(\theta) < 0$ iff $\theta \in \Theta^-$, where Θ^- is a non-empty subset of $[0,1]$. Let $(\tilde{q}(\theta), \tilde{t}(\theta)) = (q(\theta), t(\theta))$ for any $\theta \notin \Theta^-$, and $(\tilde{q}(\theta), \tilde{t}(\theta)) = (0,0)$ for any $\theta \in \Theta^-$. So $\tilde{t}(\theta) \geq 0$ for all $\theta \in [0,1]$. Obviously, the mechanism $(\tilde{q}(.), \tilde{t}(.))$ is individually rational and incentive compatible for all $\theta \notin \Theta^-$. Also, irrespective of her choice in $(\tilde{q}(.), \tilde{t}(.))$, any type $\theta \in \Theta^-$ makes a non-negative transfer in this mechanism, instead of a negative transfer in $(q(.), t(.))$. So, the mechanism $(\tilde{q}(.), \tilde{t}(.))$ is strictly/weakly more profitable for the principal than $(q(.), t(.))$ if Θ^- has a positive/zero measure. *Q.E.D*

Lemma 4 *The principal's maximization problem (1) subject to (2) and (3) has a solution.*

Proof of Lemma 4: By Lemma 3 we can restrict consideration to mechanisms $(q(\theta), t(\theta))$ s.t. $t(\theta) \geq 0$. Therefore, $q(\theta) \in [0, \bar{Q}]$ where $\bar{Q} = \max\{q|u(q,1) \geq 0\}$ (by Assumption 1(iii) $\bar{Q} < \infty$). Indeed, if $q(\theta) > \bar{Q}$, then $t(\theta) < 0$ by individual rationality. Also, individual rationality implies that $t(\theta) \leq \max_q u(q,1)$.

So, the admissible space of mechanisms is a set of bounded, measurable, and hence integrable, functions $(t(\theta), q(\theta)) : [0,1]^2 \mapsto [0, u(q^{fb}(1), 1)] \times [0, \bar{Q}]$. Endowed with pointwise convergence topology, this space is compact by Tychonoff Theorem. The objective (1) is continuous on this space. Furthermore, the subset of this space satisfying the constraints (2) and (3) is compact and non-empty. In particular, it includes all increasing $q(.)$ coupled with transfer functions that implement such $q(.)$ in the case with no fixed costs. So by Weierstrass Theorem, there exists $(q^*(.), t^*(.))$ solving (1) subject to (2) and (3). *Q.E.D.*

The next Lemma establishes continuity of $V(.)$, $t(.)$ and $q(.)$. in an optimal mechanism.

Lemma 5 *In an optimal mechanism $V(\theta) \equiv u(q(\theta), \theta) - t(\theta)$ is nondecreasing and Lipschitz continuous on $[0,1]$. There exists an optimal mechanism $(q(.), t(.))$ such that $q(.)$ and $t(.)$ are continuous at any $\theta \in [0,1]$.*

Proof of Lemma 5: Suppose that $(q(.), t(.))$ is an optimal mechanism.

(i) $V(.)$ **is increasing.** Note that $V(\theta) \geq 0$ by individual rationality. First, suppose that $V(\theta) > 0$. Then there exists a sequence θ_n s.t. $V(\theta) = \lim_{n\to\infty} u(q(\theta_n), \theta) - t(\theta_n) - C$. For, suppose otherwise. Then there exists $\epsilon > 0$ s.t. $V(\theta) > u(q(\theta'), \theta) - t(\theta') - C + \epsilon$ for all $\theta' \in [0,1]$. But then the mechanism cannot be optimal since the principal can increase her profits by raising $t(\theta)$ by $\frac{\epsilon}{2}$. Now consider some θ' s.t. $\theta' > \theta$. We have

$V(\theta') \geq \lim_{n\to\infty} u(q(\theta_n), \theta') - t(\theta_n) - C > \lim_{n\to\infty} u(q(\theta_n), \theta) - t(\theta_n) - C = V(\theta)$. So $V(.)$ is increasing at θ.

Now suppose that $V(\theta) = 0$. The previous argument establishes that we must have $V(\theta') = 0$ for all $\theta' < \theta$, so $V(.)$ is weakly increasing at θ in this case.

(ii) Lipschitz continuity of $V(.)$: There exists $L > 0$ s.t. $|V(\theta) - V(\theta')| \leq L|\theta - \theta'|$.

Since $V(\theta)$ is increasing, it is sufficient to consider the case $\theta > \theta'$ and $V(\theta) > 0$. As shown in part (i), $V(\theta) = \lim_{n\to\infty} u(q(\theta_n), \theta) - t(\theta_n) - C$ for some sequence θ_n. Since $V(\theta') \geq u(q(\theta_n), \theta') - t(\theta_n) - C$ for all n, $V(\theta) - V(\theta') \leq \lim_{n\to\infty}(u(q(\theta_n), \theta) - u(q(\theta_n), \theta')) \leq u(\bar{Q}, \theta) - u(\bar{Q}, \theta') \leq \max_{\theta''\in[0,1]} u_\theta(\bar{Q}, \theta'')(\theta - \theta')$. Taking $L = \max_{\theta''\in[0,1]} u_\theta(\bar{Q}, \theta'')$ establishes this Claim.

(iii) Continuity of $t(.)$: Suppose that there exists $\theta' \in (0, 1]$, a sequence θ_n s.t. $\lim_{n\to\infty} \theta_n = \theta'$ and $t^* = \lim_{n\to\infty} t(\theta_n)$ s.t. $|t(\theta') - t^*| > \beta$ for some $\beta > 0$. By continuity of $V(.)$, it follows that $V(\theta') = u(q^*, \theta') - t^*$ where $q^* = \lim_{n\to\infty} q(\theta_n)$ (Passing to a subsequence if necessary, the latter limit exists because, as shown above, $q(\theta)$ is bounded in an optimal mechanism).

First, suppose that $t^* > t(\theta') + \beta$. Consider an alternative mechanism in which the principal assigns the allocation (q^*, t^*) to type θ' instead of the allocation $(q(\theta'), t(\theta'))$, while all other allocations remain the same. This alternative mechanism is weakly more profitable for the principal because $t^* > t(\theta')$. It is individually rational and satisfies $IC(\theta', \theta)$ for all $\theta \in [0, 1]$ since $V(\theta)$ remains unchanged for all $\theta \in [0, 1]$. It remains to show that the alternative mechanism satisfies $IC(\theta, \theta')$. The proof is by contradiction, so suppose that $IC(\theta, \theta')$ fails for some θ in the modified mechanism i.e., $V(\theta) < u(q^*, \theta) - t^* - C$. Then, since $(q^*, t^*) = \lim_{n\to\infty}(q(\theta_n), t(\theta_n))$, there exists θ_n for n large enough that $V(\theta) < u(q_n, \theta) - t_n - C$. So the original mechanism is not incentive compatible. Contradiction.

Finally, suppose that $t(\theta') > t^* + \beta$. By continuity of $V(.)$, $u(q^*, \theta') = \lim_{n\to\infty} u(q(\theta_n), \theta_n) < u(q(\theta'), \theta')$, and so $q^* < q(\theta')$. Let $\tilde{q} = \frac{q^* + q(\theta')}{2}$, and consider a new mechanism $(\tilde{q}(.), \tilde{t}(.))$ which differs from the original mechanism $(q(.), t(.))$ only at θ_n for $n \geq N$ where N is sufficiently large so that $q(\theta_n) < \tilde{q}$. For such n, set $\tilde{t}(\theta_n) = u(\tilde{q}, \theta_n) - V(\theta_n) > t(\theta_n)$ and $\tilde{q}(\theta_n) = \tilde{q}$. So, the new mechanism $(\tilde{q}(.), \tilde{t}(.))$ is more profitable for the seller than $(q(.), t(.))$.

To check that the mechanism $(\tilde{q}(.), \tilde{t}(.))$ is individually rational and incentive compatible, let $\tilde{V}(\theta) = u(\tilde{q}(\theta), \theta) - \tilde{t}(\theta)$. By construction, $\tilde{V}(\theta) = V(\theta)$ for all $\theta \in [0, 1]$, so $(\tilde{q}(.), \tilde{t}(.))$ is individually rational. Also, since $(\tilde{q}(.), \tilde{t}(.))$ differs from $(q(.), t(.))$ only for types θ_n, $n \geq N$, it satisfies incentive constraints for all pairs (θ, θ'') s.t. $\theta, \theta'' \in [0,1]$, $\theta'' \neq \theta_n$, $n \geq N$.

Next, consider incentive constraints for (θ, θ_n), $n \geq N$. Since $(q(.), t(.))$ is incentive compatible and (for the second inequality) $\lim_{n\to\infty}(t(\theta_n), q(\theta_n)) = (t^*, q^*)$ and $u(.)$ is continuous, it follows that $V(\theta) \geq u(q(\theta'), \theta) - u(q(\theta'), \theta') + V(\theta') - C$. and $V(\theta) \geq u(q^*, \theta) - u(q^*, \theta') + V(\theta') - C$ for any θ for all $\theta \in [0,1]$. So, $V(\theta) \geq \max\{u(q(\theta'), \theta) - u(q(\theta'), \theta') + V(\theta') - C, u(q^*, \theta) - u(q^*, \theta') + V(\theta') - C\} > u(\tilde{q}, \theta) - u(\tilde{q}, \theta') + V(\theta') - C$, where the last inequality holds because $u_{q\theta}(q, \theta) > 0$ and $\tilde{q} \in (\min\{q^*, q(\theta')\}, \max\{q^*, q(\theta')\})$.

Finally, since $\lim_{n\to\infty} \theta_n = \theta'$, we have $\lim_{n\to\infty} \tilde{t}(\theta_n) = \lim_{n\to\infty}(u(\tilde{q}, \theta_n) - V(\theta_n)) = u(\tilde{q}, \theta') - V(\theta')$. So, $\tilde{V}(\theta) = V(\theta) > u(\tilde{q}, \theta) - \tilde{t}(\theta_n) - C$ for $n \geq N$ when N is sufficiently large. Therefore, in $(\tilde{q}(.), \tilde{t}(.))$, $IC(\theta, \theta_n)$ hold for all $\theta \in [0,1]$, θ_n, $n \geq N$ when N is large.

(iv) The continuity of $q(.)$ follows from the continuity of $V(.)$ and $t(.)$. *Q.E.D*

Lemma 6 *In an optimal mechanism, the correspondence $\tau(\theta)$ is upper hemicontinuous and compact-valued.*

Proof of Lemma 6: To establish the upper-hemicontinuity of $\tau(.)$, let (θ_n, θ'_n) be a sequence of type pairs such that $\theta'_n \in \tau(\theta_n)$ for all $n = 1, 2, ..., \infty$ and $\lim_{n\to\infty}(\theta_n, \theta'_n) = (\tilde{\theta}, \tilde{\theta}')$. We need to show that $\tilde{\theta}' \in \tau(\tilde{\theta})$. Define $\Delta U(\theta, \theta') = V(\theta) - u(q(\theta'), \theta) + t(\theta') + C$. Since $\theta'_n \in \tau(\theta_n)$, $\Delta U(\theta_n, \theta'_n) = 0$ for all n. Assumption 1 and Lemma 5 imply that $\Delta U(.)$ is continuous. Therefore, $\Delta U(\tilde{\theta}, \tilde{\theta}') = \lim_{n\to\infty} \Delta U(\theta_n, \theta'_n) = 0$ i.e., $\tilde{\theta}' \in \tau(\tilde{\theta})$.

The compact-valuedness of $\tau(.)$ follows because $\theta'' \in \tau(\theta)$ iff

$$\theta'' \in \arg\max_{\theta' \in [0,1]} u(q(\theta'), \theta) - t(\theta') - C.$$

The set of such maximizers is compact by Berge's Maximum Theorem because $q(.)$ and $t(.)$ are continuous functions by Lemma 5. *Q.E.D.*

The next Lemma shows the existence of a positive threshold $\hat{\theta}$ such that only types above $\hat{\theta}$ have binding incentive constraints and get a positive surplus.

Lemma 7 *For any $C > 0$, there exists $\hat{\theta} > 0$ s.t. $\tau(\theta) = \emptyset$ iff $\theta \in [0, \hat{\theta})$ and $V(\theta) = 0$ iff $\theta \in [0, \hat{\theta}]$.*

Proof of Lemma 7: Since $u(q, 0) = 0$ for all q, we must have $t(0) = 0$ and $V(0) = 0$ in an optimal mechanism. Then, since $V(.)$ is continuous and non-decreasing by Lemma 5, it follows that there exists $\hat{\theta} \in [0,1]$ such that $V(\theta) = 0$ $\forall \theta \leq \hat{\theta}$ and $V(\theta) > 0$ $\forall \theta > \hat{\theta}$.

To show that $\tau(\theta) = \emptyset \; \forall \theta < \hat{\theta}$, suppose that there exists $\theta < \hat{\theta}$ and θ' such that $\theta' \in \tau(\theta)$, so $V(\theta) = u(q(\theta'), \theta) - t(\theta') - C \geq 0$. But then $V(\hat{\theta}) \geq u(q(\theta'), \hat{\theta}) - t(\theta') - C > 0$ because $u_\theta > 0$, which contradicts $V(\hat{\theta}) = 0$.

Now suppose that $\tau(\theta) = \emptyset$ for some $\theta > \hat{\theta}$. Then the continuity of $V(.)$, $q(.)$ and $t(.)$ established in Lemma 5 imply that there exists $\epsilon > 0$ such that $V(\theta) > u(q(\theta'), \theta) - t(\theta') - C + \epsilon$ for all $\theta' \in [0, 1]$. Since $V(\theta) > 0$, the seller can increase her profit by raising $t(\theta)$ by $\min\{\epsilon, V(\theta)\}$. This modification does not violate any IR or IC constraints. Therefore $\tau(\theta) \neq \emptyset \; \forall \theta > \hat{\theta}$. The upper hemicontinuity of $\tau(.)$ established in Lemma 6 implies that $\tau(\hat{\theta}) \neq \emptyset$.

Finally, $V(\hat{\theta}) = u(q(\theta), \hat{\theta}) - t(\theta) - C = 0$ for $\theta \in \tau(\hat{\theta})$. So, since $C > 0$ and $t(\theta) \geq 0$, it must be the case that $\hat{\theta} > 0$. *Q.E.D*

Lemma 8 establishes a monotonicity property of binding incentive constraints.

Lemma 8 *Consider an incentive compatible mechanism, and suppose that $\theta_1 > \theta_2$, $\theta_1' \in \tau(\theta_1)$ and $\theta_2' \in \tau(\theta_2)$. Then $q(\theta_1') \geq q(\theta_2')$.*

Proof of Lemma 8: Since $\theta_1' \in \tau(\theta_1)$, $V(\theta_1) = u(q(\theta_1'), \theta_1) - t(\theta_1') - C \geq u(q(\theta_2'), \theta_1) - t(\theta_2') - C$. Similarly, $V(\theta_2) = u(q(\theta_2'), \theta_2) - t(\theta_2') - C \geq u(q(\theta_1'), \theta_2) - t(\theta_1') - C$. Combining these two inequalities yields: $u(q(\theta_1'), \theta_1) - u(q(\theta_2'), \theta_1) \geq t(\theta_1') - t(\theta_2') \geq u(q(\theta_1'), \theta_2) - u(q(\theta_2'), \theta_2)$. Since $\theta_1 > \theta_2$ and $u_{q\theta} > 0$, it must be that $q(\theta_1') \geq q(\theta_2')$. *Q.E.D*

Lemma 9 shows that optimal quantities never exceed the first-best level, and only downward incentive constraints can be binding. To state it, let $\tau^{-1}(\theta) = \{\theta' : \tau(\theta') = \theta\}$.

Lemma 9 *In an optimal mechanism for any $\theta \in [0, 1]$, $q(\theta) \leq q^{fb}(\theta)$. If $\tau^{-1}(\theta)$ is non-empty, then $\tau^{-1}(\theta) \subseteq (\theta, 1]$. If $\tau^{-1}(\theta)$ is empty, then $q(\theta) = q^{fb}(\theta)$.*

Proof of Lemma 9:

Claim 1: If $\tau^{-1}(\theta)$ is non-empty, then either $\tau^{-1}(\theta) \subseteq [0, \theta)$ or $\tau^{-1}(\theta) \subseteq (\theta, 1]$.

From the definition of $\tau(.)$ in (4) and the fact that $C > 0$ it follows that $\theta \notin \tau^{-1}(\theta)$. Now suppose that contrary to the Claim, there exists $\theta, \theta_1, \theta_2 \in [0, 1]$ such that $\theta_1 < \theta < \theta_2$ and $\theta_1, \theta_2 \in \tau^{-1}(\theta)$. Since $\tau(\theta_1) \neq \emptyset$ and $\theta > \theta_1$, Lemma 7 implies there exists $\theta' \in \tau(\theta)$, and so $u(q(\theta), \theta) - t(\theta) = u(q(\theta'), \theta) - t(\theta') - C$. Lemma 8 implies that $q(\theta') \geq q(\theta)$ and $q(\theta') \leq q(\theta)$, so $q(\theta') = q(\theta)$, and hence $t(\theta') = t(\theta) - C$. But then we cannot have $\theta_1, \theta_2 \in \tau^{-1}(\theta)$ because θ_1 and θ_2 get strictly higher payoff by imitating θ' rather than θ.

Claim 2: If $q(\theta) < q^{fb}(\theta)$, then $\tau^{-1}(\theta) \subseteq (\theta, 1]$. If $q(\theta) > q^{fb}(\theta)$, then $\tau^{-1}(\theta) \subseteq [0, \theta)$.

Suppose that contrary to the first part of the claim, there exists θ s.t. $q(\theta) < q^{fb}(\theta)$ but $\theta \notin \tau(\theta')$ for any $\theta' \in (\theta, 1]$. Then $V(\theta') > u(q(\theta), \theta') - t(\theta') - C$ for all $\theta' \in [\theta, 1]$. Since $[\theta, 1]$ is compact, there exists $\delta > 0$ such that $V(\theta') > u(q(\theta), \theta) - t(\theta) - C + \delta$ for all $\theta' \in [\theta, 1]$.

Now let $\tilde{q}(\theta)$ be the solution to $u(\tilde{q}(\theta), 1) - u(q(\theta), 1) = \delta$ if such exists and satisfies $\tilde{q}(\theta) \leq q^{fb}(\theta)$, and otherwise let $\tilde{q}(\theta) = q^{fb}(\theta)$. Then the seller gets a higher payoff from an alternative mechanism $(\tilde{q}(.), \tilde{t}(.))$ which differs from the original mechanism $(q(.), t(.))$ only in the allocation of type θ set to $\tilde{q}(\theta)$ and $\tilde{t}(\theta) = t(\theta) + u(\tilde{q}(\theta), \theta) - u(q(\theta), \theta) > t(\theta)$

The mechanism $(\tilde{q}(.), \tilde{t}(.))$ satisfies $IR(\theta')$ of all θ' because the net payoff of any type in it still equals $V(\theta')$. $IC(\theta, \theta')$ still holds for any $\theta' \in [0, \theta)$ since the allocation of any $\theta' \neq \theta$ does not change. $IC(\theta', \theta)$ still holds for any $\theta' \in [0, \theta)$ because $u(\tilde{q}(\theta), \theta') - \tilde{t}(\theta) = u(\tilde{q}(\theta), \theta') - u(\tilde{q}(\theta), \theta) + u(q(\theta), \theta) - t(\theta) < u(q(\theta), \theta') - t(\theta)$. The last inequality holds because $\tilde{q}(\theta) > q(\theta)$, $\theta' < \theta$ and $u_{q\theta} > 0$. Finally $IC(\theta', \theta)$ still holds for $\theta' \in (\theta, 1)$ because $V(\theta') > u(q(\theta), \theta') - t(\theta) - C + \delta \geq u(q(\theta), \theta') - t(\theta) - C + u(\tilde{q}(\theta), \theta') - u(q(\theta), \theta') > u(\tilde{q}(\theta), \theta') - \tilde{t}(\theta) - C$, where the second inequality holds by the choice of δ.

A symmetric argument establishes the second part of the claim.

Claim 3: For any $\theta \in [0, 1]$, $q(\theta) \leq q^{fb}(\theta)$.

Suppose that $q(\theta_1) > q^{fb}(\theta_1)$ for some θ_1. Then by Claim 2, $\theta_1 \in \tau(\theta_0)$ for some $\theta_0 \in [0, \theta_1)$, and so $V(\theta_0) = u(q(\theta_1), \theta_0) - t(\theta_1) - C$. This equality and $V(\theta_1) = u(q(\theta_1), \theta_1) - t(\theta_1)$ yield: $V(\theta_1) = V(\theta_0) + u(q(\theta_1), \theta_1) - u(q(\theta_1), \theta_0) + C > C$.

Next we will show that there exists a sequence $\{\theta_n\}_{n=0}^{\infty}$ such that for any $n \geq 1$, $\theta_n \in \tau(\theta_{n-1})$, $\theta_n > \theta_{n-1}$, $q(\theta_n) \geq q^{fb}(\theta_n)$ and $V(\theta_n) \geq nC$. We have established this for $n = 1$, so it suffices to establish the following inductive step: if for some fixed $k \geq 1$ these exists θ_k satisfying these conditions, then there exists θ_{k+1} for which these conditions also hold.

Indeed, since $V(\theta_k) \geq kC$, Lemma 7 implies that there exists some $\theta_{k+1} \in \tau(\theta_k)$. Since $\theta_k \in \tau(\theta_{k-1})$ and $\theta_k > \theta_{k-1}$, Lemma 8 then implies that $q(\theta_{k+1}) \geq q(\theta_k)$. If $\theta_{k+1} < \theta_k$, then $q(\theta_{k+1}) \geq q(\theta_k) > q^{fb}(\theta_k) > q^{fb}(\theta_{k+1})$, which contradicts Claim 2. Therefore $\theta_{k+1} > \theta_k$. Then $q(\theta_{k+1}) \geq q^{fb}(\theta_{k+1})$ by Claim 2.

Since $\theta_{k+1} \in \tau(\theta_k)$, we have $V(\theta_k) = u(q(\theta_{k+1}), \theta_k) - t(\theta_{k+1}) - C$. Combining this with $V(\theta_{k+1}) = u(q(\theta_{k+1}), \theta_{k+1}) - t(\theta_{k+1})$, we get:

$$V(\theta_{k+1}) = V(\theta_k) + u(q(\theta_{k+1}), \theta_{k+1}) - u(q(\theta_{k+1}), \theta_k) + C > V(\theta_k) + C > (k+1)C.$$

This completes the proof of the existence of the sequence $\{\theta_n\}_{n=0}^{\infty}$ s.t. $V(\theta_n) \geq nC$. Since $u(q(\theta^n), \theta^n)$ is bounded from above, it follows that $t(\theta^n) < 0$ for large n, contradicting Lemma 3.

Claim 4: If $q(\theta) = q^{fb}(\theta)$, then $\nexists \theta' \in (0, \theta)$ s.t. $\theta \in \tau(\theta')$.

Suppose there exists some θ such that $q(\theta) = q^{fb}(\theta)$ and $\theta \in \tau(\theta')$ for some $\theta' \in [0, \theta)$. Then the same argument as in Claim 3 can be used to establish a contradiction.

Claims 1-4 establish the statement of the Lemma. *Q.E.D*

Relying on Lemma 9 we can now establish the uniqueness of the optimal mechanism.

Lemma 10 *Suppose that $u_{\theta qq}(q, \theta) \geq 0$ for all (q, θ). Then the optimal mechanism is unique.*

Proof of Lemma 10: By Lemma 9 only downwards incentive constrains may be binding. So it is sufficient to establish the uniqueness of the solution to the relaxed problem in which the objective (1) is maximized subject to the individual rationality constraints (3) and downwards incentive constraints i.e., (2) holding for all $\theta, \theta' \in [0, 1]$ s.t. $\theta \geq \theta'$. The proof is by contradiction. So suppose that there exist two solutions to this problem, $(q_1(.), t_1(.))$ and $(q_2(.), t_2(.))$, that differ on a set of positive measure. Let $V_i(\theta) \equiv u(q_i(\theta), \theta) - t_i(\theta)$.

Next, fix some $\lambda \in (0, 1)$ and consider the mechanism $(\lambda q_1(.) + (1-\lambda)q_2(.), t^\lambda(\theta))$ where

$$t^\lambda(\theta) \equiv u(\lambda q_1(\theta) + (1-\lambda)q_2(\theta), \theta) - (\lambda V_1(\theta) + (1-\lambda)V_2(\theta)) >$$
$$\lambda u(q_1(\theta), \theta) + (1-\lambda)u(q_2(\theta), \theta) - (\lambda V_1(\theta) + (1-\lambda)V_2(\theta)) = \lambda t_1(\theta) + (1-\lambda)t_2(\theta). \quad (44)$$

(44) implies that the principal gets a strictly higher payoff in this new mechanism than in $(q_1(.), t_1(.))$ and $(q_2(.), t_2(.))$, since her payoffs from the latter two mechanisms are equal.

To complete the proof, let us confirm that this mechanism is individually rational and incentive compatible. The IR constraints hold since the net payoff of type θ in this mechanism is equal to $\lambda V_1(\theta) + (1-\lambda)V_2(\theta)$ and $V_i(\theta) \geq 0$ for all θ and $i \in \{1, 2\}$.

An IC constraint in this mechanism can be written as $\lambda V_1(\theta) + (1-\lambda)V_2(\theta) \geq$

$$u(\lambda q_1(\theta') + (1-\lambda)q_2(\theta'), \theta) - u(\lambda q_1(\theta') + (1-\lambda)q_2(\theta'), \theta') + (\lambda V_1(\theta') + (1-\lambda)V_2(\theta')) - C. \quad (45)$$

Now, note that

$$u(\lambda q_1(\theta') + (1-\lambda)q_2(\theta'), \theta) - u(\lambda q_1(\theta') + (1-\lambda)q_2(\theta'), \theta') \leq \int_{\theta'}^{\theta} \lambda u_\theta(q_1(\theta'), t) + (1-\lambda)u_\theta(q_2(\theta'), t)dt$$
$$= \lambda(u(q_1(\theta'), \theta) - u(q_1(\theta'), \theta')) + (1-\lambda)(u(q_2(\theta'), \theta) - u(q_2(\theta'), \theta')), \quad (46)$$

where the inequality holds because $u_{\theta qq} \geq 0$, while the equality holds by integration. Combining (46) with the fact that incentive constraints (2) hold in $(q_1(.), t_1(.))$ and $(q_2(.), t_2(.))$ implies that the incentive constraints (45) also hold for all $\theta, \theta' \in [0,1]$. *Q.E.D.*

The next Lemma establishes a lower bound on the slope of $q(\theta)$.

