Optimal Mechanism with Budget Constrained Buyers*

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Abstract

The paper characterizes the optimal mechanism for selling to buyers who face budget constraints. With unequal budgets, this problem is that of asymmetric optimal mechanism design. The optimal mechanism belongs to one of two classes. When the budget differences are small, the mechanism discriminates only between high-valuation types for whom the budget constraint is binding. All low valuations buyers are treated symmetrically despite budget differences. When budget differences are sufficiently large, the optimal mechanism discriminates in favor of buyers with small budgets when the valuations are low, and in favor of buyers with larger budgets when the valuations are high.

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1 Introduction

This paper deals with the optimal mechanism design when the buyers are budget constrained. The presence of budget constraints and their effect on the buyer behavior in trading mechanisms is common and natural. In particular, bidders in consumer good auctions have limited savings and incomes which may reduce their ability to pay for the goods below their valuations, especially for big-ticket items, such as houses and cars. In the keyword search auctions run by the internet search engines such as Google and Microsoft's Bing the advertisers typically face spending limits set by the senior management. Budget constraints faced by bidders are an important practical matter in spectrum auctions. Rothkopf (2007) provides an example of a spectrum auction in which a bidder valued the asset at $85 million, but was only able to finance a bid of $65 million, and stopped bidding when the price reached this level.

Therefore, it is natural that the economic analysis of trading mechanisms and institutions should take budget constraints into account. There is now a growing literature exploring the implication of budgets constraints in these contexts. With some notable exceptions, discussed below, this literature focuses on the analysis of specific institutions such as different forms of auctions. Here we focus of optimal mechanism design maximizing the seller's revenue.

We study an environment in which a number of buyers compete for a single good and the seller acts as a mechanism designer. The buyers have private values and commonly known and unequal budgets. Importantly, buyers' budget asymmetry means that our problem is that of asymmetric mechanism design, which is significantly more complex than the one where all participants are ex-ante symmetric. In a symmetric situation -such as, for example, when all budgets are equal and the bidders' valuations are drawn from the same distribution- a mechanism designer has to construct a single allocation profile (probability of trading and transfer function) which is offered to every buyer. This affords a significant simplification in the analysis and characterization of the optimal mechanism. Yet, in the asymmetric environment, such as the one we study, the designer has to construct ex-ante asymmetric allocation profiles, one for each buyer, and do so in a consistent way.

There are several real-world environments in which the bidders' budgets are typically known by the seller and other bidders. First, in large-scale privatization auctions of state assets in Eastern Europe and other countries the bidders are/were large corporations whose financial resources were known at least by the government and the competing bidders.[1] Similarly, when governments in various countries sell publicly held stakes in corporations or tracts of natural resources the bidders are typically corporate entities whose budgets can either be inferred from their financial and other reports, or provided by the analysts. Alternatively, the sellers of the high-value assets may and often do require the bidders to qualify by disclosing their assets, liquid and other financial positions. In a somewhat different domain, a number of professional sports leagues in North America such as NHL and NFL have salary caps. So when the teams bid for free agents, their maximal budgets are the available room up to their salary caps, which are publicly known.

The optimal mechanism has a number of interesting and novel qualitative properties. First, it belongs to one of the two classes, depending on the profile of budgets.[2] If the budget differences between the buyers are sufficiently small (in the sense made precise below), the optimal mechanism is a so-called "top-auction." It is characterized by a common threshold value $\bar{x}^t$ at which the budget constraint of each bidder becomes binding. All the buyers with values below $\bar{x}^t$ are treated symmetrically: each of them gets the good when she has the highest value and pays a transfer derived by the standard envelope result.

Any bidder with value exceeding $\bar{x}^t$ pays a transfer equal to her budget, and gets the good with a probability that jumps at x^t but does not change with the bidder's value on $[\bar{x}^t, 1]$. So, all buyers with values exceeding $\bar{x}^t$ are essentially tied. The tie-breaking rule setting the probabilities, with which different bidders with values exceeding $\bar{x}^t$ get the good, plays an important role in this mechanism. In fact, it is the only instrument used by the seller to discriminate between different bidders. Particularly, such probability is higher for a richer bidder to compensate her for the higher payment, equal to her budget, to the seller.

The threshold $\bar{x}^t$ is determined by the sum of individual budgets. In turn, $\bar{x}^t$ determines the reservation value which is lower than in the standard case without budget constraints.

[1] Maskin (2000) deals with the issue of efficiency of such auctions under the assumption of equal and publicly known budgets.