Lemma 11 *In an optimal mechanism, $q(\theta) > 0$ for all $\theta > 0$, and $q(\theta_2)-q(\theta_1) \geq \delta_q(\theta_2-\theta_1)$ for any $\theta_2 > \theta_1$ where $\delta_q \equiv \min\left\{\min_{\theta\in[0,1]} \dot{q}^{fb}(\theta), \frac{\underline{K}\bar{q}}{\overline{K}}, \frac{\underline{K}}{\overline{K}^2}C\right\} > 0$ and $\overline{K}$ and $\underline{K}$ are defined in Assumption 1, while $\bar{q}$ satisfies $u(\bar{q}, 1) = C$.*

Proof of Lemma 11: By Lemma 9, $q(\theta) \leq q^{fb}(\theta)$ for any θ. If $q(\theta_2) = q^{fb}(\theta_2)$, then $q(\theta_2) - q(\theta_1) \geq q^{fb}(\theta_2) - q^{fb}(\theta_1) \geq \min_{\theta\in[0,1]} \dot{q}^{fb}(\theta)(\theta_2 - \theta_1)$, where $q^{fb}(\theta)$ satisfies $u_q(q^{fb}(\theta), \theta) = 0$ and $\dot{q}^{fb}(\theta) = -\frac{u_{q\theta}(q^{fb}(\theta),\theta)}{u_{qq}(q^{fb}(\theta),\theta)}$. So, $\min \dot{q}^{fb}(\theta) > 0$ because $u_{q\theta}(q,\theta) > 0$ and u_{qq} is bounded.

Now suppose that $q(\theta_2) < q^{fb}(\theta_2)$. Then by Lemma 9, $\theta_2 \in \tau(\tilde{\theta})$ for some $\tilde{\theta} > \theta_2$.

$$V(\tilde{\theta}) = V(\theta_2) + u(q(\theta_2), \tilde{\theta}) - u(q(\theta_2), \theta_2) - C \geq \max\{0, V(\theta_1) + u(q(\theta_1), \tilde{\theta}) - u(q(\theta_1), \theta_1) - C\}. \tag{47}$$

From (47) it follows that $u(q(\theta_2), 1) \geq u(q(\theta_2), \tilde{\theta}) \geq C$. Hence, $q(\theta_2) \geq \bar{q}$ where $u(\bar{q}, 1) = C$, which implies that $q(\theta) > 0$ for all $\theta > 0$. (47) also implies that

$$\begin{aligned} V(\theta_2) - V(\theta_1) \geq & u(q(\theta_2), \theta_2) - u(q(\theta_2), \theta_1) - [u(q(\theta_2), \tilde{\theta}) - u(q(\theta_2), \theta_1) - u(q(\theta_1), \tilde{\theta}) + u(q(\theta_1), \theta_1] \\ \geq & u(q(\theta_2), \theta_2) - u(q(\theta_2), \theta_1) - (\tilde{\theta} - \theta_1) \max\{(q(\theta_2) - q(\theta_1))\overline{K}, (q(\theta_2) - q(\theta_1))\underline{K}\}. \end{aligned} \tag{48}$$

First, suppose that $V(\theta_2) = 0$. Since $\theta_2 > \theta_1$ from Lemma 5 it follows that $V_1(\theta_1) = 0$. Using this in (48) yields:

$$q(\theta_2) - q(\theta_1) \geq \frac{\int_{\theta_1}^{\theta_2} u_\theta(q(\theta_2), \theta) d\theta}{(\tilde{\theta} - \theta_1)\overline{K}} \geq \frac{\int_{\theta_1}^{\theta_2} \int_0^{\bar{q}} u_{q\theta}(q, \theta) dq + u_\theta(0, \theta) d\theta}{(\tilde{\theta} - \theta_1)\overline{K}} \geq \frac{(\theta_2 - \theta_1)\underline{K}\bar{q}}{(\tilde{\theta} - \theta_1)\overline{K}} > \frac{(\theta_2 - \theta_1)\underline{K}\bar{q}}{\overline{K}}.$$

Now suppose that $V(\theta_2) > 0$. Then by Lemmas 7-9, there exists $\theta'_2 \in [0, \theta_2)$ s.t. $\theta'_2 \in \tau(\theta_2)$ and $q(\theta_2) \geq q(\theta'_2)$. So, $V(\theta_2) = u(q(\theta'_2), \theta_2) - t(\theta'_2) - C$ and $V(\theta_1) \geq u(q(\theta'_2), \theta_1) - t(\theta'_2) - C$. Hence, $V(\theta_2) - V(\theta_1) \leq u(q(\theta'_2), \theta_2) - u(q(\theta'_2), \theta_1)$ using which in (48) yields:

$$u(q(\theta'_2), \theta_2) - u(q(\theta'_2), \theta_1) \geq u(q(\theta_2), \theta_2) - u(q(\theta_2), \theta_1) - (\tilde{\theta} - \theta_1) \max\{(q(\theta_2) - q(\theta_1))\overline{K}, (q(\theta_2) - q(\theta_1))\underline{K}\} \tag{49}$$

Since $q(\theta_2) \geq q(\theta_2')$ and $\theta_2 > \theta_1$, (49) implies that $q(\theta_2) \geq q(\theta_1)$ and $(\tilde{\theta}-\theta_1)(q(\theta_2)-q(\theta_1))\overline{K} \geq$

$$u(q(\theta_2), \theta_2) - u(q(\theta_2), \theta_1) - u(q(\theta_2'), \theta_2) + u(q(\theta_2'), \theta_1) \geq (\theta_2 - \theta_1)(q(\theta_2) - q(\theta_2'))\underline{K} \geq 0. \tag{50}$$

Since $\theta_2 \in \tau(\tilde{\theta})$, $V(\tilde{\theta}) = u(q(\theta_2), \tilde{\theta}) - t(\theta_2) - C \geq u(q(\theta_2'), \tilde{\theta}) - t(\theta_2') - C$. Combining the last inequality with $V(\theta_2) = u(q(\theta_2'), \theta_2) - t(\theta_2') \geq u(q(\theta_2'), \theta_2) - t(\theta_2') - C$ yields:

$$(\tilde{\theta} - \theta_2)(q(\theta_2) - q(\theta_2'))\overline{K} \geq u(q(\theta_2), \tilde{\theta}) - u(q(\theta_2), \theta_2) - u(q(\theta_2'), \tilde{\theta}) + u(q(\theta_2'), \theta_2) \geq C. \tag{51}$$

Finally, (50) and (51) imply that $q(\theta_2) - q(\theta_1) \geq \frac{\underline{K}}{\overline{K}^2(\tilde{\theta}-\theta_2)(\tilde{\theta}-\theta_1)} C(\theta_2 - \theta_1) \geq \frac{\underline{K}}{\overline{K}^2} C(\theta_2 - \theta_1)$, which completes the proof of the Lemma. *Q.E.D.*

Lemmas 8 and 11 imply that in the optimal mechanism binding IC correspondence is non-decreasing. That is, if $\theta_1 > \theta_2$, $\theta_1' \in \tau(\theta_1)$ and $\theta_2' \in \tau(\theta_2)$, then $\theta_1' \geq \theta_2'$

Next, let $\tau^{-1}(\theta) = \{\theta' \in [0,1] : \theta \in \tau(\theta')\}$ and $U(\theta''|\theta) = u(q(\theta''), \theta) - t(\theta'') - C$. The correspondence $\tau^{-1}(.)$ is increasing because it is the inverse of the increasing correspondence $\tau(.)$. It is also continuous, as the next Lemma shows.

Lemma 12 *In an optimal mechanism, $\tau^{-1}(\theta)$ is either empty or a singleton for any θ. Also, there exists $\delta_\tau > 0$ such that for any $\theta'' > \theta'$ for which $\tau^{-1}(\theta')$ and $\tau^{-1}(\theta'')$ are non-empty, $\theta'' - \theta' \geq \delta_\tau[\tau^{-1}(\theta'') - \tau^{-1}(\theta')]$.*

Proof of Lemma 12: At first, let us establish the following preliminary claim.

Claim 1. *For all θ_2, θ'', θ_1 and θ_1' s.t. $\theta_1' \in \tau(\theta_1)$, $V(\theta_2) - U(\theta''|\theta_2) \geq (\theta_2 - \theta_1)(\theta_1' - \theta'')\delta_q\underline{K}$.*

Proof of Claim 1: By Incentive compatibility for θ_2,

$$V(\theta_2) - U(\theta''|\theta_2) \geq U(\theta_1'|\theta_2) - U(\theta''|\theta_2) = u(q(\theta_1'), \theta_2) - u(q(\theta''), \theta_2) - [t(\theta_1') - t(\theta'')]. \tag{52}$$

By incentive compatibility for θ_1, $V(\theta_1) \equiv U(\theta_1'|\theta_1) \geq U(\theta''|\theta_1)$ i.e., $u(q(\theta_1'), \theta_1) - u(q(\theta''), \theta_1) - [t(\theta_1') - t(\theta'')] \geq 0$. Combining this inequality with (52) yields:

$$V(\theta_2) - U(\theta''|\theta_2) \geq u(q(\theta_1'), \theta_2) - u(q(\theta''), \theta_2) - u(q(\theta_1'), \theta_1) + u(q(\theta''), \theta_1) \geq$$
$$(\theta_2 - \theta_1)(q(\theta_1') - q(\theta''))\underline{K} \geq (\theta_2 - \theta_1)(\theta_1' - \theta'')\delta_q\underline{K}$$

where the second inequality holds because $q(.)$ is increasing by Lemma 11 and $(\theta_2-\theta_1)(\theta_1'-\theta_2'')>0$, and so $\theta_2-\theta_1>0$ iff $q(\theta_1')-q(\theta'')>0$. The last inequality holds by Lemma 11.

Since $\tau(.)$ is upper hemi-continuous and compact-valued, its image $\tau([0,1])\equiv\cup_{\theta\in[0,1]}\tau(\theta)$ is compact and hence closed in $[0,1]$. Also, the set $\tau^{-1}(\theta)$ is closed by continuity of $V(.)$ and $u(.)$. So, whenever $\tau^{-1}(\theta)$ is non-empty set, it is closed and its max and min are well-defined.

To prove the second claim of the Lemma, suppose that in an optimal mechanism $(q(.),t(.))$ such δ_τ does not exist. Then there exist $\theta^\dagger,\{\theta_n\}\subseteq\tau([0,1])$ s.t. $\lim_{n\to\infty}\theta_n=\theta^\dagger$ and either $\lim_{n\to\infty}\max\tau^{-1}(\theta_n)<\max\tau^{-1}(\theta^\dagger)$, or $\lim_{n\to\infty}\min\tau^{-1}(\theta_n)>\min\tau^{-1}(\theta^\dagger)$.

We will focus on the former case, since the proof in the latter case is analogous. Then, let $\lim_{n\to\infty}\max\tau^{-1}(\theta_n)=\theta_1$ and $\max\tau^{-1}(\theta^\dagger)=\theta_2$, with $\theta_1<\theta_2$. Since $\tau^{-1}(.)$ is increasing, it follows that $\theta_n<\theta^\dagger$ for all sufficiently large n. Let $\hat{\delta}=\frac{\theta_2-\theta_1}{3}$, $\hat{\theta}_1=\theta_1+\hat{\delta}$ and $\hat{\theta}_2=\theta_2-\hat{\delta}$.

Consider an alternative mechanism $(\tilde{q},\tilde{t})$ s.t. for $\theta\in[\theta^\dagger-2\epsilon,\theta^\dagger+2\epsilon]$, $\tilde{q}(\theta)=q(\theta)-\epsilon^2$ and $\tilde{t}(\theta)=t(\theta)-u(q(\theta),\theta)+u(q(\theta)-\epsilon^2,\theta)$ and so $\tilde{V}(\theta)=V(\theta)$; for $\theta\in[\hat{\theta}_1,\hat{\theta}_2]$, $\tilde{t}(\theta)=t(\theta)+\Delta(\theta,\epsilon)$, where $\Delta(\theta,\epsilon)=\min_{\theta'\in[\theta^\dagger-2\epsilon,\theta^\dagger+2\epsilon]}u(q(\theta'),\theta)-u(q(\theta')-\epsilon^2,\theta)-u(q(\theta'),\theta')+u(q(\theta')-\epsilon^2,\theta')$ and so $\tilde{V}(\theta)=V(\theta)-\Delta(\theta,\epsilon)$. The rest of the mechanism is the same as the original one. Note that $q(\theta^\dagger)>0$ since $IC(\theta_2,\theta^\dagger)$ is binding, so $\tilde{q}(\theta)>0$ for all $\theta\in[\theta^\dagger-2\epsilon,\theta^\dagger+2\epsilon]$, and small enough ϵ.

Let us show that $(\tilde{q},\tilde{t})$ satisfies incentive and individual rationality constraints, $\tilde{IR}$ and $\tilde{IC}$, respectively. If $\theta\in[0,1]\setminus[\hat{\theta}_1,\hat{\theta}_2]$, then $\tilde{IR}(\theta)$ holds since $\tilde{V}(\theta)=V(\theta')$. If $\theta\in[\hat{\theta}_1,\hat{\theta}_2]$, then $V(\theta)>0$ by Lemma 7 since $\tau(\theta_1)$ is non-empty and $\theta_1<\hat{\theta}_1$. So $\tilde{V}(\theta)=V(\theta)-\Delta(\theta,\epsilon)>0$ for small enough ϵ i.e., $\tilde{IR}(\theta)$ holds.

Next, if $\theta\le\theta'$ then $IC(\theta,\theta')$ is slack by Lemma 9. So by continuity $\tilde{IC}(\theta,\theta')$ is slack for small enough ϵ. If $\theta>\theta'$ and $\theta\notin[\hat{\theta}_1,\hat{\theta}_2]$ or $(\theta,\theta')\in[\hat{\theta}_1,\hat{\theta}_2]^2$, $\tilde{IC}(\theta,\theta')$ holds because $IC(\theta,\theta')$ holds. If $\theta\in[\hat{\theta}_1,\hat{\theta}_2]$, and $\theta'\in[\theta^\dagger-2\epsilon,\theta^\dagger+2\epsilon]$, $\tilde{IC}(\theta,\theta')$ holds because:

$$\begin{aligned}\tilde{V}(\theta)&=V(\theta)-\Delta(\theta,\epsilon)\ge V(\theta)-[u(q(\theta'),\theta)-u(q(\theta')-\epsilon^2,\theta)-u(q(\theta'),\theta')+u(q(\theta')-\epsilon^2,\theta')]\\&\ge u(q(\theta'),\theta)-t(\theta')-C-[u(q(\theta'),\theta)-u(q(\theta')-\epsilon^2,\theta)-u(q(\theta'),\theta')+u(q(\theta')-\epsilon^2,\theta')]\\&=u(q(\theta'),\theta)-\tilde{t}(\theta')-C-u(q(\theta'),\theta)+u(q(\theta')-\epsilon^2,\theta)=u(\tilde{q}(\theta'),\theta)-\tilde{t}(\theta')-C,\end{aligned}$$

where the first inequality holds by definition of $\Delta(\theta,\epsilon)$, the second inequality holds by $IC(\theta,\theta')$, the second equality holds by definition of $\tilde{t}(\theta')$, the last equality holds by definition of $\tilde{q}(\theta')$.

If $\theta \in [\hat{\theta}_1, \hat{\theta}_2]$ and $\theta' < \theta^\dagger - 2\epsilon$, then $\theta - \theta_1 \geq \hat{\delta}$. Also, $\max \tau(\theta_1) \geq \theta^\dagger - \epsilon$ for all $\epsilon > 0$ since $\theta_1 = \lim_{n\to\infty} \max \tau^{-1}(\theta_n)$. Therefore, by Lemma 11, $V(\theta) - U(\theta'|\theta) \geq (\theta - \theta_1)(\theta^\dagger - \epsilon - \theta') > \epsilon\hat{\delta}\delta_q\underline{K}$.

If $\theta \in [\hat{\theta}_1, \hat{\theta}_2]$ and $\theta' > \theta^\dagger + 2\epsilon$, then $\theta + \hat{\delta} \leq \theta_2 = \max \tau^{-1}(\theta^\dagger)$. So, by Claim (i) of this Lemma 11, $V(\theta) - U(\theta'|\theta) \geq (\theta_2 - \theta)(\theta' - \theta^\dagger) > 2\epsilon\hat{\delta}\delta_q\underline{K}$.

Therefore, $\tilde{V}(\theta) - U(\theta'|\theta) = V(\theta) - \Delta(\theta,\epsilon) - U(\theta'|\theta) > \epsilon\hat{\delta}\delta_q\underline{K} - \Delta(\theta,\epsilon) = \epsilon\left(\hat{\delta}\delta_q\underline{K} - \frac{\Delta(\theta,\epsilon)}{\epsilon}\right)$. Since $\lim_{\epsilon\to 0}\frac{\Delta(\theta,\epsilon)}{\epsilon} = 0$, it follows that $\tilde{V}(\theta) - U(\theta'|\theta) > 0$ for small ϵ. So $\tilde{IC}(\theta, \theta')$ hold.

The change in seller's profits from switching to the mechanism $(\tilde{q}, \tilde{t})$ is $\int_{\hat{\theta}_1}^{\hat{\theta}_2} \Delta(\theta,\epsilon)dF(\theta) - \int_{\theta^\dagger-2\epsilon}^{\theta^\dagger+2\epsilon} u(q(\theta),\theta') - u(q(\theta)-\epsilon^2,\theta)dF(\theta)$. Note that

$$\int_{\hat{\theta}_1}^{\hat{\theta}_2} \Delta(\theta,\epsilon)dF(\theta) \geq \epsilon^2(\hat{\theta}_1 - \theta^\dagger - 2\epsilon)\underline{K},$$

$$\int_{\theta^\dagger-2\epsilon}^{\theta^\dagger+2\epsilon} u(q(\theta),\theta') - u(q(\theta)-\epsilon^2,\theta)dF(\theta) \leq \epsilon^2 u_q(q(\theta^\dagger) - 2\epsilon - \epsilon^2, \theta^\dagger + 2\epsilon)(F(\theta^\dagger + 2\epsilon) - F(\theta^\dagger - 2\epsilon)).$$

Since $\hat{\theta}_1 > \theta_1 > \theta^\dagger$ (the latter inequality holds by Lemma 9), $(\hat{\theta}_1 - \theta^\dagger - 2\epsilon)\underline{K} > u_q(q(\theta^\dagger) - 2\epsilon - \epsilon^2, \theta^\dagger + 2\epsilon)(F(\theta^\dagger + 2\epsilon) - F(\theta^\dagger - 2\epsilon))$, and so $\int_{\hat{\theta}_1}^{\hat{\theta}_2} \Delta(\theta,\epsilon)dF(\theta) - \int_{\theta^\dagger-2\epsilon}^{\theta^\dagger+2\epsilon} u(q(\theta),\theta') - u(q(\theta)-\epsilon^2,\theta)dF(\theta) > 0$ when ϵ is sufficiently small. Therefore, the alternative mechanism $(\tilde{q}, \tilde{t})$ generates higher profit for sufficiently small ϵ, contradiction.

Finally, suppose there exists θ' such that $\tau^{-1}(\theta')$ is multi-valued, i.e. $\max \tau^{-1}(\theta') - \min \tau^{-1}(\theta') = \hat{\delta} > 0$. Define $\theta'' = \theta' + \frac{\hat{\delta}\delta_\tau}{2}$. Then, as shown above, $\delta_\tau(\max \tau^{-1}(\theta'') - \min \tau^{-1}(\theta')) \leq \theta'' - \theta' = \frac{\hat{\delta}\delta_\tau}{2}$, and so $\max \tau^{-1}(\theta'') \leq \min \tau^{-1}(\theta') + \frac{\hat{\delta}}{2}$, and thus $\max \tau^{-1}(\theta'') < \max \tau^{-1}(\theta')$, which contradicts that τ is non-decreasing. *Q.E.D.*

Lemma 12 implies that the correspondence $\tau(.)$ is strictly increasing.

Corollary 1 *Let $\theta_1 > \theta_2$. Suppose $\theta_1' \in \tau(\theta_1)$, $\theta_2' \in \tau(\theta_2)$, then $\theta_1' > \theta_2'$.*

The next Lemma provides a lower bound on the loss from imitating some type in $\tau([0,1])$.

Lemma 13 *In an optimal mechanism, if $\theta_1' \in \tau(\theta_1)$ for some θ_1, then for any $\theta, \theta_2 \in [0,1]$ and $\delta_V = \delta_\tau\delta_q\underline{K} > 0$ we have:*

$$V(\theta_2) - U(\theta_1'|\theta_2) \geq \begin{cases} \delta_V \frac{(\theta_2-\theta_1)^2}{4} & \text{if } \frac{\theta_1+\theta_2}{2} \geq \hat{\theta} \\ \frac{\theta_1-\theta_2}{2} \min_\theta u_\theta(q(\theta_1'),\theta) & \text{if } \frac{\theta_1+\theta_2}{2} < \hat{\theta}. \end{cases} \tag{53}$$

Proof of Lemma 13: Fix $\theta_2 \in [0,1]$ and let $\tilde{\theta} = \frac{\theta_1+\theta_2}{2}$. If $\tilde{\theta} \geq \hat{\theta}$, then there exists $\tilde{\theta}' \in \tau(\tilde{\theta})$. Then $V(\theta_2) - U(\theta_1'|\theta_2) \geq \delta_q \underline{K}(\theta_2 - \tilde{\theta})(\tilde{\theta}' - \theta_1') \geq \delta_\tau \delta_q \underline{K}(\theta_2 - \tilde{\theta})(\tilde{\theta} - \theta_1) = \delta_\tau \delta_q \underline{K}\frac{(\theta_2-\theta_1)^2}{4}$, where the first and second inequalities hold by Claim 1 of Lemma 12 and Lemma 12, respectively.

If $\tilde{\theta} < \hat{\theta}$, then $\theta_2 < \hat{\theta}$ since $\hat{\theta} \leq \theta_1$. So, $V(\tilde{\theta}) = V(\theta_2) = 0$ by Lemma 7, and $U(\theta_1'|\theta_2) \leq 0$ and $U(\theta_1'|\tilde{\theta}) \leq 0$. Thus $V(\theta_2) - U(\theta_1'|\theta_2) = -U(\theta_1'|\theta_2) = -U(\theta_1'|\tilde{\theta}) + u(q(\theta_1'), \tilde{\theta}) - u(q(\theta_1'), \theta_2) \geq (\tilde{\theta} - \theta_2) \min_\theta u_\theta(q(\theta_1'), \theta) = \frac{(\theta_1-\theta_2)}{2} \min_\theta u_\theta(q(\theta_1'), \theta)$. *Q.E.D.*

The next Lemma establishes the positive lower bound on the value of $\theta - \tau(\theta)$:

Lemma 14 *For all $\theta \in [0,1]$, $\theta - \max \tau(\theta) \geq \frac{C}{q^{fb}(1) \times \max_{(q,\theta)} u_{\theta q}(q,\theta)}$. So, $\tau(\theta) = \emptyset$ when $C \geq q^{fb}(1) \times \max_{(q,\theta)} u_{\theta q}(q,\theta)$.*

Proof of Lemma 14: Take any θ such that $\tau(\theta) \neq \emptyset$ and assume that $\tau(\theta)$ is a singleton without loss of generality. We have: $V(\theta) = u(q(\tau(\theta)), \theta) - u(q(\tau(\theta)), \tau(\theta)) + V(\tau(\theta)) - C$. Using $V(\theta) = \int_{\hat{\theta}}^{\theta} u_\theta(q(\tau(s)), s) ds$ in the last equation and rearranging yields:

$$\int_{\tau(\theta)}^{\theta} u_\theta(q(\tau(\theta)), s) ds - \int_{\tau(\theta)}^{\theta} u_\theta(q(\tau(s)), s) ds = \int_{\tau(\theta)}^{\theta} \int_{q(\tau(s))}^{q(\tau(\theta))} u_{\theta q}(q, s) dq ds = C.$$

Since $q(\theta) \leq q^{fb}(1)$ and $u_{\theta q}(q,\theta) \leq \overline{K}$ for all θ and q, the previous equation implies that $\theta - \tau(\theta) \geq \frac{C}{\overline{K} q^{fb}(1)}$, which establishes the claim of the Lemma. *Q.E.D.*

Now we can show that types in $[\min \tau(\theta), \max \tau(\theta)]$ are assigned first-best quantities:

Lemma 15 *In an optimal mechanism, if $\theta_1', \theta_2' \in \tau(\breve{\theta})$ for some $\breve{\theta}$ and $\theta_1' < \theta_2'$, then $q(\theta') = q^{fb}(\theta')$ for any $\theta' \in [\theta_1', \theta_2']$.*

Proof of Lemma 15:

Suppose to the contrary that $q(\theta) < q^{fb}(\theta)$ for some $\theta \in [\theta_1', \theta_2']$. Then by continuity of $q(.)$ there exist $\breve{\theta}_1', \breve{\theta}_2'$ such that $\theta_1' < \breve{\theta}_1' < \breve{\theta}_2' < \theta_2'$ and for any $\theta' \in [\breve{\theta}_1', \breve{\theta}_2']$, $q(\theta') < q^{fb}(\theta')$. Then by Lemma 9 and Corollary 1, $\tau^{-1}(\theta) = \{\breve{\theta}\}$ for all $\theta \in [\breve{\theta}_1', \breve{\theta}_2']$.

Define $\tau^{-k}(.) = \tau^{-1}(\tau^{-(k-1)}(.))$ where k is a positive integer k. By Lemmas 12 and 14 there exists $M \geq 0$ such that $\tau^{-k}(\theta)$ is a singleton for $k \leq M$ and empty for $k > M$. So, if $M \geq 1$, then for $k \in \{1, ..., M\}$ let us define $\theta^k = \tau^{-k}(\theta)$.

For any $\epsilon > 0$ and $k = 0, ..., M$, let $\Theta_k(\epsilon) = [\breve{\theta}^k - (\frac{1}{\delta_\tau} + 1)^k \epsilon, \breve{\theta}^k + (\frac{1}{\delta_\tau} + 1)^k \epsilon]$.

Now consider an alternative mechanism $(\tilde{q}(.), \tilde{t}(.))$ which differs from the original mechanism $(q(.), t(.)))$ only as follows: for $\theta \in [\breve{\theta}_1', \breve{\theta}_2']$, $\tilde{q}(\theta) = q(\theta) + \epsilon^3$ and $\tilde{t}(\theta) = t(\theta) +$

$u(q(\theta)+\epsilon^3,\theta)-u(q(\theta),\theta)$, and for $\theta\in\cup_{k=0}^M\Theta_k(\epsilon)$, $\tilde{t}(\theta)=t(\theta)-\Delta(\epsilon)$, where $\Delta(\epsilon)\equiv\max_{\theta'\in[\breve{\theta}_1',\breve{\theta}_2']}u(q(\theta')+\epsilon^3,1)-u(q(\theta'),1)-u(q(\theta')+\epsilon^3,\theta')+u(q(\theta'),\theta')$. We will show that all IC and IR are satisfied in the new contract.

First, IR constraints hold in $(\tilde{q}(.),\tilde{t}(.))$ because $\tilde{V}(\theta)>V(\theta)$ for $\theta\in\cup_{k=0}^M\Theta_k(\epsilon)$, and $\tilde{V}(\theta)=V(\theta)$ for all other types θ.

Now consider incentive constraints. For $\theta\in\cup_{k=0}^M\Theta_k(\epsilon)$ and $\theta'\in[\breve{\theta}_1',\breve{\theta}_2']$,

$$\begin{aligned}&\tilde{V}(\theta)=V(\theta)+\Delta(\epsilon)\geq u(q(\theta'),\theta)-t(\theta')-C+\\&[u(q(\theta')+\epsilon^3,\theta)-u(q(\theta'),\theta)-u(q(\theta')+\epsilon^3,\theta')+u(q(\theta'),\theta')]=u(\tilde{q}(\theta'),\theta)-\tilde{t}(\theta')-C,\end{aligned}$$

where the first equality holds by definition of $\tilde{t}(\theta)$; the first inequality holds by $IC(\theta,\theta')$, definition of $\Delta(\theta,\epsilon)$ and $u_{\theta q}>0$; the last equality holds by definitions of $\tilde{q}(\theta')$ and $\tilde{t}(\theta')$.

The case $\theta\in\cup_{k=0}^M\Theta_k(\epsilon)$, $\theta'\in\cup_{k=0}^M\Theta_k(\epsilon)$ is similar to the previous one and therefore omitted.

For $\theta\notin\cup_{k=0}^M\Theta_k(\epsilon)$, $\theta'\in\cup_{k=0}^M\Theta_k(\epsilon)$ and small enough ϵ,

$$\begin{aligned}&\tilde{V}(\theta)=V(\theta)>u(q(\theta'),\theta)-t(\theta')-C+\delta_V\delta_\tau\frac{\epsilon^2}{4}=u(\tilde{q}(\theta'),\theta)-\tilde{t}(\theta')-C+\delta_V\delta_\tau\frac{\epsilon^2}{2}-\Delta(\epsilon)\\&>u(\tilde{q}(\theta'),\theta)-\tilde{t}(\theta')-C,\end{aligned}$$

where the first inequality holds because $\theta'\in[\breve{\theta}^k-(\frac{1}{\delta_\tau}+1)^k\epsilon,\breve{\theta}^k+(\frac{1}{\delta_\tau}+1)^k\epsilon]$ for some k, so Lemma 12 implies $\tau^{-1}(\theta')\in[\breve{\theta}^{k+1}-\frac{1}{\delta_\tau}(\frac{1}{\delta_\tau}+1)^k\epsilon,\breve{\theta}^{k+1}+\frac{1}{\delta_\tau}(\frac{1}{\delta_\tau}+1)^k\epsilon]$, and since $|\theta-\tau^{-1}(\theta')|\geq(\frac{1}{\delta_\tau}+1)^k\epsilon\geq\epsilon$, while Lemma 13 implies that $V(\theta)-U(\theta'|\theta)\geq\delta_V\delta_\tau\frac{\epsilon^2}{4}$ for small ϵ; the second equality holds by the definitions of $\tilde{q}(\theta')$ and $\tilde{t}(\theta')$; the last inequality holds for small ϵ.

Finally, the case $\theta\notin\cup_{k=0}^M\Theta_k(\epsilon)$, $\theta'\in[\breve{\theta}_1',\breve{\theta}_2']$ is similar to the previous one and therefore omitted. Thus, all $\tilde{IC}(\theta,\theta')$ are satisfied for small enough ϵ.

The change in seller's profits from switching to the new mechanism is equal to

$$\int_{\breve{\theta}_1'}^{\breve{\theta}_2'}[u(q(\theta')+\epsilon^3,\theta')-u(q(\theta'),\theta')]f(\theta')d\theta'-F(\cup_{k=0}^M[\breve{\theta}^k-(\frac{1}{\delta_\tau}+1)^k\epsilon,\breve{\theta}^k+(\frac{1}{\delta_\tau}+1)^k\epsilon])\Delta(\epsilon).$$

Since $\lim_{\epsilon\to0}\frac{\int_{\breve{\theta}_1'}^{\breve{\theta}_2'}[u(q(\theta')+\epsilon^3,\theta')-u(q(\theta'),\theta')]f(\theta')d\theta'}{\epsilon^3}\in(0,\infty)$, $\lim_{\epsilon\to0}\frac{\Delta(\epsilon)}{\epsilon^3}\in(0,\infty)$ and $\lim_{\epsilon\to0}F(\cup_{k=0}^M[\breve{\theta}^k-(\frac{1}{\delta_\tau}+1)^k\epsilon,\breve{\theta}^k+(\frac{1}{\delta_\tau}+1)^k\epsilon])=0$, our alternative mechanism generates a higher profit when ϵ is small while satisfying IC and IR constraints, contradiction. *Q.E.D.*

The next Lemma shows that $\tau(\Theta)$ is non-empty when C is not too high, while low and high types are never targeted by another type. To state it, define:

$$G(\theta,\theta') = u(q^{fb}(\theta'),\theta) - u(q^{fb}(\theta'),\theta'), \tag{54}$$

$$\overline{C} = \max_{\theta,\theta'\in[0,1]} G(\theta,\theta') = \max_{\theta'\in[0,1]} G(1,\theta') \tag{55}$$

Lemma 16 *In an optimal mechanism: (i) $\tau([0,1]) \neq \emptyset$ if $C < \overline{C}$;*
(ii) $\tau([0,1]) = \emptyset$ if $C > \overline{C}$.
(iii) For any $C > 0$, there exists $\underline{\theta}, \overline{\theta} \in (0,1)$ such that $\theta \notin \tau([0,1])$ for any $\theta \in [0,\underline{\theta}) \cup (\overline{\theta},1]$.

Proof of Lemma 16:

(i) To prove the first claim of the Lemma we argue by contradiction. So suppose that $\tau([0,1]) = \emptyset$. Then for all $\theta \in [0,1]$, $V(\theta) = 0$ by Lemma 7, and $q(\theta) = q^{fb}(\theta)$ by Lemma 9. But then $IC(1,\theta)$ fails for some θ because $C < \overline{C} = \max_{\theta,\theta'} u(q^{fb}(\theta'),\theta) - u(q^{fb}(\theta'),\theta')$.

(ii) The proof that $\tau([0,1]) = \emptyset$ if $C > \overline{C}$ is straightforward and is therefore omitted.

(iii) If $\tau(1) = \emptyset$, then Lemma 7 implies $\tau([0,1]) = \emptyset$, so $\overline{\theta} = \underline{\theta}$. If $\tau(1) \neq \emptyset$, then let $\overline{\theta} = \max\{\theta' : \theta' \in \tau(1)\} < 1$, where the maximum exists by Lemma 6. Then Corollary 1 implies that $\theta' \leq \overline{\theta}$ for any $\theta' \in \tau([0,1])$.

Since $q^{fb}(\theta)$ is continuous and increasing in θ, $q^{fb}(0) = 0$ and $u(q^{fb}(0),1) = u(0,1) = 0$, there exists $\underline{\theta} > 0$ such that $u(q^{fb}(\theta),1) - C < 0$ for all $\theta \in [0,\underline{\theta}]$. By Lemma 9 $q(\theta) \leq q^{fb}(\theta)$ and by Lemma 3 $t(\theta) > 0$ for all $\theta \in [0,1]$. So for any $\theta' \in [0,1]$ and $\theta \in [0,\underline{\theta}]$, $u(q(\theta),\theta') - t(\theta) - C \leq u(q(\theta),1) - t(\theta) - C < 0$, which implies that $\theta \notin \tau([0,1])$. *Q.E.D.*

Lemma 17 shows that for a range of C, any type $\theta \in \tau([0,1])$ gets zero surplus.

Lemma 17 *There exists $\underline{C} \in (0,\overline{C})$, such that in the optimal mechanism for any $C \in [\underline{C},\overline{C}]$, $V(\theta') = 0$ if $\theta' \in \tau([0,1])$.*

Proof of Lemma 17: Let $G^*(\theta) = \max_{\theta'} G(\theta,\theta')$ where $G(\theta,\theta')$ is defined in (54). $G^*(\theta)$ is continuous and strictly increasing in θ since $G(.,.)$ is continuous and $u_{q\theta} > 0$. Also define:

$$\hat{\Theta}(C) = \{\theta \in [0,1] : G^*(\theta) \geq C\}.$$

For any $C \in (0,\overline{C})$, $\hat{\Theta}(C)$ is non-empty. Furthermore, since $G^*(\theta)$ is continuous and increasing in θ, there exists $\theta^C \in (0,1)$ such that $\hat{\Theta}(C) = [\theta^C,1]$, with $\lim_{C\to\overline{C}} \theta^C \to 1$. Next, we show that whenever $C \in (\underline{C},\overline{C})$, for some $\underline{C} \in (0,\overline{C})$, in an optimal mechanism:

(i) $V(\theta) = 0$ for all $\theta \notin \hat{\Theta}(C)$, (ii) $\hat{\Theta}(C) \cap \tau(\hat{\Theta}(C)) = \emptyset$.

To establish (i), suppose that $V(\theta) > 0$ for some $\theta \notin \hat{\Theta}(C)$. Then consider an alternative mechanism $(\tilde{q}(.), \tilde{t}(.))$ which differs from the original mechanism $(q(.), t(.))$ only in transfers. Particularly, $\tilde{t}(\theta) = u(q(\theta), \theta)$ for $\theta \notin \hat{\Theta}(C)$, $\tilde{t}(\theta) = \max\{u(q(\theta), \theta^C) - C, t(\theta)\}$ for $\theta \in \hat{\Theta}(C)$. So, $(\tilde{q}(.), \tilde{t}(.))$ is more profitable for the seller than $(q(.), t(.))$.

Let us verify that $(\tilde{q}(.), \tilde{t}(.))$ is individually rational. $IR(\theta)$ is binding by construction for $\theta \notin \hat{\Theta}(C)$. If $\theta \in \hat{\Theta}(C)$ and $\tilde{t}(\theta) = t(\theta)$ then $IR(\theta)$ holds in $(\tilde{q}(.), \tilde{t}(.))$ because it holds in $(q(.), t(.))$. If $\tilde{t}(\theta) = u(q(\theta), \theta^C) - C > t(\theta)$, then θ gets a payoff $u(q(\theta), \theta) - u(q(\theta), \theta^C) + C \geq 0$ which is nonnegative since $\theta \geq \theta^C$.

Next, $IC(\theta, \theta')$ holds in $(\tilde{q}(.), \tilde{t}(.))$ for $(\theta, \theta') \in ([0, 1] \setminus \hat{\Theta}(C)) \times \hat{\Theta}(C)$ because the payoff $\tilde{V}(.)$ in this mechanism satisfies $\tilde{V}(\theta) = 0$ for all $\theta \in [0, 1] \setminus \hat{\Theta}(C)$. At the same time, $\tilde{t}(\theta') \geq u(q(\theta'), \theta^C) - C$. So by imitating θ', θ gets at most $u(q(\theta'), \theta) - u(q(\theta'), \theta^C) \leq 0$.

For $(\theta, \theta') \in ([0, 1] \setminus \hat{\Theta}(C)) \times ([0, 1] \setminus \hat{\Theta}(C))$, $IC(\theta, \theta')$ holds because:

$$u(q(\theta'), \theta) - \tilde{t}(\theta') - C = u(q(\theta'), \theta) - u(q(\theta'), \theta') - C \leq \max\{0, u(q^{fb}(\theta'), \theta) - u(q^{fb}(\theta'), \theta') - C\} \leq 0,$$

where the first inequality holds because $q(\theta') \leq q^{fb}(\theta')$, and the second holds by assumption.

Now consider a pair $(\theta, \theta') \in \hat{\Theta}(C) \times ([0, 1] \setminus \hat{\Theta}(C))$. If $\tilde{t}(\theta) = t(\theta)$, then $IC(\theta, \theta')$ holds in $(\tilde{q}(.), \tilde{t}(.))$ because it holds in $(q(.), t(.))$ and $\tilde{t}(\theta') \geq t(\theta')$. If $\tilde{t}(\theta) = u(q(\theta), \theta^C) - C > t(\theta)$, then $IC(\theta, \theta')$ can be rewritten as: $u(q(\theta), \theta) - u(q(\theta), \theta^C) + C \geq u(q(\theta'), \theta) - u(q(\theta'), \theta') - C$. The last inequality holds if $C \geq \frac{\overline{C}}{2}$ because $q(\theta') \leq q^{fb}(\theta')$.

Finally, consider a pair $(\theta, \theta') \in \hat{\Theta}(C) \times \hat{\Theta}(C)$. $IC(\theta, \theta')$ holds iff

$$u(q(\theta'), \theta) - u(q(\theta), \theta) - \tilde{t}(\theta') + \tilde{t}(\theta) \leq C. \tag{56}$$

Recall that $q(.)$ and $t(.)$ are continuous by Lemma 5. So, $\tilde{t}(.)$ is also continuous. As shown above, $\lim_{C \to \overline{C}} \theta^C = 1$. Since $\min\{\theta, \theta'\} \geq \theta^C$, we have $\lim_{C \to \overline{C}} |\theta - \theta'| = 0$. This and the fact that $\max\{q(\theta), q(\theta')\} \leq q^{fb}(1) < \infty$ imply that the left-hand side of (56) converges to zero as C increases to $\bar{C}$. So the inequality (56) holds strictly when C is sufficiently close to $\bar{C}$, and $IC(\theta, \theta')$ is slack. Hence, the mechanism $(\tilde{q}(.), \tilde{t}(.))$ is incentive compatible.

Finally, to establish claim (ii), suppose that there exists $\theta' \in \hat{\Theta}(C) \cap \tau(\hat{\Theta}(C))$. Then $IC(\theta, \theta')$ is binding for some $\theta \in \hat{\Theta}(C)$. But this cannot be true when C is close to $\overline{C}$, since in this case (56) holds strictly. *Q.E.D.*

7 Appendix B

This Appendix contains the proofs of Theorems 5, 6, 7, 9, 10, and Lemmas 1 and 2.

Proof of Theorem 5: Fix an admissible triple $(q(.), \tau(.), \hat{\theta})$ and let $(q(.), t(.))$ be the mechanism corresponding to it, so that $t(.)$ is given by (8). The mechanism $(q(.), t(.))$ is incentive compatible iff $D(\theta, \theta') \geq 0$ for all $\theta, \theta' \in [0, 1]$, where

$$\begin{aligned} D(\theta, \theta') &\equiv \int_{\max\{\theta', \hat{\theta}\}}^{\max\{\theta, \hat{\theta}\}} u_\theta(q(\max \tau(s)), s) ds - u(q(\theta'), \theta) + u(q(\theta'), \theta') + C \qquad (57) \\ &= V(\theta) - V(\theta') - u(q(\theta'), \theta) + u(q(\theta'), \theta') + C. \end{aligned}$$

First, let us show that $D(\theta, \theta') \geq 0$ for all $\theta \in [\hat{\theta}, 1]$ and $\theta' \in [\max \tau(\hat{\theta}), \theta)$.

From (57) we obtain that $D_\theta(\theta, \theta') = u_\theta(q(\max \tau(\theta)), \theta) - u_\theta(q(\theta'), \theta)$. Since $q(.)$ is increasing, it follows that $D_\theta(\theta, \theta') > 0$ if $\theta' < \max \tau(\theta)$; $D_\theta(\theta, \theta') < 0$ if $\theta' > \max \tau(\theta)$ and $D_\theta(\theta, \theta') = 0$ if $\theta' = \max \tau(\theta)$. So, $D(\theta, \theta')$ has a unique minimum in θ at θ s.t. $\theta' \in \tau(\theta)$.

Further, differentiating (57) with respect to θ' yields:

$$D_{\theta'}(\theta, \theta') = u_\theta(q(\theta'), \theta') - 1(\theta' \geq \hat{\theta}) u_\theta(q(\tau(\theta'), \theta') - [u_q(q(\theta'), \theta) - u_q(q(\theta'), \theta')] \frac{dq(\theta')}{d\theta'}. \qquad (58)$$

From (73) and (58) it follows that $D_{\theta'}(\theta, \theta') = 0$ for almost all θ and $\theta' \in \tau(\theta)$. This and $D_\theta(\theta, \theta') = 0$ for $\theta' = \max \tau(\theta)$ implies that $\frac{dD(\theta, \tau(\theta))}{d\theta} = 0$ for almost all θ.

At the same time, $D(\hat{\theta}, \tau(\hat{\theta})) = -u(q(\tau(\hat{\theta})), \hat{\theta}) + u(q(\tau(\hat{\theta})), \tau(\hat{\theta})) + C = 0$ by part (iv) of Definition 1. Therefore, $D(\theta, \tau(\theta)) = 0$ for all $\theta \in [\hat{\theta}, 1]$. This and the fact that $D(\theta, \theta')$ has a unique minimum in θ at θ s.t. $\theta' \in \tau(\theta)$ imply that $D(\theta, \theta') \geq 0$ for all $\theta \in [\hat{\theta}, 1]$, $\theta' \in [\max \tau(\hat{\theta}), \theta)$.

Now consider $\theta \in [\hat{\theta}, 1]$, and $\theta' \in [0, \tau(\hat{\theta}))$. We have:

$$\begin{aligned} D(\theta, \theta') &\equiv \int_{\hat{\theta}}^{\theta} u_\theta(q(\max \tau(s)), s) ds - u(q(\theta'), \theta) + u(q(\theta'), \theta') + C \geq \\ &\int_{\hat{\theta}}^{\theta} u_\theta(q(\max \tau(s)), s) - u_\theta(q(\theta'), s) ds \geq 0, \qquad (59) \end{aligned}$$

where the first inequality holds because $u(q(\theta'), \hat{\theta}) - u(q(\theta'), \theta') \leq C$ for all $\theta' \in [0, \hat{\theta}]$ and the last inequality holds because $q(.)$ and $\tau(.)$ are increasing.

Now, consider $\theta \in [0, \hat{\theta}]$ and $\theta' \in [0, \hat{\theta}]$. then $D(\theta, \theta') = -u(q(\theta'), \theta) + u(q(\theta'), \theta') + C \geq 0$ because $\theta < \hat{\theta}$ and $-u(q(\theta'), \hat{\theta}) + u(q(\theta'), \theta') + C \geq 0$.

Finally, if $\theta' > \theta$, then $D(\theta, \theta') > 0$ follows from (57) since $q(.)$ is increasing and $\tau(\theta) < \theta$ for all θ. This completes the proof of incentive compatibility of $(q(.), t(.))$.

To establish the second claim of the Theorem, first, suppose that $(q(.), t(.))$ is an optimal mechanism. Let us show that corresponding triple $(q(.), \tau(.), \hat{\theta})$ must be admissible. Indeed, By claim 1 of Theorem 3 and claims 2,3 and 6 of Theorem 4, $(q(.), \tau(.), \hat{\theta})$ satisfies (i) and (ii) of Definition 1. Since $(q(.), t(.))$ is incentive compatible, $(q(.), \tau(.))$ satisfy equation (73) a.e. on $[\hat{\theta}, 1]$. To show that it satisfies (iv) in Definition 1, suppose that $u(q^{fb}(\theta'), \hat{\theta}) - u(q^{fb}(\theta'), \theta') = C$ for some $\theta' \in [0, \tau(\hat{\theta})]$. Then $q(\theta) = q^{fb}(\theta)$ and $G(\hat{\theta}, \theta) = u(q^{fb}(\theta), \hat{\theta}) - u(q^{fb}(\theta), \theta) > C$ for all $\theta \in [\theta', \tau(\hat{\theta})]$, the latter by quasiconcavity of $G(\theta, \theta')$ in θ'. But this contradicts the incentive compatibility of $(q(.), t(.))$. Hence, $(q(.), \tau(.), \hat{\theta})$ is *admissible.*

Finally, if a triple $(q(.), \tau(.), \hat{\theta})$ does not maximize (10), then by construction its corresponding mechanism does not solve the problem (1)-(3). So, an admissible triple maximizing (10) corresponds to an optimal mechanism. *Q.E.D.*

Proof of Theorem 6: Since $\tau(.)$ is single-valued and differentiable a.e. on $[\hat{\theta}, 1]$, so is a higher order attracted type $\tau^k(.)$, for all k. Therefore, there is a unique chain of targeted types $(\theta, \tau(\theta), ..., \tau^k(\theta))$ emanating from θ for almost all $\theta \in [\hat{\theta}, 1]$.

The proof of (13) uses a perturbation method and proceeds by contradiction. In particular, suppose that for some $\tilde{\theta} \in (\underline{\theta}, \overline{\theta})$ and $s \in \{1, ..., M\}$ there exists $\mu > 0$ s.t.

$$u_q(q(\tau^s(\tilde{\theta}))), \tau^s(\tilde{\theta}))f(\tau^s(\tilde{\theta}))\dot{\tau}^s(\tilde{\theta}) - [u_q(q(\tau^s(\tilde{\theta})), \tau^{s-1}(\tilde{\theta})) - u_q(q(\tau^s(\tilde{\theta})), \tau^s(\tilde{\theta}))] \sum_{k=1}^{s} f(\tau^{s-k}(\tilde{\theta}))\dot{\tau}^{s-k}(\tilde{\theta})$$
$$> \mu. \tag{60}$$

(The proof of the opposite case is similar and will therefore be omitted.)

As explained in the text below the statement of the Theorem, the left-hand side of (84) is the marginal efficiency gain of raising q on a neighborhood of $\tau^s(\tilde{\theta})$, while its right hand side is the associated expected marginal increase in the surplus of the types in the neighborhoods around the predecessors of $\tau^s(\tilde{\theta})$ in the chain of targeted types, $\tau^{s-k}(\tilde{\theta})$ for $k = 1, ..., s$. So, if (84) holds, the principal can get higher profits by raising the quantities of the types around $\tau^s(\tilde{\theta})$. The rest of the proof formalizes this intuition via three steps.

Step 1. Constructing an Alternative Mechanism.

For $\epsilon > 0$ and $k = 0, ..., s$, let $\Theta_k(\epsilon) = [\tau^k(\tilde{\theta} - \epsilon) - (\frac{\delta_\tau}{2})^k \epsilon^2, \tau^k(\tilde{\theta} + \epsilon) + (\frac{\delta_\tau}{2})^k \epsilon^2]$ where δ_τ satisfies the conditions of Lemma 12.

The alternative mechanism differs from the original mechanism $(q(.), t(.)))$ only as follows: for $\theta \in \Theta_s(\epsilon)$, $\tilde{q}(\theta) = q(\theta) + \epsilon^5$ and $\tilde{t}(\theta) = t(\theta) + u(q(\theta) + \epsilon^5, \theta) - u(q(\theta), \theta)$, and for $\theta \in \cup_{k=0}^{s-1}\Theta_k(\epsilon)$, $\tilde{t}(\theta) = t(\theta) - \Delta(\epsilon)$, where $\Delta(\epsilon) \equiv \max_{\theta' \in \Theta_s(\epsilon)} u(q(\theta') + \epsilon^5, \overline{\theta}_{s-1}) - u(q(\theta'), \overline{\theta}_{s-1}) - u(q(\theta') + \epsilon^5, \theta') + u(q(\theta'), \theta')$ and $\overline{\theta}_{s-1} = \max \Theta_{s-1}(\epsilon)$.

Step 2. Establishing that the mechanism $(\tilde{q}(.), \tilde{t}(.))$ is more profitable for the principal than the original mechanism.

The change in seller's profits from switching to $(\tilde{q}(.), \tilde{t}(.))$ from $(q(.), t(.))$ is equal to

$$\Pi(\epsilon) = \int_{\Theta_s(\epsilon)} [u(q(\theta) + \epsilon^5, \theta) - u(q(\theta), \theta)] f(\theta) d\theta - \Delta(\epsilon) \sum_{k=0}^{s-1} \int_{\Theta_k(\epsilon)} f(\theta) d\theta.$$

Letting $H(\theta', \theta) = u_q(q(\theta'), \theta) - u_q(q(\theta'), \theta')$ we may then compute:

$$\begin{aligned}
\lim_{\epsilon \to 0} \frac{\Pi(\epsilon)}{\epsilon^6} &= \lim_{\epsilon \to 0} \frac{\sum_{k=0}^{s-1} \int_{\Theta_k(\epsilon)} f(\theta) d\theta]}{\epsilon} \left(\int_{\Theta_s(\epsilon)} u_q(q(\theta), \theta) f(\theta) d\theta - \max_{\theta' \in \Theta_s(\epsilon)} H(\theta', \overline{\theta}_{s-1}) \right) \\
&= \lim_{\epsilon \to 0} \frac{\sum_{k=0}^{s-1} \dot{\tau}^k(\theta) f(\tau^k(\theta))}{\epsilon} \left(\int_{\tilde{\theta}-\epsilon}^{\tilde{\theta}+\epsilon} u_q(q(\tau^s(\theta)), \tau^s(\theta)) f(\tau^s(\theta)) \dot{\tau}^s(\theta) d\theta - \max_{\theta' \in \Theta_s(\epsilon)} H(\theta', \overline{\theta}_{s-1} \right) \\
&= 2 \left(u_q(q(\tau^s(\tilde{\theta}), \tau^s(\tilde{\theta})) f(\tau^s(\tilde{\theta})) \dot{\tau}^s(\tilde{\theta}) - H(\tau^s(\tilde{\theta}), \tau^{s-1}(\tilde{\theta})) \right) \sum_{k=0}^{s-1} \dot{\tau}^k(\tilde{\theta}) f(\tau^k(\tilde{\theta}))] > 2\mu > 0,
\end{aligned}$$

where the first equality holds by definition of Δ; the second equality holds by eliminating the second order residuals in $\cup_{k=0}^{s}\Theta_k(\epsilon)$ and making a change of variables; the third equality holds since $\overline{\theta}_{s-1} \to \tau^{s-1}(\tilde{\theta})$ and $\Theta_s(\epsilon) \to \{\tau^s(\tilde{\theta})\}$ as $\epsilon \to 0$; and the first inequality holds by (84). So, $\Pi(\epsilon) > 0$ for small enough ϵ, contradicting the optimality of $(q(.), t(.))$.

Step 3. Establishing individual rationality and incentive compatibility of the alternative mechanism $(\tilde{q}(.), \tilde{t}(.))$.

IR constraints hold in $(\tilde{q}(.), \tilde{t}(.))$ because they hold in $(q(.), t(.))$, and $\tilde{V}(\theta) \geq V(\theta)$ for all θ.

Next, let us show that incentive constraints in $(\tilde{q}(.), \tilde{t}(.))$, $\tilde{IC}(\theta, \theta')$, hold. We will need to consider several subsets of $[0,1]^2$. First, if $\theta \in [0,1]$ and $\theta' \notin \cup_{k=0}^{s}\Theta_k(\epsilon)$, then $\tilde{IC}(\theta, \theta')$ holds because $\tilde{V}(\theta) \geq V(\theta)$, $\tilde{q}(\theta') = q(\theta')$, $\tilde{t}(\theta') = t(\theta')$ and $IC(\theta, \theta')$ holds in $(q(.), t(.))$.

Second, if $\theta \in [0,1]$ and $\theta' \in \Theta_0(\epsilon)$, then for small enough ϵ, $\tau^{-1}(\theta') = \emptyset$ since $\Theta_0(\epsilon) \subset (\underline{\theta}, \overline{\theta}) \subset [\tau(1), 1]$. So, in the original mechanism incentive constraints $IC(\theta, \theta')$ are slack on this set of types, with minimal slack $\delta > 0$ over all $\theta \in [0,1]$ and all $\theta' \in \Theta_0(\epsilon)$. In the mechanism, $(\tilde{q}_j(.), \tilde{t}_j(.))$, $\tilde{V}(\theta) \geq V(\theta)$ for all $\theta \in [0,1]$, and $\tilde{V}(\theta') = V(\theta') + \Delta(\epsilon)$ for $\theta' \in \Theta_0(\epsilon)$. Therefore, $\tilde{IC}(\theta, \theta')$ holds for sufficiently small ϵ i.e., when $\Delta(\epsilon) \leq \delta$.

Third, consider $IC(\theta, \theta')$ where $\theta \in \Theta_{s-1}(\epsilon)$ and $\theta' \in \Theta_s(\epsilon)$. Recall that $U(\theta, \theta') = u(q(\theta'), \theta) - t(\theta') - C$. So we have:

$$\begin{aligned} &\tilde{V}(\theta) = V(\theta) + \Delta(\epsilon) \geq U(\theta, \theta') + \Delta(\epsilon) \geq u(q(\theta'), \theta) - t(\theta') - C \\ &+ [u(q(\theta') + \epsilon^5, \theta) - u(q(\theta') + \epsilon^5, \theta')] - [u(q(\theta'), \theta) - u(q(\theta'), \theta')] = u(\tilde{q}(\theta'), \theta) - \tilde{t}(\theta') - C, \end{aligned}$$

where the first equality holds by definition of $\tilde{t}(\theta)$; the first inequality holds by incentive compatibility of the original mechanism; the second inequality holds by definition of $\Delta(\epsilon)$, $\theta \leq \overline{\theta}_{s-1}$ and $u_{\theta q} > 0$; the last equality holds by definitions of $\tilde{q}(\theta')$ and $\tilde{t}(\theta')$.

The remaining cases are similar to the third one and are therefore omitted. *Q.E.D.*

Proof of Lemma 1: Part (i): Suppose to the contrary that there exists θ s.t. $\tau(\theta)$ is multi-valued. Let $\theta_2 = \max \tau(\theta) > \theta_1 = \min \tau(\theta)$. θ_1 and θ_2 exist by Lemma 6. By assumption $V(\theta_2) = V(\theta_1) = 0$. By Lemma 5, $V(\theta') = 0$ $\forall \theta' \in [\theta_1, \theta_2]$, and by Lemma 15, $q(\theta') = q^{fb}(\theta')$ for all $\theta' \in [\theta_1, \theta_2]$. Hence, $u(q(\theta'), \theta) - t(\theta') = G(\theta, \theta')$ for all $\theta' \in [\theta_1, \theta_2]$. By quasi-concavity, $G(\theta, \theta') > \min\{G(\theta, \theta_1), G(\theta, \theta_2)\}$ for all $\theta' \in (\theta_1, \theta_2)$. So, $u(q^{fb}(\theta'), \theta) - t(\theta') > \min\{u(q^{fb}(\theta_1), \theta) - t(\theta_1), u(q^{fb}(\theta_2), \theta) - t(\theta_2)\}$, contradicting $\theta_1, \theta_2 \in \tau(\theta)$.

Part (ii): Suppose to the contrary that there exists θ such that $\tau(\theta)$ is multi-valued. If $V(\max \tau(\theta)) = 0$, then the same argument as in part (i) establishes a contradiction, so it must be that $V(\max \tau(\theta)) > 0$. Let $\Theta_m = \{\theta : \tau(\theta) \text{ is multi-valued}\}$ and $\underline{\theta}_m = \inf \Theta_m$. Fix some $\tilde{\theta} \in \Theta_m$ such that $\tilde{\theta} - \underline{\theta}_m < \frac{C}{\overline{K} q^{fb}(1)}$ where $\overline{K} = \max_{\theta \in [0,1], q(\theta) \in [0, q^{fb}(\theta)]} u_{\theta q}(q(\theta), \theta)$. Then by Lemma 14 in Appendix B, $\max \tau(\tilde{\theta}) < \underline{\theta}_m$.

Now, let $\theta_2 = \max \tau(\tilde{\theta})$ and $\theta_1 = \min \tau(\tilde{\theta})$. Then $V(\theta_2) > 0$ by part (i) of this Lemma, and so $\tau(\theta_2)$ is non-empty by Lemma 7. As shown above, $\theta_2 < \underline{\theta}_m$. So, $\tau(\theta_2)$ is single-valued.

Let $\tilde{G}(\theta, \theta') = G(\theta, \theta') + V(\theta') = u(q^{fb}(\theta'), \theta) - u(q^{fb}(\theta'), \theta') + \int_{\hat{\theta}}^{\max\{\theta', \hat{\theta}\}} u_\theta(q(\tau(s)), s) ds$, where the second equality holds by definition of G and equation (7). By Lemma 15, $q(\theta') = q^{fb}(\theta')$ for all $\theta' \in [\theta_1, \theta_2]$, which implies that $u(q(\theta'), \tilde{\theta}) - t(\theta') = \tilde{G}(\tilde{\theta}, \theta')$ for all $\theta' \in [\theta_1, \theta_2]$.

Since $IC(\tilde{\theta}, \theta)$ holds, $\tilde{G}(\tilde{\theta}, \theta_2) \geq \tilde{G}(\tilde{\theta}, \theta)$ for θ slightly below θ_2, implying that

$$\frac{\partial \tilde{G}(\tilde{\theta}, \theta_2)}{\partial \theta_2} = u_q(q^{fb}(\theta_2), \tilde{\theta}) \dot{q}^{fb}(\theta_2) - u_\theta(q^{fb}(\theta_2), \theta_2) + u_\theta(q(\tau(\theta_2)), \theta_2) \geq 0. \tag{61}$$

Next, we have:

$$
\begin{aligned}
&(\tilde{\theta}-\theta_2)u_{\theta q}(q^{fb}(\theta_2),\tilde{\theta})-\int_{\theta_2}^{\tilde{\theta}}(s-\theta_2)u_{\theta\theta q}(q^{fb}(\theta_2),s)ds=\int_{\theta_2}^{\tilde{\theta}}u_{\theta q}(q^{fb}(\theta_2),s)ds=u_q(q^{fb}(\theta_2),\tilde{\theta})\\
&\geq\int_{\tau(\theta_2)}^{\theta_2}u_{\theta q}(q^{fb}(s),\theta_2)\frac{\dot{q}^{fb}(s)}{\dot{q}^{fb}(\theta_2)}ds+\int_{q(\tau(\theta_2))}^{q^{fb}(\tau(\theta_2))}u_{\theta q}(q,\theta_2)\frac{1}{\dot{q}^{fb}(\theta_2)}dq\\
&=\int_{\tau(\theta_2)}^{\theta_2}u_{\theta q}(q^{fb}(s),\theta_2)\frac{u_{\theta q}(q^{fb}(s),s)u_{qq}(q^{fb}(\theta_2),\theta_2)}{u_{\theta q}(q^{fb}(\theta_2),\theta_2)u_{qq}(q^{fb}(s),s)}ds+\int_{q(\tau(\theta_2))}^{q^{fb}(\tau(\theta_2))}u_{\theta q}(q,\theta_2)\frac{-u_{qq}(q^{fb}(\theta_2),\theta_2)}{u_{\theta q}(q^{fb}(\theta_2),\theta_2)}dq\\
&\geq\int_{\tau(\theta_2)}^{\theta_2}u_{\theta q}(q^{fb}(s),\theta_2)ds+\int_{q(\tau(\theta_2))}^{q^{fb}(\tau(\theta_2))}[-u_{qq}(q^{fb}(\theta_2),\theta_2)]dq=
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\tau(\theta_2)}^{\theta_2}u_{\theta q}(q^{fb}(\theta_2),\tilde{\theta})ds-\int_{\tau(\theta_2)}^{\theta_2}\int_{\theta_2}^{\tilde{\theta}}u_{\theta\theta q}(q^{fb}(\theta_2),r)drds+\int_{q(\tau(\theta_2))}^{q^{fb}(\tau(\theta_2))}[-u_{qq}(q^{fb}(\theta_2),\theta_2)]dq=\\
&(\theta_2-\tau(\theta_2))u_{\theta q}(q^{fb}(\theta_2),\tilde{\theta})-\int_{\theta_2}^{\tilde{\theta}}(\theta_2-\tau(\theta_2))u_{\theta\theta q}(q^{fb}(\theta_2),s)ds-\int_{\tau(\theta_2)}^{\theta_2}(s-\tau(\theta_2))u_{\theta\theta q}(q^{fb}(\theta_2),s)ds\\
&+[q^{fb}(\tau(\theta_2))-q(\tau(\theta_2))][-u_{qq}(q^{fb}(\theta_2),\theta_2)]. \qquad (62)
\end{aligned}
$$

To establish the first equality use integration by parts. The second equality holds because $u_q(q^{fb}(\theta_2),\theta_2)=0$. The first inequality holds by (61). The third equality holds because $\dot{q}^{fb}(\theta)=\frac{-u_{\theta q}(q^{fb}(\theta),\theta)}{u_{qq}(q^{fb}(\theta),\theta)}$. The second inequality holds because $u_{\theta qq}\leq 0$, $\dot{q}^{fb}(\theta)>0$, and $u_{qqq}\leq 0$, so $u_{qq}(q^{fb}(\theta_2),\theta_2)\leq u_{qq}(q^{fb}(s),s)<0$ for $\theta_2>s$. The fourth inequality holds by the Fundamental Theorem of Calculus. The last equality holds by computation.

Now, let $A=\tilde{\theta}-\theta_2$, $B=\theta_2-\tau(\theta_2)$ and $D=[q^{fb}(\tau(\theta_2))-q(\tau(\theta_2))][-u_{qq}(q^{fb}(\tau(\theta_2)),\theta_2)]$. Then from (62) we have:

$$
(A-B)u_{\theta q}(q^{fb}(\theta_2),\tilde{\theta})+\int_{\theta_2}^{\tilde{\theta}}(2\theta_2-\tau(\theta_2)-s)u_{\theta\theta q}(q^{fb}(\theta_2),s)ds+\int_{\tau(\theta_2)}^{\theta_2}(s-\tau(\theta_2))u_{\theta\theta q}(q^{fb}(\theta_2),s)ds\geq D \qquad (63)
$$

Next, Corollary 2 implies that for almost all $\theta'\in[\hat{\theta},1]$:

$$
\begin{aligned}
\dot{\tau}(\theta')&\geq\frac{f(\theta')[u_q(q(\tau(\theta')),\theta')-u_q(q(\tau(\theta')),\tau(\theta'))]}{f(\tau(\theta'))u_q(q(\tau(\theta')),\tau(\theta'))}\\
&\geq\frac{u_q(q(\tau(\theta')),\theta')-u_q(q(\tau(\theta')),\tau(\theta'))}{u_q(q(\tau(\theta')),\tau(\theta'))}>\frac{u_q(q(\tau(\theta')),\theta')-u_q(q(\tau(\theta')),\theta_2)}{u_q(q(\tau(\theta_2)),\tau(\theta'))}. \qquad (64)
\end{aligned}
$$

Note that the second inequality in (64) holds because $\theta'>\tau(\theta')$ and $f'\geq 0$, and the last inequality holds because $\theta_2>\theta_1\in\tau(\tilde{\theta}))$ and therefore $\theta_1\geq\tau(\theta')$ and $q(\tau(\theta_2))\leq q(\tau(\theta'))$.

Now we have:

$$[\theta_2-\tau(\theta_2)][q^{fb}(\tau(\theta_2))-q(\tau(\theta_2))][-u_{qq}(q^{fb}(\theta_2),\theta_2)]+\frac{(\theta_2-\tau(\theta_2))^2}{2}u_{\theta q}(q^{fb}(\theta_2),\tilde{\theta})$$
$$-[\int_{\theta_2}^{\tilde{\theta}}\frac{(\theta_2-\tau(\theta_2))^2}{2}u_{\theta\theta q}(q^{fb}(\theta_2),s)ds+\int_{\tau(\theta_2)}^{\theta_2}(\theta_2-\frac{s+\tau(\theta_2)}{2})(s-\tau(\theta_2))u_{\theta\theta q}(q^{fb}(\theta_2),s)ds]=$$
$$[\theta_2-\tau(\theta_2)][q^{fb}(\tau(\theta_2))-q(\tau(\theta_2))][-u_{qq}(q^{fb}(\theta_2),\theta_2)]+\int_{\tau(\theta_2)}^{\theta_2}u_q(q^{fb}(\theta_2),s)-u_q(q^{fb}(\theta_2),\tau(\theta_2))ds\geq$$
$$[\theta_2-\tau(\theta_2)]\left(u_q(q(\tau(\theta_2)),\theta_2)-u_q(q^{fb}(\tau(\theta_2)),\theta_2)\right)+\int_{\tau(\theta_2)}^{\theta_2}u_q(q^{fb}(\theta_2),s)-u_q(q^{fb}(\theta_2),\tau(\theta_2))ds\geq$$

$$\int_{\tau(\theta_2)}^{\theta_2}u_q(q(\tau(\theta_2)),\theta_2)+u_q(q^{fb}(\theta_2),s)ds=\int_{\tau(\theta_2)}^{\theta_2}u_q(q(\tau(\theta_2)),\theta_2)-\int_{s}^{\theta_2}u_{q\theta}(q^{fb}(\theta_2),x)dxds$$
$$>\int_{\tau(\theta_2)}^{\theta_2}u_q(q(\tau(\theta_2)),\theta')d\theta'\geq\int_{\theta_2}^{\tilde{\theta}}u_q(q(\tau(\theta_2)),\tau(\theta'))\dot{\tau}(\theta')d\theta'\geq\int_{\theta_2}^{\tilde{\theta}}[u_q(q(\tau(\theta')),\theta')-u_q(q(\tau(\theta')),\theta_2)]d\theta'$$
$$=\int_{\theta_2}^{\tilde{\theta}}\int_{\theta_2}^{\theta'}u_{\theta q}(q(\tau(\theta')),s)dsd\theta'\geq\int_{\theta_2}^{\tilde{\theta}}\int_{\theta_2}^{\theta'}u_{\theta q}(q^{fb}(\theta_2),s)dsd\theta'=\int_{\theta_2}^{\tilde{\theta}}u_q(q^{fb}(\theta_2),\theta')-u_q(q^{fb}(\theta_2),\theta_2)d\theta'$$
$$=\frac{(\tilde{\theta}-\theta_2)^2}{2}u_{\theta q}(q^{fb}(\theta_2),\tilde{\theta})-\int_{\theta_2}^{\tilde{\theta}}(\tilde{\theta}-\frac{s+\theta_2}{2})(s-\theta_2)u_{\theta\theta q}(q^{fb}(\theta_2),s)ds, \tag{65}$$

where the first equality holds by simple computation, the first inequality holds because $u_q(q^{fb}(\tau(\theta_2)),\tau(\theta_2))=0$, $u_{qqq}(q,\theta)\leq 0$, $q(\tau(\theta_2))\leq q^{fb}(\tau(\theta_2)$ and $u_{\theta q}>0$. The second inequality holds because $\frac{du_q^2(q^{fb}(\theta+x),\theta-x)}{dx^2}=u_{qqq}\left(\dot{q}^{fb}(\theta)\right)^2+2u_{qqq}\ddot{q}^{fb}(\theta)+u_{qq\theta}\dot{q}^{fb}(\theta)+u_{q\theta\theta}<0$, and therefore $u_q(q^{fb}(\tau(\theta_2)),\tau(\theta_2))+u_q(q^{fb}(\tau(\theta_2)),\theta_2)\leq 2u_q(q^{fb}(\frac{\tau(\theta_2)+\theta_2}{2}),\frac{\tau(\theta_2)+\theta_2}{2})=0$. The second equality holds by the Fundamental Theorem of Calculus. The third inequality holds because $u_{qq\theta}\leq 0$ and by the Fundamental Theorem of Calculus. To get the fourth inequality we make a change of variable and note that $\tau((\theta_2,\tilde{\theta}))\subseteq(\tau(\theta_2),\theta_2)$. The fifth inequality holds by (64). The third and fourth equalities hold by the Fundamental Theorem of Calculus. The last inequality holds because $u_{\theta qq}\leq 0$, and $\tau(\theta')\leq\theta_2$ and so $q(\tau(\theta'))\leq q^{fb}(\tau(\theta_2))$ for all $\theta'\in[\theta_2,\tilde{\theta}]$. The last equality holds by simple computation.

Using A, B and D defined above, (65) implies:

$$BD>\frac{A^2-B^2}{2}u_{\theta q}(q^{fb}(\theta_2),\tilde{\theta})-\int_{\theta_2}^{\tilde{\theta}}\left(\tilde{\theta}-\frac{s+\theta_2}{2}\right)(s-\theta_2)-\frac{(\theta_2-\tau(\theta_2))^2}{2}u_{\theta\theta q}(q^{fb}(\theta_2),s)ds$$
$$-\int_{\tau(\theta_2)}^{\theta_2}\left(\theta_2-\frac{s+\tau(\theta_2)}{2}\right)(s-\tau(\theta_2))u_{\theta\theta q}(q^{fb}(\theta_2),s)ds. \tag{66}$$

Combining (63) and (66) yields:

$$\int_{\theta_2}^{\tilde{\theta}} k_1(s) u_{\theta\theta q}(q^{fb}(\theta_2), s) ds + \int_{\tau(\theta_2)}^{\theta_2} (2\theta_2 - \frac{s + 3\tau(\theta_2)}{2})(s - \tau(\theta_2)) u_{\theta\theta q}(q^{fb}(\theta_2), s) ds >$$
$$\frac{(A-B)^2}{2} u_{\theta q}(q^{fb}(\theta_2), \tilde{\theta}) \geq 0, \tag{67}$$

where $k_1(s) = (\tilde{\theta} + \tau(\theta_2) - \frac{s+3\theta_2}{2})(s - \theta_2) + \frac{(\theta_2 - \tau(\theta_2))^2}{2}$. Since $k_1''(s) = -1 < 0$ for any $s \in [\theta_2, \tilde{\theta}]$, $k_1(s) \geq \min\{k_1(\theta_2), k_1(\tilde{\theta})\} = \min\{\frac{B^2}{2}, \frac{(A-B)^2}{2}\} \geq 0$. Since $u_{\theta\theta q} \leq 0$, the left-hand side of (67) is negative. A contradiction. Therefore, τ must be single-valued. *Q.E.D.*

Proof of Theorem 7: First, let us show that (21) holds. The proof of this claim relies on the optimality condition (13) in Theorem 6. Specifically, let $A^k(\theta) = f(\theta) + \sum_{s=1}^{k} f(\tau^s(\theta))\dot{\tau}^s(\theta)$. Then using (13) yields:

$$\dot{\tau}^k = \frac{u_q(Q^k, \tau^{k-1}) - u_q(Q^k, \tau^k)}{f(\tau^k) u_q(Q^k, \tau^k)} A^{k-1}(\theta), \tag{68}$$

$$A^k(\theta) = A^{k-1}(\theta) + f(\tau^k)\dot{\tau}^k = A^{k-1}(\theta) \frac{u_q(Q^k, \tau^{k-1})}{u_q(Q^k, \tau^k)} = f(\theta) \prod_{s=1}^{k} \frac{u_q(Q^s, \tau^{s-1})}{u_q(Q^s, \tau^s)}. \tag{69}$$

Combining (68) and (69) yields equation (21). Further, equation (22) follows immediately from (15) and (21). Finally, note that since $\tau(.)$ is strictly increasing and continuous on $[\hat{\theta}, 1]$, $\tau^k(.)$ must be strictly increasing on $[\tau(1), 1]$ for $k \in \{1, ..., M(\theta)\}$. *Q.E.D.*

Proof of Lemma 2: First, let us rewrite the problem (1)-(3) as follows:

$$\max_{q(\theta) \geq 0, V(\theta) \geq 0} \int_0^1 [u(q(\theta), \theta) - V(\theta)] f(\theta) d\theta \tag{70}$$

subject to:

$$V(\theta) - V(\theta') \geq u(q(\theta'), \theta) - u(q(\theta'), \theta') - C \qquad \forall\, \theta, \theta' \in [0, 1] \tag{71}$$

By Lemmas 3, 5 and 9, we can without loss of generality restrict $q(.)$ to belong to the space of continuous functions from $[0, 1]$ to $[0, q^{fb}(1)]$ and $V(.)$ to belong to the space of continuous functions from $[0, 1]$ to $[0, u(q^{fb}(1), 1)]$. Let $C([0, 1])^T$ be the space of continuous functions from $[0, 1]$ to $[0, T]$. Below we will consider $T_q = q^{fb}(1)$ and $T_u = u(q^{fb}(1), 1)$.

Let us endow $C([0,1])^T$ with weak-* topology.[11] By Alaoglu Theorem the space $C([0,1])^T$ is compact in the weak* topology, and by Tychonoff's Theorem the product $C([0,1])^{T_q} \times C([0,1])^{T_u}$ is compact in the product topology generated by the weak* topology. Further, for every fixed cost $C \geq 0$, the set of nonnegative functions $(q(.), V(.)) \in C([0,1])^{T_q} \times C([0,1])^{T_u}$ that satisfy (71) is a closed subset of $C([0,1])^{T_q} \times C([0,1])^{T_u}$, and is therefore compact in the product topology generated by the weak* topology, and varies continuously with the fixed cost C. So, the correspondence mapping the fixed costs C into $\{(q(.), V(.)) \in C([0,1])^{T_q} \times C([0,1])^{T_u} : (q(.), V(.)) \text{ satisfy } (71)\}$ specifying the set of admissible quantity and surplus functions is continuous in C and compact valued.

Let $(q(.|C)), V(.|C)))$ be the solution to problem (70)-(71). By Theorem 2 the solution exists and is unique. Since the objective function (70) is continuous in $q(.)$, $V(.)$ and C, by Berge's Maximum Theorem $(q(.|C), V(.|C))$ is upper hemicontinuous in C. This implies that $\lim_{C\downarrow 0}(q(\theta|C), V(\theta|C)) = (q(\theta|0), V(\theta|0)) \equiv (q^{sb}(\theta), V^{sb}(\theta))$ for all $\theta \in [0,1]$.

Further, $(q^{sb}(\theta), V^{sb}(\theta)) = (q(\theta|0)), V(\theta|0))$ is the standard second-best solution to our problem for $C = 0$. Note that $q^{sb}(\theta)$ is continuous and $q^{sb}(0) = 0 < q^{sb}(1) = q^{fb}(1)$. Therefore, there exist $\underline{\theta}, \overline{\theta} \in [0,1]$, $\underline{\theta} < \overline{\theta}$, such that $q^{sb}(\theta)$ is strictly increasing and $V^{sb}(\theta) > 0$ on $[\underline{\theta}, \overline{\theta}]$. Since $\lim_{C\downarrow 0} V(\theta|C) = V^{sb}(\theta) > 0$ for $\theta \in [\underline{\theta}, \overline{\theta}]$, Lemma 7 implies that there exists $\hat{C} > 0$ such that $\tau(\theta|C) \neq \emptyset$ for all $C \in (0, \hat{C})$ and $\theta \in [\underline{\theta}, \overline{\theta}]$.

Next, to show that $\lim_{C\downarrow 0} M(C) = \infty$, where $M = \max\{k : \tau^k(1) \neq \emptyset\}$, fix any pair (θ, θ') s.t. $\theta \in (\underline{\theta}, \overline{\theta}]$ and $\theta' < \theta$, and consider the corresponding incentive constraint (71). Putting all terms on one side and taking the limit as $C \to 0$ we get:

$$\lim_{C\downarrow 0} (V(\theta|C) - V(\theta'|C) + C - u(q(\theta'|C), \theta) + u(q(\theta'|C), \theta')) =$$

$$V^{sb}(\theta) - V^{sb}(\theta') - u(q^{sb}(\theta'), \theta) - u(q^{sb}(\theta'), \theta') = \int_{\theta'}^{\theta} u_\theta(q^{sb}(s), s)ds - u(q^{sb}(\theta'), \theta) + u(q^{sb}(\theta'), \theta') > 0$$

where the last inequality holds because $q^{sb}(.)$ is increasing, strictly on $(\underline{\theta}, \overline{\theta})$. So, $\tau(\theta|C) > \theta'$ for any $\theta \in (\underline{\theta}, \overline{\theta}]$ and $\theta' < \theta$ when C is sufficiently small. Hence, $\lim_{C\downarrow 0} \tau(\theta|C) = \theta$ for $\theta \in [\underline{\theta}, \overline{\theta}]$.

Finally, fix some integer $N > 0$ and $\epsilon_N = \frac{\overline{\theta} - \underline{\theta}}{N}$. Since $\lim_{C\downarrow 0} \tau(\theta|C) = \theta$ for $\theta \in [\underline{\theta}, \overline{\theta}]$, there exists $C_N > 0$ such that $\tau^{k-1}(\overline{\theta}|C) - \tau^k(\overline{\theta}|C) \leq \epsilon_N$ for any $k = 1, ..., N$ and hence

[11] $x_n(\theta)$ converges to $x(\theta)$ in the weak* topology iff $\int_0^1 x_n(\theta)y(\theta)dF(\theta) \to \int_0^1 x(\theta)y(\theta)dF(\theta)$ for all $y \in L^2$.

$\tau^N(\overline{\theta}|C) \geq \underline{\theta}$ for all $C \in (0, C_N]$. By Corollary 1, $\tau^N(1|C) > \tau^N(\overline{\theta}|C) \geq \underline{\theta}$ for $C \in (0, C_N]$. Thus, for any integer $N > 0$, $\tau^N(1|C) \neq \emptyset$ when C is sufficiently small. *Q.E.D.*

Proof of Theorem 9: Consider an admissible triple $(\tau(\theta), q(\theta), \hat{\theta})$ and let $Q(\theta) = q(\tau(\theta))$ for $\theta \in [\hat{\theta}, 1]$. Let us show that $(\tau(\theta), Q(\theta), \hat{\theta})$ is an increasing solution to the relaxed program.

Since $C_i \in (\underline{C}, \overline{C})$, Theorem 8 implies that $\tau(\theta) < \hat{\theta}$ for all $\theta \in [\hat{\theta}, 1]$ or, equivalently, $M(\theta) = 1$ for all $\theta \in [\hat{\theta}, 1]$, and so (73) can be rewritten as: $\dot{Q}(\theta) = \frac{u_\theta(Q(\theta), \tau(\theta))}{u_q(Q(\theta), \theta) - u_q(Q(\theta), \tau(\theta))} \dot{\tau(\theta)}$ for all $\theta \in [\hat{\theta}, 1]$, while (21) can be rewritten as (23). In combination with $\dot{Q}(\theta) = \frac{u_\theta(Q(\theta), \tau(\theta))}{u_q(Q(\theta), \theta) - u_q(Q(\theta), \tau(\theta))} \dot{\tau}(\theta)$ this yields (24).

To complete the proof it is sufficient to show that an increasing solution to (23)-(24) with boundary conditions (18)-(20) is unique. We establish this via a sequence of Claims.

First, fix some $\hat{\theta}_i$ and C_j where $i, j \in \{1, 2\}$ and let Let $\Gamma(\hat{\theta}_i, C_j) = \{\theta' : G(\hat{\theta}_i, \theta') \equiv u(q^{fb}(\theta'), \hat{\theta}) - u(q^{fb}(\theta'), \theta') = C_j\}$. Suppose that $\Gamma(\hat{\theta}_i, C_j) \neq \emptyset$ for $i, j \in \{1, 2\}$. Note that $\Gamma(\hat{\theta}_i, C_j)$ contains at most two elements because $G(\hat{\theta}_i, \theta') \equiv u(q^{fb}(\theta'), \hat{\theta}) - u(q^{fb}(\theta'), \theta')$ is strictly quasi-concave in θ'.

Claim 1: If $\hat{\theta}_1 > \hat{\theta}_2$ and $C_1 < C_2$, then $\min \Gamma(\hat{\theta}_1, C_i) < \min \Gamma(\hat{\theta}_2, C_i)$, and $\min \Gamma(\hat{\theta}_i, C_1) < \min \Gamma(\hat{\theta}_i, C_2)$ for $i \in \{1, 2\}$.

Proof of Claim 1: Since $G(.)$ is strictly quasi-concave and $G(\hat{\theta}_i, min\Gamma_i) > G(\hat{\theta}_i, 0) = 0$, it follows that $G_2(\hat{\theta}_i, min\Gamma_i) \geq 0$. On the other hand, $G_1(\hat{\theta}_i, min\Gamma_i) = u_\theta(q^{fb}(min\Gamma_i), \hat{\theta}_i) > 0$. The last two inequalities together imply Claim 1. ∎

Claim 2: *Suppose that there exist $(\hat{\theta}_1, \hat{\tau}_1)$ and $(\hat{\theta}_2, \hat{\tau}_2)$ such that for $i = 1, 2$, $(Q_i(\theta)), \tau_i(\theta))$ is an increasing solution to the system of differential equations (23) and (24) on $[\hat{\theta}_i, 1]$ that satisfies boundary conditions $\tau_i(\hat{\theta}_i) = \hat{\tau}_i$, $Q_i(\hat{\theta}_i) = q^{fb}(\hat{\tau}_i)$ and $Q_i(1) = q^{fb}(\tau_i(1))$. Let $q_i(\theta) = Q_i(\tau_i^{-1}(\theta))$ for $\theta \in [\hat{\tau}_i, \tau_i(1)]$.*

Then the following "no-crossing" property holds:

For some $i, j \in \{1, 2\}$, $i \neq j$, $q_i(\theta) \leq q_j(\theta)$ for all $\theta \in [\max\{\hat{\tau}_1, \hat{\tau}_2\}, \min\{\tau_1(1), \tau_2(1)\}]$.

Proof of Claim 2: The proof is by contradiction, so suppose that there exists $\theta' \in [max\{\hat{\tau}_1, \hat{\tau}_2\}, min\{\tau_1(1), \tau_2(1)\}]$ such that $q_2(\theta') = q_1(\theta') \equiv q'$ and $\dot{q}_2(\theta') \neq \dot{q}_1(\theta')$. Without loss of generality we can assume $\dot{q}_2(\theta') > \dot{q}_1(\theta')$. Differential equations (23) and (24) and $\dot{q}_i = \frac{\dot{Q}_i}{\dot{\tau}_i}$ imply $\frac{u_\theta(q', \theta')}{u_q(q', \tau_2^{-1}(\theta')) - u_q(q', \theta')} > \frac{u_\theta(q', \theta')}{u_q(q', \tau_1^{-1}(\theta')) - u_q(q', \theta')}$. Since $u_{\theta q} > 0$, $\tau_1^{-1}(\theta') > \tau_2^{-1}(\theta')$.

Let $\tilde{\theta}' = \tau_1^{-1}(\theta')$. Consider the following two cases:

Case 1: $q_2(\theta) > q_1(\theta)$ for $\theta \in (\theta', min\{\tau_1(1), \tau_2(1)\}]$.

First note that (24) i.e., $\dot{Q}_i(\theta) = \frac{f(\theta)u_\theta(Q_i,\tau)}{f(\tau)u_q(Q_i,\tau)}$ and $\dot{Q}_i(\theta) > 0$ in combination imply that $q_i(\theta) \leq q^{fb}(\theta)$ for all $\theta \in (\hat{\tau}_i, \tau_i(1))$. It follows that $\tau_1(1) > \tau_2(1)$, for otherwise $q_2(\tau_1(1)) > q_1(\tau_1(1)) = q^{fb}(\tau_1(1))$, where the inequality holds by case assumption, and the equality holds by boundary condition (18), violating $q_2(.) \leq q^{fb}(.)$.

At the same time $\tau_1(\tilde{\theta}') = \theta' < \tau_2(\tilde{\theta}')$ since $\tau_1^{-1}(\theta') > \tau_2^{-1}(\theta')$. This and the inequality $\tau_1(1) > \tau_2(1)$ imply that there exists $\tilde{\theta}'' \in (\tilde{\theta}', 1)$ such that $\tau_1(\tilde{\theta}'') = \tau_2(\tilde{\theta}'') \equiv \theta''$ and $\dot{\tau}_1(\tilde{\theta}'') > \dot{\tau}_2(\tilde{\theta}'')$. By (23) the latter is equivalent to $\frac{f(\tilde{\theta}'')(u_q(Q_1(\tilde{\theta}''),\tilde{\theta}'')-u_q(Q_1(\tilde{\theta}''),\theta''))}{f(\theta'')u_q(Q_1(\tilde{\theta}''),\theta'')} > \frac{f(\tilde{\theta}'')(u_q(Q_2(\tilde{\theta}''),\tilde{\theta}'')-u_q(Q_2(\tilde{\theta}''),\theta''))}{f(\theta'')u_q(Q_2(\tilde{\theta}''),\theta'')}$. Then from $u_{qq} < 0$ and $u_{\theta qq} \geq 0$ it follows that $Q_1(\tilde{\theta}'') > Q_2(\tilde{\theta}'')$, or equivalently $q_1(\theta'') > q_2(\theta'')$. However, this contradicts the case assumption $q_2(\theta) > q_1(\theta)$ for all $\theta \in (\theta', min\{\tau_1(1), \tau_2(1)\}]$.

Case 2: There exists $\theta'' \in (\theta', \min\{\tau_1(1), \tau_2(1)\}]$ such that $q_2(\theta) > q_1(\theta)$ for $\theta \in (\theta', \theta'')$, $q_2(\theta'') = q_1(\theta'') \equiv q''$ and $\dot{q}_2(\theta'') < \dot{q}_1(\theta'')$.

Given $\dot{q}_2(\theta'') < \dot{q}_1(\theta'')$, a similar argument to that in Case 1 yields that $\tau_1^{-1}(\theta'') < \tau_2^{-1}(\theta'')$, and $\tau_2(\tau_1^{-1}(\theta'')) < \tau_2(\tau_2^{-1}(\theta'')) = \theta'' = \tau_1(\tau_1^{-1}(\theta''))$. Let $\tilde{\theta}'' = \tau_1^{-1}(\theta'')$. Note that $\tilde{\theta}'' > \tilde{\theta}'$ as $\theta'' > \theta'$. Since $\tau_1(\tilde{\theta}') < \tau_2(\tilde{\theta}')$ and $\tau_2(\tilde{\theta}'') < \tau_1(\tilde{\theta}'')$, there exists $\tilde{\theta}''' \in [\tilde{\theta}', \tilde{\theta}'']$ such that $\tau_1(\tilde{\theta}''') = \tau_2(\tilde{\theta}''') \equiv \theta'''$ and $\dot{\tau}_1(\tilde{\theta}''') > \dot{\tau}_2(\tilde{\theta}''')$. A similar argument to the in Case 1 yields $Q_1(\tilde{\theta}''') > Q_2(\tilde{\theta}''')$, or equivalently $q_1(\theta''') > q_2(\theta''')$. But by the case assumption $q_1(\theta''') < q_2(\theta''')$. Contradiction.

Claim 3: *If there exists* $\tilde{\theta}' \in [\max\{\hat{\theta}_1, \hat{\theta}_2\}, 1]$ *such that* $\tau_2(\tilde{\theta}') < \tau_1(\tilde{\theta}')$*, then* $\tau_2(\theta) < \tau_1(\theta)$ *for all* $\theta \in [\max\{\hat{\theta}_1, \hat{\theta}_2\}, 1]$.

Proof of Claim 3: The proof is by contradiction, so suppose the Claim is not true. Then there exists a "crossing point" $\tilde{\theta}' \in [max\{\hat{\theta}_1, \hat{\theta}_2\}, 1]$ such that $\tau_2(\tilde{\theta}') = \tau_1(\tilde{\theta}') \equiv \theta'$ and $\dot{\tau}_1(\tilde{\theta}') \neq \dot{\tau}_2(\tilde{\theta}')$. Without loss of generality we can assume $\dot{\tau}_1(\tilde{\theta}') > \dot{\tau}_2(\tilde{\theta}')$. Then from the differential equation (23) it follows that $Q_1(\tilde{\theta}') > Q_2(\tilde{\theta}')$, or equivalently $q_1(\theta') > q_2(\theta')$.

Note that $\tilde{\theta}' < 1$ for otherwise we would have $\theta' = \tau_1(1) = \tau_2(1)$ and $q^{fb}(\theta') = q_1(\theta') = q_2(\theta')$ which contradicts $q_1(\theta') > q_2(\theta')$.

Now consider the following two cases:

Case 1: $\tau_1(\theta) > \tau_2(\theta)$ for $\theta \in (\tilde{\theta}', 1]$.

Since $\tau_1(1) > \tau_2(1)$, we have $q_1(\tau_2(1)) \leq q^{fb}(\tau_2(1)) = q_2(\tau_2(1))$, which combined with $q_1(\theta') > q_2(\theta')$ violates Claim 2, the no-crossing property of q.

Case 2: There exists $\tilde{\theta}'' \in (\tilde{\theta}', 1]$ such that $\tau_1(\theta) > \tau_2(\theta)$ for $\theta \in (\tilde{\theta}', \tilde{\theta}'')$, $\tau_1(\tilde{\theta}'') = \tau_2(\tilde{\theta}'') \equiv \theta''$ and $\dot{\tau}_1(\tilde{\theta}'') < \dot{\tau}_2(\tilde{\theta}'')$.

Using $\dot{\tau}_1(\tilde{\theta}'') < \dot{\tau}_2(\tilde{\theta}'')$ and $\tau_1(\tilde{\theta}'') = \tau_2(\tilde{\theta}'')$ in differential equation (23) yields $Q_1(\tilde{\theta}'') < Q_2(\tilde{\theta}'')$, or equivalently $q_1(\theta'') < q_2(\theta'')$, which combined with $q_1(\theta') > q_2(\theta')$ violates Claim 2, the no-crossing property of q.

Claim 4: *Suppose that there exist $(\hat{\theta}_1, \hat{\tau}_1) \neq (\hat{\theta}_2, \hat{\tau}_2)$ such that for $i = 1, 2$, $(Q_i(\theta)), \tau_i(\theta))$ is an increasing solution to the system of differential equations (23) and (24) on $[\hat{\theta}_i, 1]$ that satisfies boundary conditions $\tau_i(\hat{\theta}_i) = \hat{\tau}_i$, $Q_i(\hat{\theta}_i) = q^{fb}(\hat{\tau}_i)$ and $Q_i(1) = q^{fb}(\tau_i(1))$. Then $\hat{\theta}_2 > \hat{\theta}_1$ if and only if $\hat{\tau}_2 > \hat{\tau}_1$.*

Proof of Claim 4: Suppose not, then without loss of generality we have $\hat{\theta}_2 \geq \hat{\theta}_1$ and $\hat{\tau}_1 \geq \hat{\tau}_2$ with at least one strict inequality. Then $\tau_1(\hat{\theta}_2) \geq \tau_1(\hat{\theta}_1)$ and $\tau_1(\hat{\theta}_1) \geq \tau_2(\hat{\theta}_2)$ with at least one strict inequality, from which it immediately follows that $\tau_1(\hat{\theta}_2) > \tau_2(\hat{\theta}_2)$, and so $q_1(\tau_1(\hat{\theta}_1)) = q^{fb}(\tau_1(\hat{\theta}_1)) \geq q_2(\tau_1(\hat{\theta}_1))$. By Claim 3 (the "no-crossing" property of τ), $\tau_1(1) > \tau_2(1)$, and therefore $q_2(\tau_2(1)) = q^{fb}(\tau_2(1)) > q_1(\tau_2(1))$. The last inequality together with $q_1(\tau_1(\hat{\theta}_1)) \geq q_2(\tau_1(\hat{\theta}_1))$ contradict the no-crossing property of q in Claim 2.

Uniqueness. To establish the uniqueness of the solution to the relaxed program, we rely on Claims 1-4. Again the proof is by contradiction, so suppose the solution is not unique. Then there exist $\hat{\theta}_1$ and $\hat{\theta}_2$, $\hat{\theta}_1 \neq \hat{\theta}_2$ s.t. $(Q_1(\theta), \tau_1(\theta))$ and $(Q_2(\theta), \tau_2(\theta)))$ solve the system of differential equations (23) and (24) with boundary conditions (18)- (20) where $\hat{\tau}_i \equiv \tau_i(\hat{\theta}_i) = \min \Gamma_i$. Without loss of generality suppose that $\hat{\theta}_1 > \hat{\theta}_2$. Claim 4 implies that $\hat{\tau}_1 > \hat{\tau}_2$. However, this contradicts Claim 1. *Q.E.D.*

Proof of Theorem 10:

Parts 1 and 2. Suppose that contrary to claim in part 1), $\hat{\theta}_2 \leq \hat{\theta}_1$. Then by Claim 4 in the proof of Theorem 9, $\tau_2(\hat{\theta}_2) \leq \tau_1(\hat{\theta}_1)$. But since $C_2 > C_1$, this contradicts Claim 1 in the proof of Theorem 9. Therefore, we must have $\hat{\theta}_2 > \hat{\theta}_1$ and $\tau_2(\hat{\theta}_2) > \tau_1(\hat{\theta}_1)$.

Parts 3 and 4. By Part 2 and boundary condition (20), $q_2(\tau_2(\hat{\theta}_2)) = q^{fb}(\tau_2(\hat{\theta}_2)) > q_1(\tau_2(\hat{\theta}_2))$. Part 4 then follows from Claim 2 in the proof of Theorem 9. Next, since $q_1(\tau_2(1)) < q_2(\tau_2(1)) = q^{fb}(\tau_2(1))$, it must be the case that $\tau_1(1) > \tau_2(1)$. Part 3 then follows from Claim 3 in Proof of Theorem 9 (the no-crossing property of $\tau(.)$). *Q.E.D.*

Online Appendix to "Screening Under A Fixed Cost of Misrepresentation"

8 Appendix. Multi-valued targeted type

This appendix characterize the optimal mechanism when the targeted type $\tau(.)$ may be multi-valued. Two additional issues needs to be addressed in this case. First, with multi-valued targeted types, $\tau(.)$ is a correspondence which can be equivalently represented as a discontinuous function with upwards jumps. Such jumps cannot be characterized by the following differential equation derived in Theorem 7 in the main paper:

$$\dot{\tau}^k(\theta) = \frac{f(\theta)[u_q(Q^k,\tau^{k-1}) - u_q(Q^k,\tau^k)]}{f(\tau^k)u_q(Q^k,\tau^k)} \prod_{s=1}^{k-1} \frac{u_q(Q^s,\tau^{s-1})}{u_q(Q^s,\tau^s)}(\theta), \quad k \in \{1, ..., M(\theta)\}, \tag{72}$$

which applies only where $\tau(.)$ is continuous. Second, the image $\tau(\theta)$ need not be convex for all θ, and so one would have to determine the boundaries of subintervals in $[\min\tau(\hat{\theta}), \max\tau(1)]$ where the law of motion is:

$$[u_q(q(\tau(\theta)),\theta) - u_q(q(\tau(\theta)),\tau(\theta))]\dot{q}(\tau(\theta)) = u_\theta(q(\tau(\theta)),\tau(\theta)) - 1(\tau(\theta) \geq \hat{\theta})u_\theta(q(\tau(\tau(\theta))),\tau(\theta)). \tag{73}$$

In order to tackle these issues, we introduce and work with a concept of an "attracted type," a generalized inverse of τ. Specifically, let $\underline{\tau} = \min\tau(\hat{\theta})$ and $\overline{\tau} = \max\tau(1)$. The attracted type function $\beta : [\underline{\tau}, \overline{\tau}] \to [\hat{\theta}, 1]$ is defined as follows:

$$\beta(\theta) = \theta' \ \text{ if } \theta \in [\min\tau(\theta'), \max\tau(\theta')].$$

This definition implies that $\beta(\theta) = \tau^{-1}(\theta)$ if $\tau^{-1}(\theta)$ is non-empty. If $\tau^{-1}(\theta)$ is empty, then $\beta(\theta)$ a unique θ' s.t. $\min\tau(\theta') < \theta < \max\tau(\theta')$. Since $\tau(\theta)$ is strictly increasing and upper hemicontinuous by Theorem 4, $\beta(\theta)$ is well-defined, weakly increasing and continuous.[12]

To describe the chains of attracted types connected by binding incentive constraints, we use the concept of higher-order attracted types in a similar fashion to higher-order targeted

[12] To illustrate the relationship between τ and β, consider the following example: $\hat{\theta} = 0.6$, $\tau(\theta) = \theta - 0.3$ if $\theta \in [0.6, 0.8)$, $\tau(\theta) = \{0.5, 0.6\}$ if $\theta = 0.8$, $\tau(\theta) = \theta - 0.2$ if $\theta \in (0.8, 1]$. The corresponding β function is: $\beta(\theta) = \theta + 0.3$ if $\theta \in [0.3, 0.5)$, $\beta(\theta) = 0.8$ if $\theta \in [0.5, 0.6]$, $\beta(\theta) = \theta + 0.2$ if $\theta \in (0.6, 0.8]$. Particularly, note that a type in $(0.5, 0.6)$ is not in the image of $\tau(.)$, but $\beta(\theta) = 0.8$ for all $\theta \in [0.5, 0.6]$.

types. Specifically, for $\theta \in [\underline{\tau}, \hat{\theta}]$ let $\beta^0(\theta) = \theta$ and $\beta^k(\theta) = \beta(\beta^{k-1}(\theta))$ for $k \geq 1$. Let $R(\theta)$ be the number of elements in the chain of attracted types, so that $\beta^k(\theta)$ exists for $k = 1, ..., R(\theta) - 1$. The maximal length of the chain of attracted types is $R = R(\underline{\tau})$. Since $\beta(.)$ is continuous and increasing, it maps the interval $[\beta^{k-1}(\theta), \beta^k(\theta)]$ onto the adjacent interval $[\beta^k(\theta), \beta^{k+1}(\theta)]$.

Then the following condition must hold for $\theta \in [\underline{\tau}, \overline{\tau}]$ in the optimal mechanism:

$$u_q(q(\theta), \theta) f(\theta) = [u_q(q(\theta), \beta(\theta)) - u_q(q(\theta), \theta)] \sum_{k=1}^{s} f(\beta^k(\theta)) \dot{\beta}^k(\theta), \tag{74}$$

where s is such that $\quad \beta^s(\theta) \in (\max \tau(1), 1]$.

A formal proof of this claim is provided in the proof of Theorem 11. Condition (74) is the same as the optimality condition (13) in Theorem 6 in the paper, but restated using attracted type function $\beta(.)$. Intuitively, this condition reflects the optimal tradeoff between the marginal efficiency gain from raising $q(\theta)$ and the marginal cost of information rent that the principal has to provide to the types in every predecessor of θ in the chain of attracted types $\beta^k(\theta)$ for $k = 1, ..., s$.

Our next step is to generalize the optimal "law of motion" of $q(\theta)$ to the current case. Note that $\beta^{-1}(\theta)$ is well-defined as the convex hull of $\tau(\theta)$. If $\theta \in \tau(\beta(\theta))$ i.e., the incentive constraint $IC(\beta(\theta), \theta)$ is binding, for all θ in some open interval, then the corresponding law of motion, which we denote by $\dot{q}^{IC}(.)$, is obtained by rewriting (73) which yields:

$$\dot{q}^{IC}(\theta) \equiv \frac{u_\theta(q(\theta), \theta) - 1(\theta \geq \hat{\theta}) u_\theta(q(\min \beta^{-1}(\theta)), \theta)}{u_q(q(\theta), \beta(\theta)) - u_q(q(\theta), \theta)}. \tag{75}$$

On the other hand, by part 4 of Theorem 4 in the main paper, $q(\theta) = q^{fb}(\theta)$ for all $\theta \in [\theta_1, \theta_2]$ where θ_1 and θ_2 are the boundaries of the maximal interval on which $\beta(.)$ is constant (put otherwise, $\tau(\beta(\theta))$ is multi-valued. So, $\dot{q}(\theta) = \dot{q}^{fb}(\theta)$.

Thus, $\dot{q}(\theta) = \dot{q}^{IC}(\theta)$ when $IC(\beta(\theta), \theta)$ is binding, and $\dot{q}(\theta) = \dot{q}^{fb}(\theta)$ when it is not binding. To identify which of these two cases applies, consider the payoff of type θ when she imitates type θ', $U(\theta', \theta) = u(q(\theta), \theta') - u(q(\theta), \theta) - C + \int_{\hat{\theta}}^{\max\{\theta, \hat{\theta}\}} u_\theta(q(\min \beta^{-1}(s)), s) ds$. Then for $\theta \in [\underline{\tau}, \max \tau(1)]$, let $I(\theta) = \int_{\underline{\tau}}^{\theta} U_2(\beta(x), x) dx =$

$$= \int_{\underline{\tau}}^{\theta} [u_q(q(x), \beta(x)) - u_q(q(x), x)] \dot{q}(x) - u_\theta(q(x), x) + 1(x \geq \hat{\theta}) u_\theta(q(\min \beta^{-1}(x)), x) dx. \tag{76}$$

As shown in the proof of Theorem 11 stated below, $I(\theta)$ tracks the slackness of $IC(\beta(\theta), \theta)$. Specifically, if $I(\theta) = 0$, then $IC(\beta(\theta), \theta)$ is binding; if $I(\theta) < 0$, $IC(\beta(\theta), \theta)$ is slack. Therefore, the optimal law of motion of $q(\theta)$ can be stated as follows:

$$\dot{q}(\theta) = \begin{cases} \dot{q}^{IC}(\theta) & \text{if } q(\theta) < q^{fb}(\theta), \\ \dot{q}^{fb}(\theta) & \text{if } q(\theta) = q^{fb}(\theta) \text{ and } I(\theta) < 0, \\ \min\{\dot{q}^{IC}(\theta), \dot{q}^{fb}(\theta)\} & \text{if } q(\theta) = q^{fb}(\theta) \text{ and } I(\theta) = 0. \end{cases} \tag{77}$$

The logic behind (77) is that, when $q(\theta)$ is below the first-best, the incentive constraint $IC(\beta(\theta), \theta)$ must be binding, and so $\dot{q}(\theta) = \dot{q}^{IC}(\theta)$. On the other hand, if $I(\theta) < 0$, then $IC(\beta(\theta), \theta)$ is slack and the optimal quantity must stay at the first-best level in a neighborhood of θ. This case arises when $\tau(.)$ is non-convex.

When $I(\theta) = 0$ and $q(\theta) = q^{fb}(\theta)$, we are in a boundary situation with binding $IC(\beta(\theta), \theta)$. In this case, if $\dot{q}^{IC}(\theta) > \dot{q}^{fb}(\theta)$, the types in a neighborhood of θ do not have IC constraints binding towards them and the quantities remains at the first-best level. On the other hand, if $\dot{q}^{IC}(\theta) < \dot{q}^{fb}(\theta)$, the types in a neighborhood of θ do have IC constraints binding towards , and the law of motion of q is given by (75).

Further, the boundary conditions for $\beta(.)$ and $q(.)$ on the interval $[\underline{\tau}, \overline{\tau}]$ where $\underline{\tau} = \min\{\theta : \beta(\theta) \neq \emptyset\}$, $\overline{\tau} = \max\{\theta : \beta(\theta) = 1\}$, are as follows:

$$\beta^k(\underline{\tau}) = \beta^{k-1}(\beta(\underline{\tau})), \tag{78}$$

$$q(\underline{\tau}) = q^{fb}(\underline{\tau}), \tag{79}$$

$$q(\overline{\tau}) = q^{fb}(\overline{\tau}), \tag{80}$$

$$u(q^{fb}(\underline{\tau}), \beta(\underline{\tau})) - u(q^{fb}(\underline{\tau}), \underline{\tau}) - C = 0. \tag{81}$$

The necessary conditions for optimality are presented in the following Theorem:

Theorem 11 *The following conditions must hold in an optimal mechanism* $(q(.), t(.))$*:*
(i) The optimality condition (74);
(ii) The law of motion (77);
(iii) The boundary conditions (78) - (81).

We can now use the optimality conditions of Theorem 11, in particular, (74), to obtain the differential equations characterizing the attracted type functions $\beta^k(.)$ and the corresponding quantities. To state them, let $\hat{\theta} = \beta(\underline{\tau})$, $G^k(\theta) = q(\beta^k(\theta))$ for $\theta \in [\underline{\tau}, \hat{\theta}]$, so that $G^k(.)$ is the quantity received by the k-th order attracted type $\beta^k(\theta)$. Also, with a slight abuse of notation, let $L(\theta, k) = \prod_{i=1}^{k-1} \frac{u_q(G^i(\theta)),\beta^i(\theta))}{u_q(G^i(\theta),\beta^{i+1}(\theta))}$, with $L(\theta, 1) = 1$ by convention. Then we have:

Corollary 2 *In an optimal mechanism, for $\theta \in [\underline{\tau}, \hat{\theta}]$ and $s(\theta) \in \mathbf{N}$ such that $\beta^{s(\theta)}(\theta) \in [\min \tau(1), 1]$ we have:*

$$\dot{\beta}^k(\theta) = \begin{cases} \frac{f(\theta)u_q(G^0(\theta),\theta))[u_q(G^k(\theta),\beta^{k+1}(\theta))-u_q(G^k(\theta),\beta^k(\theta))]}{f(\beta^k(\theta))u_q(G^k(\theta),\beta^{k+1}(\theta))[u_q(G^0(\theta),\beta^1(\theta))-u_q(G^0(\theta),\theta)]} L(\theta, k) & \textit{if } k < s(\theta); \\ \frac{f(\theta)u_q(G^0(\theta),\theta)}{f(\beta^k(\theta))[u_q(G^0(\theta),\beta^1(\theta))-u_q(G^0(\theta),\theta)]} L(\theta, k) & \textit{if } k = s(\theta); \end{cases} \tag{82}$$

Also, for $k = 0, ..., s(\theta) - 1$, $\dot{G}^k(\theta) =$

$$\begin{cases} \frac{u_\theta(G^0(\theta),\theta))}{u_q(G^0(\theta),\beta^1(\theta))-u_q(G^0(\theta),\theta)} & \text{if } G^0(\theta) < q^{fb}(\theta)),\ k = 0; \\ \frac{-u_{\theta q}(G^k(\theta),\beta^k(\theta))}{u_{qq}(G^k(\theta),\beta^k(\theta))} & G^0(\theta) = q^{fb}(\theta), I(\theta) < 0, k = 0; \\ \frac{f(\theta)u_q(G^0(\theta),\theta))[u_\theta(G^k(\theta),\beta^k(\theta))-u_\theta(G^{k-1}(\theta),\beta^k(\theta))]}{f(\beta^k(\theta))[u_q(G^0(\theta),\beta^1(\theta))-u_q(G^0(\theta),\theta)]u_q(G^k(\theta),\beta^{k+1}(\theta))} L(\theta, k) & G^k(\theta) < q^{fb}(\beta^k(\theta)),\ k \geq 1; \\ \frac{-f(\theta)u_q(G^0(\theta),\theta))[u_q(G^k(\theta),\beta^{k+1}(\theta))-u_q(G^k(\theta),\beta^k(\theta))]u_{\theta q}(G^k(\theta),\beta^k(\theta))}{f(\beta^k(\theta))[u_q(G^0(\theta),\beta^1(\theta))-u_q(G^0(\theta),\theta)]u_q(G^k(\theta),\beta^{k+1}(\theta))u_{qq}(G^k(\theta),\beta^k(\theta))} L(\theta, k) & G^k(\theta) = q^{fb}(\beta^k(\theta)), I(\beta^k(\theta)) < 0, k \geq 1; \\ \dot{\beta}^k(\theta) \min\{\dot{q}^{IC}(\beta^k(\theta)), \dot{q}^{fb}(\beta^k(\theta))\} & G^k(\theta) = q^{fb}(\beta^k(\theta)), I(\beta^k(\theta)) = 0. \end{cases} \tag{83}$$

Differential equations (82) and (83) describe the laws of motion of the high-order attracted types β^k and their corresponding quantities G^k. Together with the boundary conditions (78)- (81), these differential equations provide a characterization of the optimal mechanism when multi-valued targeted types exist. Particularly, consider the law of motion of quantities (83). Its first two cases specify the law of motion $\dot{q}^{IC}$ that applies when the quantities are below the first-best and is derived from the binding incentive constraint towards the respective types. The next two cases in (83) specify the law of motion for types who do not have "attracted types." In these cases, the law of motion is the rate that keeps the quantities at the first-best level. The last case of condition (83) specifies the law of motion for such θ where both the quantity is at the first-best and there is a type attracted to θ. The incentive constraints are binding, which is the smaller of $\dot{q}^{IC}$ and $\dot{q}^{fb}$, as the quantities cannot exceed the first-best.

Proof of Theorem 11 and Corollary 2:

First, the boundary conditions in part (iii) hold by the definitions of $\underline{\tau}$, $\overline{\tau}$, and $\hat{\theta}$.

Next, we derive the optimality condition (74) in the following Lemma. Note that this condition is equivalent to condition (13).

Lemma 18 *In an optimal mechanism, $\beta^k(.)$ is differentiable at θ for all $k \in \{1, ..., s\}$ and equation (74) holds for any $\theta \in [\min \tau(\hat{\theta}), \max \tau(1)]$ and s such that $\beta^s(\theta) \in [\min \tau(1), 1]$.*

Proof of Lemma 18: The proof is by contradiction. So, suppose that there exists $\tilde{\theta} \in [\min \tau(\hat{\theta}), \max \tau(1)]$ with differentiable $\beta^k(\tilde{\theta})$, $k = 1, ..., s$, such that

$$u_q(q(\tilde{\theta}), \tilde{\theta}) f(\tilde{\theta}) > [u_q(q(\tilde{\theta}), \beta(\tilde{\theta})) - u_q(q(\tilde{\theta}), \tilde{\theta})] \sum_{k=1}^{s} f(\beta^k(\tilde{\theta})) \dot{\beta}^k(\tilde{\theta}). \tag{84}$$

We will show that in this case the mechanism is not optimal, as the principal can get a higher profit by increasing the quantities assigned to the types around $\tilde{\theta}$ and collecting the additional revenue generated thereby, while providing increased information rents to types around $\beta^k(\tilde{\theta})$, $k = 1, ..., s$. The case when this inequality has the opposite sign is similar.

The proof proceeds through three steps. In Step 1, we construct an alternative mechanism $(\tilde{q}(.), \tilde{t}(.))$. In Steps 2 and 3 we show that this alternative mechanism is incentive compatible and more profitable, respectively, for the principal than the original one, when the quantity changes for the types near $\tilde{\theta}$ are sufficiently small.

Step 1. Constructing an Alternative Mechanism $(\tilde{q}(.), \tilde{t}(.))$**.**

Inequality (84) implies that there exists $\mu > 0$ such that

$$u_q(q(\tilde{\theta}), \tilde{\theta}) f(\tilde{\theta}) - [u_q(q(\tilde{\theta}), \beta(\tilde{\theta})) - u_q(q(\tilde{\theta}), \tilde{\theta})] \sum_{k=1}^{s} f(\beta^k(\tilde{\theta})) \dot{\beta}^k(\tilde{\theta}) - \mu > 0. \tag{85}$$

Note that the inequality (85) implies that $q(\tilde{\theta}) < q^{fb}(\tilde{\theta})$.

Now, for $\epsilon > 0$ small enough and $k = 0, ..., s$, let $\Theta_k(\epsilon) = [\beta^k(\tilde{\theta} - \epsilon) - (\frac{\delta_\tau}{2})^{s-k}\epsilon^2, \beta^k(\tilde{\theta} + \epsilon) + (\frac{\delta_\tau}{2})^{s-k}\epsilon^2]$. Since $\beta^k(\tilde{\theta}) < \beta^{k+1}(\tilde{\theta})$ for all $k \in \{0, ..., s-1\}$, Lemma 12 implies that $\Theta_k(\epsilon) \cup \Theta_k + 1(\epsilon)$ for all $k \in \{0, ..., s-1\}$, which we now assume.

The alternative mechanism $(\tilde{q}(.), \tilde{t}(.))$ differs from the original one, $(q(.), t(.))$, only as follows: (i) for $\theta \in \Theta_0(\epsilon)$, $\tilde{q}(\theta) = q(\theta) + \epsilon^5$ and $\tilde{t}(\theta) = t(\theta) + u(q(\theta) + \epsilon^5, \theta) - u(q(\theta), \theta)$; (ii) for $\theta \in \cup_{k=1}^{s} \Theta_k(\epsilon)$, $\tilde{q}(\theta) = q(\theta)$ and $\tilde{t}(\theta) = t(\theta) - \Delta(\epsilon)$, where $\Delta(\epsilon) \equiv \max_{\theta' \in \Theta_0(\epsilon)} u(q(\theta') + \epsilon^5, \overline{\theta}_1) - u(q(\theta'), \overline{\theta}_1) - u(q(\theta') + \epsilon^5, \theta') + u(q(\theta'), \theta')$ and $\overline{\theta}_1 = \max \Theta_1(\epsilon)$. So, $\Delta(\epsilon) > 0$ and $\lim_{\epsilon \to 0} \Delta(\epsilon) = 0$. Let $\tilde{V}(\theta)$ be the net payoff of type θ in $(\tilde{q}(.), \tilde{t}(.))$.

Step 2. Establishing individual rationality and incentive compatibility of the alternative mechanism for small $\epsilon > 0$.

IR constraints hold in $(\tilde{q}(.), \tilde{t}(.))$ because $\tilde{V}(\theta) > V(\theta)$ for $\theta \in \cup_{k=1}^{s}\Theta_k(\epsilon)$, and $\tilde{V}(\theta) = V(\theta)$ for all $\theta \in [0,1] \setminus \cup_{k=1}^{s}\Theta_k(\epsilon)$.

Now, let us focus on incentive constraints in the mechanism $(\tilde{q}(.), \tilde{t}(.))$, which we denote by $\tilde{IC}(\theta, \theta')$ for $(\theta, \theta') \in [0,1]^2$. First, if $\theta \in [0,1]$ and $\theta' \notin \cup_{k=0}^{s}\Theta_k(\epsilon)$, then $\tilde{IC}(\theta, \theta')$ holds because $\tilde{V}(\theta) \geq V(\theta)$, $\tilde{q}(\theta') = q(\theta')$, $\tilde{t}(\theta') = t(\theta')$ and $IC(\theta, \theta')$ holds.

Second, if $\theta \in [0,1]$ and $\theta' \in \Theta_s(\epsilon)$, then for small enough ϵ, $\tau^{-1}(\theta') = \emptyset$ since $\beta^{s+1}(\tilde{\theta}) = \emptyset$. Therefore, $IC(\theta, \theta')$ is slack in the original mechanism. Let $\delta > 0$ be the minimal slack over all $\theta \in [0,1]$ and all $\theta' \in \Theta_s(\epsilon)$. Note that $\tilde{V}(\theta) \geq V(\theta)$ for all $\theta \in [0,1]$, and $\tilde{V}(\theta') = V(\theta') + \Delta(\epsilon)$ for $\theta' \in \Theta_s(\epsilon)$. So, $\tilde{IC}(\theta, \theta')$ holds for sufficiently small ϵ s.t. $\Delta(\epsilon) \leq \delta$.

Third, $\tilde{IC}(\theta, \theta')$ holds for $\theta \in \Theta_1(\epsilon)$ and $\theta' \in \Theta_0(\epsilon)$ because we have:

$$\tilde{V}(\theta) = V(\theta) + \Delta(\epsilon) \geq u(q(\theta'), \theta) - t(\theta') - C + \Delta(\epsilon) \geq u(q(\theta'), \theta) - t(\theta') - C +$$
$$[u(q(\theta') + \epsilon^5, \theta) - u(q(\theta') + \epsilon^5, \theta')] - [u(q(\theta'), \theta) - u(q(\theta'), \theta')] = u(\tilde{q}(\theta'), \theta) - \tilde{t}(\theta') - C,$$

where the first equality holds by construction; the first inequality holds by incentive compatibility of the original mechanism; the second inequality holds by definition of $\Delta(\epsilon)$, and because $\theta \leq \overline{\theta}_1$ and $u_{\theta q} > 0$; the last equality holds by definition of $\tilde{q}(\theta')$ and $\tilde{t}(\theta')$.

Fourth, if $\theta \in \cup_{k=1}^{s}\Theta_k(\epsilon)$ and $\theta' \in \cup_{k=1}^{s-1}\Theta_k(\epsilon)$, then $\tilde{V}(\theta) = V(\theta) + \Delta(\epsilon)$ and $\tilde{V}(\theta') = V(\theta') + \Delta(\epsilon)$ since $\tilde{q}(\theta) = q(\theta)$, $\tilde{t}(\theta) = t(\theta) - \Delta(\epsilon)$, $\tilde{q}(\theta') = q(\theta')$, and $\tilde{t}(\theta') = t(\theta') - \Delta(\epsilon)$. So, $\tilde{IC}(\theta, \theta')$ holds because $IC(\theta, \theta')$ holds.

Fifth, consider $\tilde{IC}(\theta, \theta')$ s.t. $\theta \notin \Theta_1(\epsilon)$, $\theta' \in \Theta_0(\epsilon)$. Now, suppose that $\frac{\theta + \beta(\theta')}{2} \geq \hat{\theta}$ in the original mechanism. Then applying Lemma 13 in the main paper, we get:

$$\tilde{V}(\theta) = V(\theta) > u(q(\theta'), \theta) - t(\theta') - C + \delta_V \left(\frac{\delta_\tau}{2}\right)^{2(s-1)} \frac{\epsilon^4}{16} = u(\tilde{q}(\theta'), \theta) - \tilde{t}(\theta') - C$$
$$+ \delta_V \left(\frac{\delta_\tau}{2}\right)^{2(s-1)} \frac{\epsilon^4}{16} - [u(q(\theta') + \epsilon^5, \theta) - u(q(\theta'), \theta) - u(q(\theta') + \epsilon^5, \theta') + u(q(\theta'), \theta')]$$
$$> u(\tilde{q}(\theta'), \theta) - \tilde{t}(\theta') - C,$$

where the first inequality holds because $\theta' \in \Theta_0(\epsilon) \equiv [\tilde{\theta} - \epsilon - (\frac{\delta_\tau}{2})^s \epsilon^2, \tilde{\theta} + \epsilon + (\frac{\delta_\tau}{2})^s \epsilon^2]$ and $\theta - \theta' \geq \delta_\tau[\beta(\theta) - \beta(\theta')]$ by Lemma 12 in the main paper. So, $\beta(\theta') \in [\beta(\tilde{\theta} - \epsilon) - \frac{1}{2}(\frac{\delta_\tau}{2})^{s-1}\epsilon^2, \beta(\tilde{\theta} + \epsilon) + \frac{1}{2}(\frac{\delta_\tau}{2})^{s-1}\epsilon^2]$. This and the fact that $\theta \notin \Theta_1(\epsilon)$ imply that $|\theta - \beta(\theta')| \geq$

$\frac{1}{2}(\frac{\delta_\tau}{2})^{s-1}\epsilon^2$. Using the latter in

$$V(\theta_2) - U(\theta_1'|\theta_2) \geq \begin{cases} \delta_V \frac{(\theta_2-\theta_1)^2}{4} & \text{if } \frac{\theta_1+\theta_2}{2} \geq \hat{\theta} \\ \frac{\theta_1-\theta_2}{2} \min_\theta u_\theta(q(\theta_1'),\theta) & \text{if } \frac{\theta_1+\theta_2}{2} < \hat{\theta}. \end{cases} \tag{86}$$

of Lemma 13 in the main paper yields $V(\theta) - U(\theta'|\theta) \geq \delta_V(\frac{\delta_\tau}{2})^{2(s-1)}\frac{\epsilon^4}{16}$ for small enough ϵ. The second equality above holds by definitions of $\tilde{q}(\theta')$ and $\tilde{t}(\theta')$. The last inequality holds for small enough ϵ.

Now, suppose that $\frac{\theta+\beta(\theta')}{2} \leq \hat{\theta}$ in the original mechanism. Since $\beta(\theta') > \hat{\theta}$, it follows that $\theta < \hat{\theta}$ and so $\tau(\theta) = \emptyset$ and $IC(\theta,\theta')$ is slack in the original mechanism. Hence, when ϵ is sufficiently small, $\tilde{IC}(\theta,\theta')$ is slack in the modified mechanism as well.

Sixth, suppose that $\theta \notin \cup_{k=1}^s \Theta_k(\epsilon)$ and $\theta' \in \Theta_r(\epsilon)$, $r = 1, ..., s-1$, and ϵ is sufficiently small. Let us start with the case when $\frac{\theta+\beta(\theta')}{2} \geq \hat{\theta}$ in the original mechanism. We have:

$$\begin{aligned} \tilde{V}(\theta) = V(\theta) &> u(q(\theta'),\theta) - t(\theta') - C + \delta_V \left(\frac{\delta_\tau}{2}\right)^{2(r-1)} \frac{\epsilon^4}{16} \\ &= u(\tilde{q}(\theta'),\theta) - \tilde{t}(\theta') - C + \delta_V\delta_\tau(\frac{\delta_\tau}{2})^{2(r-1)}\frac{\epsilon^4}{16} - \Delta(\epsilon) > u(\tilde{q}(\theta'),\theta) - \tilde{t}(\theta') - C, \end{aligned}$$

where the first inequality holds because $\theta' \in [\beta^r(\tilde{\theta}-\epsilon) - (\frac{\delta_\tau}{2})^{s-r}\epsilon^2, \beta^r(\tilde{\theta}+\epsilon) + (\frac{\delta_\tau}{2})^{s-r}\epsilon^2]$, and $\theta - \theta' \geq \delta_\tau[\beta(\theta) - \beta(\theta')]$ by Lemma 12. So, $\beta(\theta') \in [\beta^{r+1}(\tilde{\theta}-\epsilon) - \frac{1}{2}(\frac{\delta_\tau}{2})^{s-r-1}\epsilon^2, \beta^{r+1}(\tilde{\theta}+\epsilon) + \frac{1}{2}(\frac{\delta_\tau}{2})^{s-r-1}\epsilon^2]$. Therefore, $|\theta - \beta(\theta')| \geq \frac{1}{2}(\frac{\delta_\tau}{2})^{r-1}\epsilon^2$. Hence, inequality (86) in Lemma 13 implies that $V(\theta) - U(\theta'|\theta) \geq \delta_V(\frac{\delta_\tau}{2})^{2(r-1)}\frac{\epsilon^4}{16}$ for small enough ϵ. The second equality holds by definitions of $\tilde{q}(\theta')$ and $\tilde{t}(\theta')$. The last inequality holds for small enough ϵ.

Now, suppose that $\frac{\theta+\beta(\theta')}{2} \leq \hat{\theta}$ in the original mechanism. Since $\beta(\theta') > \hat{\theta}$, it follows that $\theta < \hat{\theta}$ and so $\tau(\theta) = \emptyset$ and $IC(\theta,\theta')$ is slack in the original mechanism. Hence, when ϵ is sufficiently small, $\tilde{IC}(\theta,\theta')$ is slack in the modified mechanism as well.

Seventh, suppose that $\{\theta,\theta'\} \subset \Theta_0(\epsilon)$. If $\theta > \theta'$. Since $V(.)$ and $q(.)$ are continuous, $\tilde{V}(.)$ and $\tilde{q}(.)$ are continuous on $\Theta_0(\epsilon)$, and so $\tilde{IC}(\theta,\theta')$ holds when ϵ is sufficiently small.

Step 3. Establishing that the mechanism $(\tilde{q}(.),\tilde{t}(.))$ is more profitable for the principal than the original mechanism.

The change in seller's profits from switching to the new mechanism is equal to

$$\Pi(\epsilon) = \int_{\Theta_0(\epsilon)} [u(q(\theta)+\epsilon^5,\theta) - u(q(\theta),\theta)]f(\theta)d\theta - \Delta(\epsilon)\sum_{k=1}^s \int_{\Theta_k(\epsilon)} f(\theta)d\theta.$$

Hence,

$$\lim_{\epsilon\to0}\frac{\Pi(\epsilon)}{\epsilon^6}=\lim_{\epsilon\to0}\frac{1}{\epsilon}\left(\int_{\Theta_0(\epsilon)}u_q(q(\theta),\theta)f(\theta)d\theta-\max_{\theta'\in\Theta_0(\epsilon)}[u_q(q(\theta'),\overline{\theta}_1)-u_q(q(\theta'),\theta')]\sum_{k=1}^{s}\int_{\Theta_k(\epsilon)}f(\theta)d\theta\right)$$
$$=\lim_{\epsilon\to0}\frac{1}{\epsilon}\left(\int_{\tilde{\theta}-\epsilon}^{\tilde{\theta}+\epsilon}u_q(q(\theta),\theta)f(\theta)d\theta-\max_{\theta'\in\Theta_0(\epsilon)}[u_q(q(\theta'),\overline{\theta}_1)-u_q(q(\theta'),\theta')]\sum_{k=1}^{s}\int_{\beta^k(\tilde{\theta}-\epsilon)}^{\beta^k(\tilde{\theta}+\epsilon)}f(\theta)d\theta\right)$$
$$=\lim_{\epsilon\to0}\frac{1}{\epsilon}\left(\int_{\tilde{\theta}-\epsilon}^{\tilde{\theta}+\epsilon}u_q(q(\theta),\theta)f(\theta)-\max_{\theta'\in\Theta_0(\epsilon)}[u_q(q(\theta'),\overline{\theta}_1)-u_q(q(\theta'),\theta')]\sum_{k=1}^{s}\dot{\beta}^k(\theta)f(\beta^k(\theta))d\theta\right)$$
$$=2\left(u_q(q(\tilde{\theta}),\tilde{\theta})f(\tilde{\theta})-[u_q(q(\tilde{\theta}),\beta(\tilde{\theta}))-u_q(q(\tilde{\theta}),\tilde{\theta})]\sum_{k=1}^{s}\dot{\beta}^k(\tilde{\theta})f(\beta^k(\tilde{\theta}))\right)>2\mu>0,$$

where the first equality holds by definition of $\Delta(.)$. The second equality holds because $\Theta_k(\epsilon)$ converges to $[\beta^k(\tilde{\theta}-\epsilon),\beta^k(\tilde{\theta}+\epsilon)]$ at the same rate as ϵ^2. The third equality is obtained by a change of variables. The fourth equality holds since $\overline{\theta}_1\to\beta(\tilde{\theta})$ and $\Theta_0(\epsilon)\to\tilde{\theta}$ as $\epsilon\to0$; and the first inequality holds by (85). Therefore, $\Pi(\epsilon)>0$ for small enough ϵ, which contradicts the optimality of the original mechanism. *Q.E.D.*

Next, to derive the law of motion of $q(.)$ in (77), let us prove the following Lemma.

Lemma 19 *If $IC(\beta(\theta),\theta)$, then $I(\theta)=0$. If $IC(\beta(\theta),\theta)$ is slack, then $I(\theta)<0$.*

Proof of Lemma 19: Let $S=\{\theta\in[\min\tau(\hat{\theta}),\max\tau(1)]:\theta\notin\tau(\beta(\theta))\}$. That is, S is the set of types such that $IC(\beta(\theta),\theta)$ is slack. Since τ is upper hemicontinuous and strictly increasing, $S=\cup_{i=1}^{\infty}(\underline{\theta}_i,\overline{\theta}_i)$ where $\underline{\theta}_i\le\overline{\theta}_i\le\underline{\theta}_{i+1}$, and for any $\theta\in[\min\tau(\hat{\theta}),\max\tau(1)]\setminus S_k$, $IC(\beta(\theta),\theta)$ is binding and so $U_2(\beta(\theta),\theta)=0$. Also, there exists $\tilde{\theta}_i$ such that $\beta(\theta)=\tilde{\theta}_i$ for all $\theta\in[\underline{\theta}_i,\overline{\theta}_i]$, and $IC(\tilde{\theta}_i,\underline{\theta}_i)$ and $IC(\tilde{\theta}_i,\overline{\theta}_i)$ are binding.

Note that the number of intervals $(\underline{\theta}_i,\overline{\theta}_i)$ such that $IC(\tilde{\theta}_i,\theta)$ is non-binding for $\theta\in(\underline{\theta}_i,\overline{\theta}_i)$ is at most countable, because all such intervals are pairwise disjoint, their union is contained in $[0,1]$, and, being open, each such interval contains at least one rational number, while the number of rational numbers in an interval is countable.

Therefore, $U(\tilde{\theta}_i,\underline{\theta}_i)=U(\tilde{\theta}_i,\overline{\theta}_i)>U(\tilde{\theta}_i,\theta)$ for any $\theta\in(\underline{\theta}_i,\overline{\theta}_i)$, and hence

$$\int_{\underline{\theta}_i}^{\theta}U_2(\tilde{\theta}_i,s)ds\begin{cases}<0 & \text{if } \theta\in(\underline{\theta}_i,\overline{\theta}_i)\\ =0 & \text{if } \theta=\overline{\theta}_i\end{cases}. \tag{87}$$

Let $\overline{\theta}_0 = \beta(\min\tau(\hat{\theta}))$. If $IC(\beta(\theta),\theta)$ is binding, then $\theta \in [\overline{\theta}_i, \underline{\theta}_{i+1}]$ for $i \geq 0$ and $U_2(\beta(\theta),\theta) = 0$. So using (87), we obtain $I(\theta) = \int_{\min\tau(\hat{\theta})}^{\theta} U_2(\beta(x),x)dx = 0$.

If $IC(\beta(\theta),\theta)$ is slack, then $\theta \in (\underline{\theta}_i, \overline{\theta}_i)$ for some $i \in \{1,..,N\}$, and so $I(\theta) = \int_{\min\tau(\hat{\theta})}^{\theta} U_2(\beta(x),x)dx = \int_{\underline{\theta}_i}^{\theta} U_2(\tilde{\theta}_i, s)ds < 0$, where the first equality holds by the definition of $I(.)$, the second equality holds because $U_2(\beta(\theta),\theta) = 0$ for all $\theta \in [\min\tau(\hat{\theta}), \max\tau(1)] \setminus S_k$, and the inequality holds by (87). It follows that $IC(\beta(\theta),\theta)$ is binding/non-binding if $I(\theta) = 0/I(\theta) < 0$. This completes the proof of Lemma 19. *Q.E.D.*

Now, we are in a position to complete the derivation of the law of motion for $q(.)$. We need to consider three cases.

(1) *Suppose that* $q(\theta) < q^{fb}(\theta)$. Since $q(.)$ is continuous by Theorem 3 in the main paper, it follows that there exists $\epsilon > 0$ s.t. for all $\theta' \in (\theta-\epsilon, \theta+\epsilon)$, $q(\theta') < q^{fb}(\theta')$, and so by Lemma 9 in the main paper, $IC(\beta(\theta'),\theta')$ is binding. Hence $U_2(\beta(\theta),\theta) = 0$, and (75) must hold i.e., $\dot{q}(\theta) = \dot{q}^{IC}((\theta)$.

(2) *Now suppose that* $I(\theta) < 0$. Then $IC(\beta(\theta),\theta)$ is slack by Lemma 19. By continuity of $q(.)$ and $I(.)$, there exists $\epsilon > 0$ s.t. for all $\theta' \in (\theta-\epsilon, \theta+\epsilon)$, we also have $I(\theta') < 0$ and hence $IC(\beta(\theta'),\theta')$ and $q(\theta') = q^{fb}(\theta')$ by Lemma 9. So $\dot{q}(\theta) = \dot{q}^{fb}(\theta)$.

(3) *Now suppose that* $q(\theta) = q^{fb}(\theta)$ *and* $I(\theta) = 0$. Then we must have $\dot{q}(\theta) \leq \dot{q}^{fb}(\theta)$, for otherwise $q(\theta') > q^{fb}(\theta')$ for $\theta' \in (\theta, \theta+\epsilon)$ for some $\epsilon > 0$, which would contradict Lemma 9.

Suppose also that $\dot{q}^{fb}(\theta) < \dot{q}^{IC}(\theta)$. Then $\dot{q}(\theta) < \dot{q}^{IC}(\theta)$, and so $U_2(\beta(\theta),\theta) < 0$. Hence, there exists $\epsilon > 0$ s.t. for all $\theta' \in (\theta, \theta+\epsilon)$, $I(\theta') < 0$ which implies that $IC(\beta(\theta'),\theta')$ is slack by Lemma 19, and so $q(\theta') = q^{fb}(\theta')$ by Lemma 9. Hence $\dot{q}(\theta) = \dot{q}^{fb}(\theta)$.

Now suppose that $\dot{q}^{fb}(\theta) \geq \dot{q}^{IC}(\theta)$. If $\dot{q}(\theta) > \dot{q}^{IC}(\theta)$, then $U_2(\beta(\theta),\theta) > 0$. Hence, there exists $\epsilon > 0$ s.t. for all $\theta' \in (\theta, \theta+\epsilon)$, $I(\theta') > 0$ which contradicts Lemma 19.

On the other hand, if $\dot{q}(\theta) < \dot{q}^{IC}(\theta)$, then $U_2(\beta(\theta),\theta) < 0$. Hence, there exists $\epsilon > 0$ s.t. for all $\theta' \in (\theta, \theta+\epsilon)$, $I(\theta') < 0$ and, by Lemma 19, $IC(\beta(\theta'),\theta')$ is slack, and so $q(\theta') = q^{fb}(\theta')$ by Lemma 9. But this contradicts $\dot{q}(\theta) < \dot{q}^{IC}(\theta) \leq \dot{q}^{fb}(\theta)$. Hence, $\dot{q}(\theta) = \dot{q}^{IC}(\theta)$. This completes the derivation of the law of motion of $q(.)$ in (77).

Finally, let us establish (83) and (82). For $\theta \in [\underline{\tau}, \overline{\tau}]$, let $A(\theta) = \sum_{k=1}^{s} f(\beta^k(\theta))\dot{\beta}^k(\theta)$.

Then by recursion,

$$A(\theta) = \begin{cases} \dot{\beta}(\theta)[f(\beta(\theta)) + A(\beta(\theta))] & \text{if } \beta^2(\theta) \neq \emptyset \\ \dot{\beta}(\theta) f(\beta(\theta)) & \text{if } \beta^2(\theta) = \emptyset \end{cases} \tag{88}$$

Next, let

$$B(\theta) = \frac{u_q(q(\theta), \theta)}{u_q(q(\theta), \beta(\theta)) - u_q(q(\theta), \theta)}. \tag{89}$$

The optimality condition (74) in Theorem 11 implies that

$$A(\theta) = f(\theta)B(\theta) = \begin{cases} \dot{\beta}(\theta) f(\beta(\theta))[1 + B(\beta(\theta))] & \text{if } \beta^2(\theta) \neq \emptyset, \\ \dot{\beta}(\theta) f(\beta(\theta)) & \text{if } \beta^2(\theta) = \emptyset. \end{cases} \tag{90}$$

Therefore,

$$\dot{\beta}(\theta) = \begin{cases} \frac{f(\theta)B(\theta)}{f(\beta(\theta))[1+B(\beta(\theta))]} & \text{if } \beta^2(\theta) \neq \emptyset \\ \frac{f(\theta)B(\theta)}{f(\beta(\theta))} & \text{if } \beta^2(\theta) = \emptyset \end{cases} \tag{91}$$

Since $\beta^k(\theta) = \prod_{i=0}^{k-1} \dot{\beta}(\beta^i(\theta))$, we have

$$\dot{\beta}^k(\theta) = \begin{cases} \prod_{i=0}^{k-1} \frac{f(\beta^i(\theta))B(\beta^i(\theta))}{f(\beta^{i+1}(\theta))[1+B(\beta^{i+1}(\theta))]} & \text{if } \beta^{k+1}(\theta) \neq \emptyset, \\ \frac{f(\beta^{k-1}(\theta))B(\beta^{k-1}(\theta)}{f(\beta^k(\theta))} \prod_{i=0}^{k-2} \frac{f(\beta^i(\theta))B(\beta^i(\theta))}{f(\beta^{i+1}(\theta))[1+B(\beta^{i+1}(\theta))]} & \text{if } \beta^{k+1}(\theta) = \emptyset. \end{cases}$$

Using (89) in the above and setting $Q^k(\theta) = q(\beta^k(\theta))$ yields for $\theta \in [\underline{\tau}, \hat{\theta}]$:

$$\dot{\beta}^k(\theta) = \begin{cases} \frac{f(\theta)u_q(Q^0(\theta),\theta))[u_q(Q^k(\theta),\beta^{k+1}(\theta))-u_q(Q^k(\theta),\beta^k(\theta))]}{f(\beta^k(\theta))[u_q(Q^0(\theta),\beta^1(\theta))-u_q(Q^0(\theta),\theta)]u_q(Q^k(\theta),\beta^{k+1}(\theta))} \prod_{i=1}^{k-1} \frac{u_q(Q^i(\theta)),\beta^i(\theta))}{u_q(Q^i(\theta),\beta^{i+1}(\theta))} & \text{if } k < s(\theta) \\ \frac{f(\theta)u_q(Q^0(\theta),\theta)}{f(\beta^k(\theta))[u_q(Q^0(\theta),\beta^1(\theta))-u_q(Q^0(\theta),\theta)]} \prod_{i=1}^{k-1} \frac{u_q(Q^i(\theta)),\beta^i(\theta))}{u_q(Q^i(\theta),\beta^{i+1}(\theta))} & \text{if } k = s(\theta) \end{cases} \tag{92}$$

Equations (83) are derived from (74), (77), and the definition of Q^k. *Q.E.D.*

The optimal mechanism in the quadratic-uniform case under an intermediate cost C

Consider the following system of ordinary differential equation system presented in section 4.4 in the main paper:

$$\dot{\tau} = \frac{\theta - \tau}{\tau - Q}, \tag{93}$$

$$\dot{Q} = \frac{Q}{\tau - Q}, \tag{94}$$

With boundary conditions:

$$Q(1) = \tau(1) \tag{95}$$

$$Q(\hat{\theta}) = \tau(\hat{\theta}) \tag{96}$$

$$Q(\hat{\theta})(\hat{\theta} - \tau(\hat{\theta})) = C \tag{97}$$

First, let us make a change of variables:

$$y = \tau - Q, \qquad z = \tau + Q \tag{98}$$

Then the system (93)-(94) is equivalent to the following system:

$$\dot{y}y = \theta - z \tag{99}$$

$$\dot{z} = \frac{\theta}{y} - 1 \tag{100}$$

Differentiating (99) yields:

$$\ddot{y}y + (\dot{y})^2 = 1 - \dot{z} = 2 - \frac{\theta}{y} \tag{101}$$

Let us make another change of variables: $w = \frac{y^2}{4}$. Then (101) becomes:

$$\ddot{w}(\theta) = 1 - \frac{\theta}{4\sqrt{w(\theta)}} \tag{102}$$

The general solution to the differential equation (102) is parametric. Specifically, let b_1, b_2 and b_3 be some constants and $t \in [0, \infty)$ be a parameter. Then:

$$\theta(t) = b_1 t + b_2 t^{\frac{\sqrt{5}-1}{2}} + b_3 t^{-\frac{\sqrt{5}+1}{2}} \tag{103}$$

$$\frac{y^2(t)}{4} \equiv w(t) = \left(\frac{1}{2} b_1 t + \frac{\sqrt{5}-1}{4} b_2 t^{\frac{\sqrt{5}-1}{2}} - \frac{\sqrt{5}+1}{4} b_3 t^{-\frac{\sqrt{5}+1}{2}} \right)^2 \tag{104}$$

Indeed, note that we have:

$$\frac{d\frac{y^2(t)}{4}}{dt} \equiv \frac{dw(t)}{dt} = \left(b_1 + \frac{3-\sqrt{5}}{2}b_2 t^{\frac{\sqrt{5}-3}{2}} + \frac{3+\sqrt{5}}{2}b_3 t^{-\frac{\sqrt{5}+3}{2}}\right)\left(\frac{1}{2}b_1 t + \frac{\sqrt{5}-1}{4}b_2 t^{\frac{\sqrt{5}-1}{2}} - \frac{\sqrt{5}+1}{4}b_3 t^{-\frac{\sqrt{5}+1}{2}}\right) \tag{105}$$

$$\frac{d^2\frac{y^2(t)}{4}}{dt^2} \equiv \frac{d^2w(t)}{dt^2} = \frac{1}{2}\left(b_1 + \frac{3-\sqrt{5}}{2}b_2 t^{\frac{\sqrt{5}-3}{2}} + \frac{3+\sqrt{5}}{2}b_3 t^{-\frac{\sqrt{5}+3}{2}}\right)^2$$
$$+ \left(-\frac{7-3\sqrt{5}}{2}b_2 t^{\frac{\sqrt{5}-5}{2}} - \frac{7+3\sqrt{5}}{2}b_3 t^{-\frac{\sqrt{5}+5}{2}}\right)\left(\frac{1}{2}b_1 t + \frac{\sqrt{5}-1}{4}b_2 t^{\frac{\sqrt{5}-1}{2}} - \frac{\sqrt{5}+1}{4}b_3 t^{-\frac{\sqrt{5}+1}{2}}\right) \tag{106}$$

$$\theta'(t) = b_1 + \frac{\sqrt{5}-1}{2}b_2 t^{\frac{\sqrt{5}-3}{2}} - \frac{\sqrt{5}+1}{2}b_3 t^{-\frac{\sqrt{5}+3}{2}} \tag{107}$$

$$\theta''(t) = \frac{\sqrt{5}-1}{2}\frac{\sqrt{5}-3}{2}b_2 t^{\frac{\sqrt{5}-5}{2}} + \frac{\sqrt{5}+1}{2}\frac{\sqrt{5}+3}{2}b_3 t^{-\frac{\sqrt{5}+5}{2}} \tag{108}$$

Note that $\frac{d^2w}{d\theta^2} = \frac{\ddot{w}(t)}{(\theta'(t))^2} - \dot{w}(t)\frac{\theta''}{\theta'(t)^3}$. Therefore, the ODE (102) can be rewritten as follows:

$$\frac{\ddot{w}(t)}{(\theta'(t))^2} - \dot{w}(t)\frac{\theta''(t)}{\theta'(t)^3} = 1 - \frac{\theta(t)}{4\sqrt{w(t)}} \tag{109}$$

Note that we must have $0 \le y < \theta$, since $y = \tau - Q$, $\tau < \theta$, and the optimal quantity Q cannot be greater than its first-best level, which in this case is equal to τ. So,

$$y(t) = \left|b_1 t + \frac{\sqrt{5}-1}{2}b_2 t^{\frac{\sqrt{5}-1}{2}} - \frac{\sqrt{5}+1}{2}b_3 t^{-\frac{\sqrt{5}+1}{2}}\right| \tag{110}$$

We can without loss of generality take that $\theta(1) = 1$. Indeed, if $\theta(t_1) = 1$ for some $t_1 \in (0, \infty)$, $t_1 \ne 1$, then we can replace the parameter t with the parameter $s = \frac{t}{t_1}$, and replace the constants b_1, b_2, b_3 with constants b_1', b_2', b_3' such that $b_1' = b_1 t_1$, $b_2' = b_2 t_1^{\frac{\sqrt{5}-1}{2}}$ and $b_3' = b_3 t_1^{-\frac{\sqrt{5}+1}{2}}$. Then we would have $\theta(s) = \theta(t)$ and $y(s) = y(t)$ for all $t \in [0, \infty)$, with $\theta(s)_{s=1} = 1$.

Using $\theta(1) = 1$ in (103) yields $b_1 + b_2 + b_3 = 1$. Also, $\theta(1) = 1$ and the boundary condition $\tau(1) = Q(1)$ imply that $y(1) = 0$. In turn, the latter implies that $b_1 + \frac{\sqrt{5}-1}{2}b_2 - \frac{\sqrt{5}+1}{2}b_3 = 0$. Now, we can solve for b_2 and b_3 in terms of b_1 to obtain:

$$b_2 = -b_1\frac{5+3\sqrt{5}}{10} + \frac{\sqrt{5}+1}{2\sqrt{5}}.$$

$$b_3 = b_1\frac{3\sqrt{5}-5}{10} + \frac{\sqrt{5}-1}{2\sqrt{5}}.$$

Then (103) and (110) become:

$$\theta(t) = b_1\left(t - \frac{1+3\sqrt{\frac{1}{5}}}{2}t^{\frac{\sqrt{5}-1}{2}} + \frac{3\sqrt{\frac{1}{5}}-1}{2}t^{-\frac{\sqrt{5}+1}{2}}\right) + \frac{\sqrt{5}+1}{2\sqrt{5}}t^{\frac{\sqrt{5}-1}{2}} + \frac{\sqrt{5}-1}{2\sqrt{5}}t^{-\frac{\sqrt{5}+1}{2}} \tag{111}$$

$$y(t) = \left|b_1\left(t - \frac{1+\sqrt{\frac{1}{5}}}{2}t^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}t^{-\frac{\sqrt{5}+1}{2}}\right) + \frac{1}{\sqrt{5}}t^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}}t^{-\frac{\sqrt{5}+1}{2}}\right| \tag{112}$$

At first, let us suppose that the expression under the absolute value sign on the right-hand side of (112) is positive i.e:[13]

$$y(t) = b_1\left(t - \frac{1+\sqrt{\frac{1}{5}}}{2}t^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}t^{-\frac{\sqrt{5}+1}{2}}\right) + \frac{1}{\sqrt{5}}t^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}}t^{-\frac{\sqrt{5}+1}{2}} \tag{113}$$

Next, we solve the differential equation (100) for z, which we will also parameterize by t. So, we have $z'(t) \equiv \frac{dz}{dt} = z'(\theta)\theta'(t)$. By (111) and (113), $y(t) = \theta'(t)t$. Then (100) can be rewritten as:

$$z'(t) = \left(\frac{\theta}{y} - 1\right)\theta'(t) = \frac{\theta}{\theta'(t)t}\theta'(t) - \theta'(t) = \frac{\theta}{t} - \theta'(t). \tag{114}$$

Substituting (111) for $\theta(t)$ we obtain:

$$z'(t) = b_1\left(-\frac{1}{\sqrt{5}}t^{\frac{\sqrt{5}-3}{2}} + \frac{1}{\sqrt{5}}t^{-\frac{\sqrt{5}+3}{2}}\right) + \frac{\sqrt{5}-1}{2\sqrt{5}}t^{\frac{\sqrt{5}-3}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}}t^{-\frac{\sqrt{5}+3}{2}} \tag{115}$$

Integrating (115) yields:

$$z(t) = b_1\left(-\frac{1+\sqrt{\frac{1}{5}}}{2}t^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}t^{-\frac{\sqrt{5}+1}{2}}\right) + \frac{1}{\sqrt{5}}t^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}}t^{-\frac{\sqrt{5}+1}{2}} + k \tag{116}$$

where k is a constant of integration. Now, let us show that equation (99), $y(\dot{\theta})y = \theta - z$, implies that the constant of integration k is equal to zero. Note that $y'(t) = y(\dot{\theta})\theta'(t)$. So we can rewrite (99) as $y'(t)y = (\theta - z)\theta'(t)$. Since $y = \theta'(t)(t)$, the previous equation can be rewritten as follows: $y'(t)t = (\theta - z)$

[13]Later we will show that this is, indeed, the case since the opposite case when this expression is negative leads to a contradiction.

Next, from (113) we obtain:

$$y'(t)t = b_1\left(t - \frac{1}{\sqrt{5}}t^{\frac{\sqrt{5}-1}{2}} + \frac{1}{\sqrt{5}}t^{-\frac{\sqrt{5}+1}{2}}\right) + \frac{1-\sqrt{\frac{1}{5}}}{2}t^{\frac{\sqrt{5}-1}{2}} + \frac{1+\sqrt{\frac{1}{5}}}{2}t^{-\frac{\sqrt{5}+1}{2}} \tag{117}$$

Also, (111) and (116) yield:

$$\theta(t) - z(t) = b_1\left(t - \frac{1}{\sqrt{5}}t^{\frac{\sqrt{5}-1}{2}} + \frac{1}{\sqrt{5}}t^{-\frac{\sqrt{5}+1}{2}}\right) + \frac{\sqrt{5}-1}{2\sqrt{5}}t^{\frac{\sqrt{5}-1}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}}t^{-\frac{\sqrt{5}+1}{2}} - k \tag{118}$$

Equating (117) and (118) yields $k = 0$.

Furthermore, observe that $z(t) - y(t) = -b_1 t$. Since $z(t) - y(t) = 2Q(t)$, it follows that $Q(t) = -\frac{b_1}{2}t$ and so $b_1 < 0$.

Now, let us confirm that, as claimed, the expression under the absolute value sign on the right-hand side of (112) is positive. The proof is by contradiction, so suppose otherwise i.e.,

$$y(t) = -b_1\left(t - \frac{1+\sqrt{\frac{1}{5}}}{2}t^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}t^{-\frac{\sqrt{5}+1}{2}}\right) - \frac{1}{\sqrt{5}}t^{\frac{\sqrt{5}-1}{2}} + \frac{1}{\sqrt{5}}t^{-\frac{\sqrt{5}+1}{2}} \tag{119}$$

Then (111) and (119) yield $y(t) = -\theta'(t)t$ and so, instead of (114), we now have:

$$z'(t) = \left(\frac{\theta}{y} - 1\right)\theta'(t) = \frac{\theta}{-\theta'(t)t}\theta'(t) - \theta'(t) = -\frac{\theta}{t} - \theta'(t) = \frac{\theta}{t} - \theta'(t) - 2\frac{\theta}{t}. \tag{120}$$

Substituting (111) for $\theta(t)$ in (120) and integrating yields:

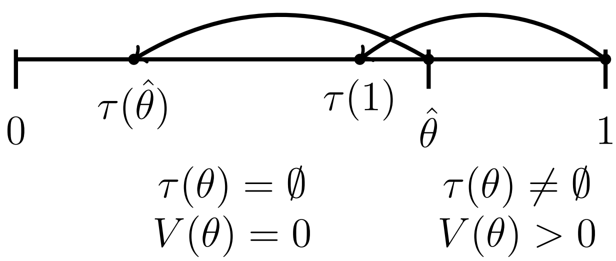
$$\begin{aligned} z(t) = {} & b_1\left(-\frac{1+\sqrt{\frac{1}{5}}}{2}t^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}t^{-\frac{\sqrt{5}+1}{2}}\right) + \frac{1}{\sqrt{5}}t^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}}t^{-\frac{\sqrt{5}+1}{2}} \\ & - 2b_1\left(t - \frac{1+3\sqrt{\frac{1}{5}}}{\sqrt{5}-1}t^{\frac{\sqrt{5}-1}{2}} - \frac{3\sqrt{\frac{1}{5}}-1}{\sqrt{5}+1}t^{-\frac{\sqrt{5}+1}{2}}\right) + \frac{\sqrt{5}+1}{\sqrt{5}(\sqrt{5}-1)}t^{\frac{\sqrt{5}-1}{2}} - \frac{\sqrt{5}-1}{\sqrt{5}(\sqrt{5}-1)}t^{-\frac{\sqrt{5}+1}{2}} + k_2 \end{aligned} \tag{121}$$

where k_2 is a constant of integration.

Since in this case $y = -\theta'(t)t$, the equation $y'(t)y = (\theta - z)\theta'(t)$ (i.e., equation (99) parameterized by t) can be rewritten as $-y'(t)t = (\theta - z)$. This equation can be rewritten as follows using (111) and (121) and differentiating (119):

$$-2b_1\left(t - \frac{1+3\sqrt{\frac{1}{5}}}{\sqrt{5}-1}t^{\frac{\sqrt{5}-1}{2}} - \frac{3\sqrt{\frac{1}{5}}-1}{\sqrt{5}+1}t^{-\frac{\sqrt{5}+1}{2}}\right) + \frac{\sqrt{5}+1}{\sqrt{5}(\sqrt{5}-1)}t^{\frac{\sqrt{5}-1}{2}} - \frac{\sqrt{5}-1}{\sqrt{5}(\sqrt{5}-1)}t^{-\frac{\sqrt{5}+1}{2}} + k_2 = 0$$

Figure 4: Structure of targeted types $\tau(.)$ and informational rents $V(.)$ under intermediate costs of lying



which cannot hold on any neighborhood of t.

Thus, we have confirmed that $y(t)$ is given by (113), and hence $y(t) = \theta'(t)t$. Since $y(t) \geq 0$, it follows that $\theta'(t) > 0$.

So, to complete the solution, it remains to characterize b_1 and $\hat{t}$ such that $\hat{t} < 1$ and $y(\hat{t}) = 0$ and $y(t) \geq 0$ for all $t \in [\hat{t}, 1]$. We will then have $\hat{\theta} = \theta(\hat{t}) < 1$. For this, we need to compute $y'(t)$ and $y''(t)$. We have:

$$y'(t) = b_1 + \frac{(\sqrt{5}-1) - 2b_1}{2\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-3}{2}} + \frac{(\sqrt{5}+1) + 2b_1}{2\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+3}{2}} \tag{122}$$

$$y''(t) = -\frac{3-\sqrt{5}}{2}\frac{(\sqrt{5}-1) - 2b_1}{2\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-5}{2}} - \frac{\sqrt{5}+3}{2}\frac{(\sqrt{5}+1) + 2b_1}{2\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+5}{2}} \tag{123}$$

Using (113) and (117) we obtain:

$$\begin{aligned}
y(t) - ty'(t) = b_1 &\left(t - \frac{1+\sqrt{\frac{1}{5}}}{2}t^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{1}{\sqrt{5}}t^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}}t^{-\frac{\sqrt{5}+1}{2}} \\
&- b_1\left(t - \frac{1}{\sqrt{5}}t^{\frac{\sqrt{5}-1}{2}} + \frac{1}{\sqrt{5}}t^{-\frac{\sqrt{5}+1}{2}} \right) - \frac{1-\sqrt{\frac{1}{5}}}{2}t^{\frac{\sqrt{5}-1}{2}} - \frac{1+\sqrt{\frac{1}{5}}}{2}t^{-\frac{\sqrt{5}+1}{2}} = \\
&b_1\left(-\frac{\sqrt{5}-1}{2\sqrt{5}}t^{\frac{\sqrt{5}-1}{2}} - \frac{\sqrt{5}+1}{2\sqrt{5}}t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{3-\sqrt{5}}{2\sqrt{5}}t^{\frac{\sqrt{5}-1}{2}} - \frac{3+\sqrt{5}}{2\sqrt{5}}t^{-\frac{\sqrt{5}+1}{2}}
\end{aligned} \tag{124}$$

As established above, $b_1 < 0$. In fact, let us show that $b_1 \in [-\frac{\sqrt{5}+1}{2}, -1)$.

First, let us rule out $b_1 < -\frac{\sqrt{5}+1}{2}$. Observe that if $b_1 < -\frac{\sqrt{5}+1}{2}$, then by (122) $y'(t) < 0$ for all $t \geq 1$. Since $y(1) = 0$, it follows that $y(t) < 0$ for all $t > 1$ and $y(1-\epsilon) > 0$ for sufficiently small $\epsilon > 0$. Further, observe from (113) that $y(t) > 0$ when t is sufficiently small, with $\lim_{t\to 0+} y(t) = \infty$. Finally, (124) implies that $y'(t) < 0$ if $y(t) = 0$. So, if $b_1 < -\frac{\sqrt{5}+1}{2}$ then there does not exist $\hat{t} \neq 1$ such that $y(\hat{t}) = 0$.

Consider now $b_1 \in [-\frac{\sqrt{5}+1}{2}, 0]$. Note that in this case: (i) by (123), $y''(t) < 0$ for all t; (ii) $y(t) < 0$ when t is sufficiently small, with $\lim_{t\to 0+} y(t) = -\infty$, (iii) $y(t) < 0$ when t is sufficiently large, with $\lim_{t\to\infty} y(t) = -\infty$. (iv) By (122) $y'(1) = b_1 + 1$.

So, if $b_1 \in (-1, 0]$, then $y'(1) > 0$. This, in combination with (i)-(iii) above, implies that if $b_1 \in (-1, 0]$, then there exists a unique $\hat{t}$, $\hat{t} \neq 1$ such that $y(\hat{t}) = 0$ and, moreover, $\hat{t} > 1$ and $y(t) > 0$ for all $t \in (1, \hat{t})$. But we also have $y(t) = \theta'(t)t$ and $\theta(1) = 1$. So $\theta(t) > 1$ for all $t \in (1, \hat{t})$. This contradicts the fact that $\theta(t) \in [0, 1]$. Hence, we can rule out $b_1 \in (-1, 0]$.

Similarly, we can rule out $b_1 = -1$ because in this case $y(t) = 1$ only if $t = 1$.

Finally, if $b_1 \in [-\frac{\sqrt{5}+1}{2}, -1)$, then (i)-(iv) above imply that there exists $\hat{t}$, $\hat{t} < 1$ such that $y(\hat{t}) = 0$, and $y(t) > 0$ for all $t \in (\hat{t}, 1)$. Also, since $y(t) = \theta'(t)t$ and $\theta(1) = 1$, it follows that $\theta(t) \in [0, 1)$ for all $t \in (\hat{t}, 1)$. Moreover,

$$\begin{aligned}\theta(t) - y(t) &= b_1\left(-\frac{1}{\sqrt{5}}t^{\frac{\sqrt{5}-1}{2}} + \frac{1}{\sqrt{5}}t^{-\frac{\sqrt{5}+1}{2}}\right) + \frac{\sqrt{5}-1}{2\sqrt{5}}t^{\frac{\sqrt{5}-1}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}}t^{-\frac{\sqrt{5}+1}{2}} = \\ &= \frac{\sqrt{5}-1-2b_1}{2\sqrt{5}}t^{\frac{\sqrt{5}-1}{2}} + \frac{\sqrt{5}+1+2b_1}{2\sqrt{5}}t^{-\frac{\sqrt{5}+1}{2}} \end{aligned} \tag{125}$$

To summarize, $\theta(t) - y(t) > 0$ and $\theta(t) \leq 1$ for all $t \in [\hat{t}, 1]$ when $b_1 \in [-\frac{\sqrt{5}+1}{2}, -1]$, as required for the solution. We conclude that $b_1 \in [-\frac{\sqrt{5}+1}{2}, -1)$.

Thus, the two remaining parameters completing the solution are $\hat{t} \in (0, 1)$ and $b_1 \in [-\frac{\sqrt{5}+1}{2}, -1)$. They are jointly determined as the solutions to the two equations: $y(\hat{t}) = 0$ where $y(\hat{t})$ is given by (119) and the boundary condition $Q(\hat{t})(\theta(\hat{t}) - \tau(\hat{t})) = C$.

Setting (119) to zero at $\hat{t}$ yields:

$$b_1 = -\frac{\frac{1}{\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+1}{2}}}{\hat{t} - \frac{1+\sqrt{\frac{1}{5}}}{2}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}\hat{t}^{-\frac{\sqrt{5}+1}{2}}} \tag{126}$$

Differentiating (126) we obtain for $\hat{t} \in (0,1)$:

$$
\begin{aligned}
\frac{\partial b_1}{\partial \hat{t}} &= -\frac{\frac{\sqrt{5}-1}{2\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-3}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+3}{2}}}{\hat{t} - \frac{1+\sqrt{\frac{1}{5}}}{2}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}\hat{t}^{-\frac{\sqrt{5}+1}{2}}} + \frac{\left(\frac{1}{\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+1}{2}}\right)\left(1 - \frac{1}{\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-3}{2}} + \frac{1}{\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+3}{2}}\right)}{\left(\hat{t} - \frac{1+\sqrt{\frac{1}{5}}}{2}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}\hat{t}^{-\frac{\sqrt{5}+1}{2}}\right)^2} \\
&= \frac{\frac{3-\sqrt{5}}{2\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{3+\sqrt{5}}{2\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+1}{2}} + \hat{t}^{-2}}{\left(\hat{t} - \frac{1+\sqrt{\frac{1}{5}}}{2}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}\hat{t}^{-\frac{\sqrt{5}+1}{2}}\right)^2} > 0 \qquad (127)
\end{aligned}
$$

where the last inequality follows from the fact that $\frac{3-\sqrt{5}}{2\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{3+\sqrt{5}}{2\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+1}{2}} + \hat{t}^{-2} = 0$ for $\hat{t} = 1$ and $\frac{\partial\left(\frac{3-\sqrt{5}}{2\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{3+\sqrt{5}}{2\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+1}{2}} + \hat{t}^{-2}\right)}{\partial \hat{t}} = \frac{(3-\sqrt{5})(\sqrt{5}-1)}{4\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-3}{2}} + \frac{(\sqrt{5}+1)(3+\sqrt{5})}{4\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+3}{2}} - 2\hat{t}^{-3} < 0$ for $\hat{t} \in (0,1)$.

Recall that $Q(\hat{t}) = \tau(\hat{t}) = -\frac{b_1}{2}\hat{t}$. Also, since $y(\hat{t}) = 0$, $\theta(\hat{t})$ is given by the right-hand side of (125). Using this, we can rewrite the boundary condition $Q(\hat{t})(\theta(\hat{t}) - \tau(\hat{t})) = C$ as follows:

$$
F(b_1, \hat{t}, C) \equiv -\frac{b_1}{2}\left(b_1\left(\frac{\hat{t}^2}{2} - \frac{1}{\sqrt{5}}\hat{t}^{\frac{\sqrt{5}+1}{2}} + \frac{1}{\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}-1}{2}}\right) + \frac{\sqrt{5}-1}{2\sqrt{5}}\hat{t}^{\frac{\sqrt{5}+1}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}-1}{2}}\right) - C = 0 \qquad (128)
$$

Next, from (127) and (128) we get $\frac{dF}{dC} = -1 < 0$ and

$$
\frac{dF(b_1(\hat{t}), \hat{t}, C)}{d\hat{t}} = -\frac{b_1}{2}y(\hat{t}) - \frac{\partial b_1}{\partial \hat{t}}\left(\frac{b_1}{2}\hat{t}^2 + \frac{\sqrt{5}-1-4b_1}{4\sqrt{5}}\hat{t}^{\frac{\sqrt{5}+1}{2}} + \frac{\sqrt{5}+1+4b_1}{4\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}-1}{2}}\right) > 0.
$$

The last inequality holds since: (i) $y(\hat{t}) = 0$; (ii) $\frac{\partial b_1}{\partial \hat{t}} > 0$ as shown in (127); (iii) the multiplier of $\frac{\partial b_1}{\partial \hat{t}}$, $\frac{b_1}{2}\hat{t}^2 + \frac{\sqrt{5}-1-4b_1}{4\sqrt{5}}\hat{t}^{\frac{\sqrt{5}+1}{2}} + \frac{\sqrt{5}+1+4b_1}{4\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}-1}{2}}$, is negative when $\hat{t} = 1$ and $b_1 < -1$ and is increasing in $\hat{t}$ at any $\hat{t} \in (0,1)$ and $b_1 < -1$.

Next, applying l'Hospital's rule to (126) we obtain:

$$
\lim_{\hat{t}\to 1} b_1(\hat{t}) = -\frac{\lim_{\hat{t}\to 1}\left(\frac{\sqrt{5}-1}{2\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-3}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+3}{2}}\right)}{\lim_{\hat{t}\to 1}\left(1 - \frac{1}{\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-3}{2}} + \frac{1}{\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+3}{2}}\right)} = -1.
$$

So, $\lim_{\hat{t}\to 1} F(b_1(\hat{t}), \hat{t}, C) = \frac{1}{4} - C$.

On the other hand, $\lim_{\hat{t}\to 0} b_1(\hat{t}) = -\frac{\sqrt{5}+1}{2}$, and so $\lim_{\hat{t}\to 0} F(b_1(\hat{t}), \hat{t}, C) = -C$, From the above we conclude that for $C \in (0, \frac{1}{4})$ there exist a unique solution $\hat{t} \in (0,1)$ to the equation $F(b_1(\hat{t}), \hat{t}, C) = 0$ and that $\frac{d\hat{t}}{dC} > 0$.

Now let us establish the interval of C on which our solution applies. The upper bound of C is equal to $\frac{1}{4}$, since for $C > \frac{1}{4}$ no incentive constraints are binding. To establish the lower bound of C, $\underline{C}_1$, note that our solution applies when $\hat{\theta} \geq \tau(1)$. At $\underline{C}_1$ we then have $\hat{\theta} = \tau(1) = Q(1)$. Let $\hat{t}_m$, and $b_{1,m}$ denote the parameter values where the latter condition holds. Then we can rewrite the boundary condition $Q(\hat{\theta})(\hat{\theta} - \tau(\hat{\theta})) = C$ as follows:

$$\begin{aligned} Q(\hat{t}_m)(Q(1) - Q(\hat{t}_m)) &= \underline{C}_1 \\ \frac{(b_{1,m})^2}{4}\hat{t}_m(1-\hat{t}_m) &= \underline{C}_1 \end{aligned} \tag{129}$$

So, $\underline{C}_1$, $\hat{t}_m$, and $b_{1,m}$ are determined by (126), (129) and condition $\theta(\hat{t}_m) = \tau(1) = Q(1)$ Since $\tau(1) = Q(1) = -\frac{b_{1,m}}{2}$, we can equate the latter to $\theta(\hat{t}_m)$ as given by (125), since $y(\hat{t}_m) = 0$, to obtain:

$$b_{1,m}\left(-\frac{1}{2} + \frac{1}{\sqrt{5}}\hat{t}_m^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}}\hat{t}_m^{-\frac{\sqrt{5}+1}{2}}\right) = \frac{\sqrt{5}-1}{2\sqrt{5}}\hat{t}_m^{\frac{\sqrt{5}-1}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}}\hat{t}_m^{-\frac{\sqrt{5}+1}{2}} \tag{130}$$

Using (126) in (130) and simplifying yields:

$$\begin{aligned} &-\left(\frac{1}{\sqrt{5}}\hat{t}_m^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}}\hat{t}_m^{-\frac{\sqrt{5}+1}{2}}\right)\left(-\frac{1}{2} + \frac{1}{\sqrt{5}}\hat{t}_m^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}}\hat{t}_m^{-\frac{\sqrt{5}+1}{2}}\right) = \\ &\left(\frac{\sqrt{5}-1}{2\sqrt{5}}\hat{t}_m^{\frac{\sqrt{5}-1}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}}\hat{t}_m^{-\frac{\sqrt{5}+1}{2}}\right)\left(\hat{t}_m - \frac{1+\sqrt{\frac{1}{5}}}{2}\hat{t}_m^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}\hat{t}_m^{-\frac{\sqrt{5}+1}{2}}\right) \end{aligned} \tag{131}$$

The last equation simplifies to:

$$\hat{t}_m^{\sqrt{5}+1}(1-\sqrt{5}) + \hat{t}_m^{\sqrt{5}} - \hat{t}_m(1+\sqrt{5}) + 2\sqrt{5}\hat{t}_m^{\frac{\sqrt{5}-1}{2}} - 1 = 0 \tag{132}$$

The approximate root of the last equation in $(0,1)$ is $\hat{t}_m = 0.187169$. Then from (126) we obtain $b_{1,m} \approx -1.554$ and from (129), $\underline{C}_1 \approx 0.0918$.

Let us now establish some useful comparative statics results. First, we have:

$$\frac{d\hat{\theta}}{dC} = \frac{\partial\hat{\theta}}{\partial b_1}\frac{\partial b_1}{\partial\hat{t}}\frac{d\hat{t}}{dC} + \theta'(\hat{t})\frac{d\hat{t}}{dC} = \left(\hat{t} - \frac{1+3\sqrt{\frac{1}{5}}}{2}\hat{t}^{\frac{\sqrt{5}-1}{2}} + \frac{3\sqrt{\frac{1}{5}}-1}{2}\hat{t}^{-\frac{\sqrt{5}+1}{2}}\right)\frac{\partial b_1}{\partial\hat{t}}\frac{d\hat{t}}{dC} > 0 \tag{133}$$

The second equality follows from the fact that $\theta'(\hat{t}) = \frac{y(\hat{t})}{\hat{t}} = 0$ and (111), while the last inequality holds because, as established above, $\frac{\partial b_1}{\partial \hat{t}} > 0$, $\frac{d\hat{t}}{dC} > 0$, and $\hat{t} - \frac{1+3\sqrt{\frac{1}{5}}}{2}\hat{t}^{\frac{\sqrt{5}-1}{2}} + \frac{3\sqrt{\frac{1}{5}}-1}{2}\hat{t}^{-\frac{\sqrt{5}+1}{2}} = 0$ if $\hat{t} = 1$ and $\frac{\partial\left(\hat{t} - \frac{1+3\sqrt{\frac{1}{5}}}{2}\hat{t}^{\frac{\sqrt{5}-1}{2}} + \frac{3\sqrt{\frac{1}{5}}-1}{2}\hat{t}^{-\frac{\sqrt{5}+1}{2}}\right)}{\partial \hat{t}} < 0$ for any $\hat{t} \in (0,1)$.

We can now confirm that $\hat{\theta} > \tau(1)$ for $C \in (\underline{C}_1, \frac{1}{4})$. We have shown above that $\frac{d\hat{\theta}}{dC} > 0$. Next, since $\tau(1) = Q(1) = -\frac{b_1}{2}$, we have $\frac{d\tau(1)}{dC} = -\frac{1}{2}\frac{db_1}{d\hat{t}} < 0$ where b_1 is given by (126). So, since $\hat{\theta} = \tau(1)$ at $C = \underline{C}_1$, it follows that $\hat{\theta} > \tau(1)$ when $C \in (\underline{C}_1, \frac{1}{4})$, as required.

To obtain the comparative statics for $\tau(\hat{\theta})$, recall that $\tau(\hat{\theta}) = Q(\hat{\theta}) = -\frac{b_1}{2}\hat{t}$. Therefore, $\frac{d\tau(\hat{\theta})}{dC} = \frac{d\tau(\hat{\theta})}{d\hat{t}}\frac{d\hat{t}}{dC} = \left(-\frac{b_1}{2} - \frac{1}{2}\hat{t}\frac{\partial b_1}{\partial \hat{t}}\right)\frac{d\hat{t}}{dC}$. Using (126) and (127) we obtain:

$$
\begin{aligned}
-\frac{b_1}{2} - \frac{1}{2}\hat{t}\frac{\partial b_1}{\partial \hat{t}} &= \frac{1}{2}\frac{\frac{1}{\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+1}{2}}}{\hat{t} - \frac{1+\sqrt{\frac{1}{5}}}{2}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}\hat{t}^{-\frac{\sqrt{5}+1}{2}}} - \frac{1}{2}\hat{t}\frac{\frac{3-\sqrt{5}}{2\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{3+\sqrt{5}}{2\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+1}{2}} + \hat{t}^{-2}}{(\hat{t} - \frac{1+\sqrt{\frac{1}{5}}}{2}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}\hat{t}^{-\frac{\sqrt{5}+1}{2}})^2} \\
&= \frac{1}{2}\frac{(\frac{1}{\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+1}{2}})(\hat{t} - \frac{1+\sqrt{\frac{1}{5}}}{2}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}\hat{t}^{-\frac{\sqrt{5}+1}{2}}) - \hat{t}\left(\frac{3-\sqrt{5}}{2\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{3+\sqrt{5}}{2\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+1}{2}} + \hat{t}^{-2}\right)}{\left(\hat{t} - \frac{1+\sqrt{\frac{1}{5}}}{2}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}\hat{t}^{-\frac{\sqrt{5}+1}{2}}\right)^2} \\
&= \frac{\frac{\sqrt{5}-1}{2\sqrt{5}}\hat{t}^{\frac{\sqrt{5}+1}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}-1}{2}} - \frac{\sqrt{5}+1}{10}\hat{t}^{\sqrt{5}-1} + \frac{\sqrt{5}-1}{10}\hat{t}^{-(\sqrt{5}+1)} - \frac{4}{5}\hat{t}^{-1}}{(\hat{t} - \frac{1+\sqrt{\frac{1}{5}}}{2}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2}\hat{t}^{-\frac{\sqrt{5}+1}{2}})^2}
\end{aligned}
\tag{134}
$$

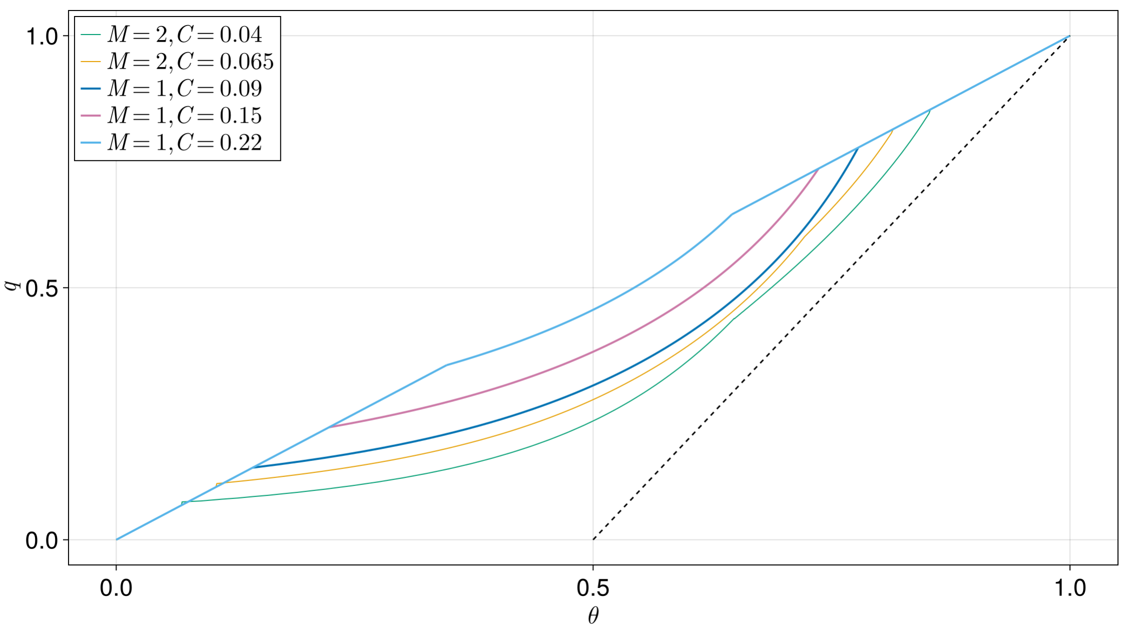
Let $G(\hat{t})$ be the numerator of the last equation in (134). Note that $G(1) = 0$, and $\frac{\partial G}{\partial \hat{t}} = \frac{1}{\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+1}{2}} - \frac{4}{10}\hat{t}^{\sqrt{5}-2} - \frac{4}{10}\hat{t}^{-(\sqrt{5}+2)} + \frac{4}{5}\hat{t}^{-2} = \frac{1}{\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+1}{2}} - \frac{4}{10}\hat{t}^{-2}(\hat{t}^{-\sqrt{5}} - 1)(1 - \hat{t}^{\sqrt{5}})$. So, $\frac{\partial G}{\partial \hat{t}} < 0$ for all $\hat{t} \in (0,1)$. Hence, $G(\hat{t}) > 0$ for all $\hat{t} \in (0,1)$, which by (134) means that $-\frac{b_1}{2} - \frac{1}{2}\hat{t}\frac{\partial b_1}{\partial \hat{t}} > 0$ for $\hat{t} \in (0,1)$. Since $\frac{d\hat{t}}{dC} > 0$, we conclude that $\frac{d\tau(\hat{\theta})}{dC} > 0$.

Finally, let us show that Q is convex in τ. Note that $\tau(t) = y(t) + Q(t) = y(t) - \frac{b_1}{2}t$. Therefore, $\frac{dy}{d\tau} = \frac{y'(t)}{\tau'(t)} = \frac{y'(t)}{y'(t) - \frac{b_1}{2}}$ and $\frac{d^2y}{d\tau^2} = \frac{\frac{d\frac{dy}{d\tau}}{dt}}{\tau'(t)} = \frac{-\frac{b_1}{2}y''(t)}{(y'(t) - \frac{b_1}{2})^3}$. From (122), $\tau'(t) = y'(t) - \frac{b_1}{2} = \frac{1}{2}b_1 + \frac{(\sqrt{5}-1)-2b_1}{2\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-3}{2}} + \frac{(\sqrt{5}+1)+2b_1}{2\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+3}{2}}$ which is equal to $\frac{b_1}{2} + 1 > 0$ when $t = 1$. Since $y''(t) < 0$ by (123), it follows that $y'(t) - \frac{b_1}{2} > 0$ for $t \in (0,1)$. So, $\frac{d^2y}{d\tau^2} < 0$. Since $Q = \tau - y$, we have $\frac{dQ}{d\tau} = 1 - \frac{dy}{d\tau} = \frac{-\frac{b_1}{2}}{y'(t) - \frac{b_1}{2}} > 0$ and $\frac{d^2Q}{d\tau^2} = -\frac{d^2y}{d\tau^2} > 0$.

Also, since $Q(t) = q(\tau(t))$, we have $Q'(t) = q'(\tau(t))\tau'(t)$. So, since $Q'(t) > 0$ and $\tau'(t) > 0$, it follows that $q'(\theta) \equiv q'(\tau(t)) > 0$ for $\theta = \tau(t)$. Finally, differentiating $Q'(t) = q'(\tau(t))\tau'(t)$ we get: $0 = Q''(t) = q''(\tau(t))(\tau'(t))^2 + q'(\tau(t))\tau''(t)$. Since $\tau''(t) = y''(t) < 0$,

Figure 5: Optimal mechanism in quadratic-uniform case

(a) Optimal quantities



(b) Optimal targeted types

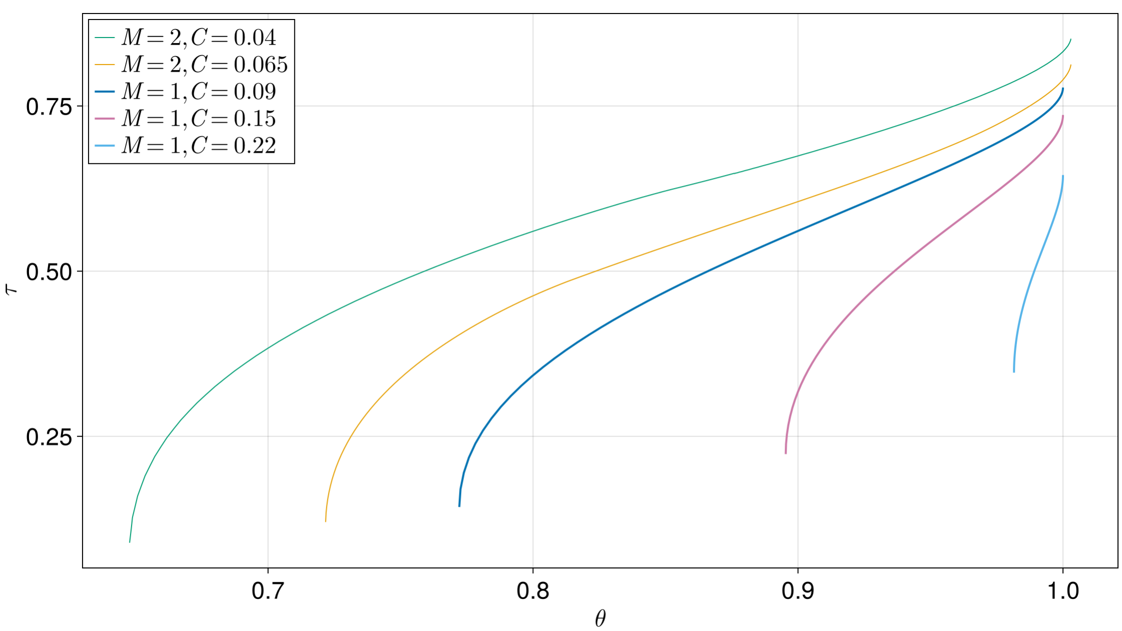
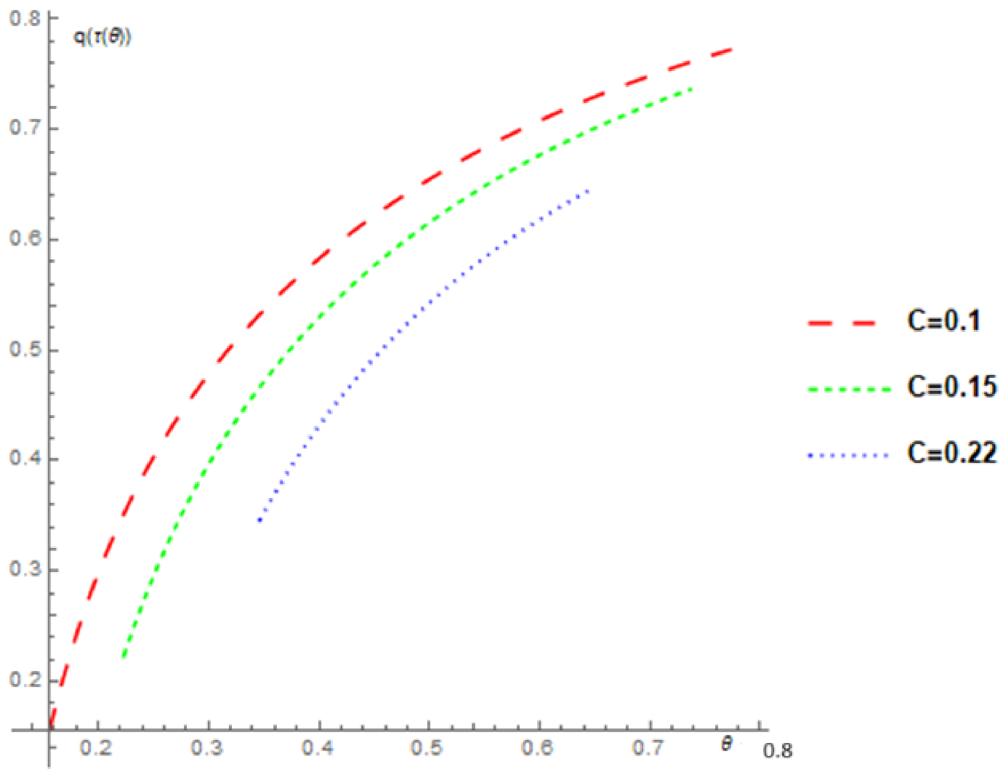


Figure 6: Quantity $q(\tau(\theta))$ of the targeted type $\tau(\theta)$



we conclude that $q''(\theta) \equiv q''(\tau(t)) > 0$ for $\theta \in (\tau(\hat{\theta}), \tau(1))$. So $q(\theta)$ is strictly increasing and convex for $\theta \in (\tau(\hat{\theta}), \tau(1))$. This completes the analysis of the quadratic-uniform case.