[2] We assume that each bidder's budget is sufficiently small so that it becomes binding at higher values. A sufficient condition for this is that each budget is less than the price set by a seller facing a single bidder.

This happens because the bidders with values above $\bar{x}^t$ pay their budgets, and the seller cannot extract more surplus from them. Therefore, the tradeoff between higher efficiency and leaving greater surplus to the bidders shifts to higher efficiency at lower values.

When the buyers' budgets are sufficiently different, the "top auction" is infeasible because the seller can no longer achieve necessary differentiation between the buyers with different budgets by discriminating only "at the top." In particular, it becomes impossible to allocate the good to the buyers with valuations above an (endogenous) threshold $\bar{x}^t$ in such a way that the budget constraint of every buyer is binding. The optimal mechanism for this case is what we call a "budget-handicap auction" in which the seller uses two kinds of discrimination between the buyers. First, she sets different thresholds for different buyers or groups of buyers. Not surprisingly, richer buyers have higher thresholds. Not all thresholds have to be different: there may be clusters of buyers with the same threshold. But there is more than one threshold across bidders.

Importantly, in the budget-handicap auction the seller also discriminates between buyers with low values. In particular, a poorer bidder with a lower threshold has a lower reservation value than a a richer bidder with a higher threshold. The poorer bidder also gets the good with a higher probability than the richer bidder when they both have the same value below the threshold of the poorer bidder. This handicapping of richer bidders creates more competition for them from poorer ones, and allows the seller to extract more surplus from the former. But it also introduces an additional inefficiency into the mechanism.

The optimal mechanism is unique, and the necessary and sufficient conditions for it are provided in Theorem 2. On the way towards this result, we show that the profile of the bidders' thresholds determines all elements of the optimal mechanism, except for the tie-breaking allocation rule for types above the threshold when this threshold is the same for several bidders. Interestingly, the optimality conditions for a profile of thresholds are essentially the feasibility conditions ensuring consistency between the allocation probabilities at the thresholds and the binding budget constraints at the thresholds. We provide the intuition for these conditions in the discussion following Theorem 2.

Building on this result, Theorem 3 provides the conditions for the optimality of the "top auction." The "budget-handicap" auction is optimal in the complementary case, when these conditions fail. The most challenging part in computing the "budget-handicap" auction is determining the "clusters" of bidders who share the same threshold. This problem does not present analytical difficulties as it only involves checking the conditions of Theorem 2 for a given cluster configuration. However, one may have to go through all such configurations to determine the optimal/feasible one, which is a combinatorial problem that can be solved computationally. We provide an illustration by computing the optimal mechanism with two and three bidders, the latter - under uniform type distribution. The example with three bidders is particularly telling as it shows that every possible configuration of clusters is optimal for a set of budget profiles of a positive measure.

The optimal mechanism can be implemented via an indirect bidding mechanism which combines the features of an all-pay auction and a lottery. Precisely, a bidder is offered a choice between buying a lottery ticket by paying her whole budget, and participating in the all pay-auction. A bidder chooses to buy a lottery ticket if her value is above her respective threshold, and participates in the all-pay auction otherwise. The difference between the top auction and the budget handicap auction is that in the former the all-pay auction is symmetric: a bidder gets the good if her bid is the highest and noone has chosen to buy a lottery ticket. In contrast, in the budget-handicap mechanism the all-pay auction is asymmetric and handicaps richer bidders. So, a bidder gets the good if her bid exceeds the bids of poorer bidders with lower thresholds by a certain margin, and also exceeds the bids lowered by a certain margin from richer bidders with higher thresholds.

A natural question is how the variability of budgets among the bidders affects the seller's profits. This question has a simple answer. The seller (weakly) prefers less budget variability and, with a fixed aggregate budget, she gets the highest expected profits when each bidder has the same budget (Lemma 10). However, the seller's revenue does not change after sufficiently small redistributions of the aggregate budget between the bidders after which the profile of thresholds remains unchanged. In particular, this means that the seller gets the same expected revenue under two budget profiles with the same aggregate value, if the top auction is an optimal mechanism under both profiles.

Technically, our paper contains a number of interesting aspects. Among them - the equivalence between the optimality and feasibility conditions for the mechanism. Another interesting aspect is the uncovered strong connection between the threshold values at which budget constraints become binding and the Lagrange multipliers associated with budget constraints. Not only there is a one-to-one relationship between them, as demonstrated by Theorem 1, but also the strong duality property between them ultimately allows us to derive the optimal mechanism.

Finally, although we focus on the case in which all budget constraints are binding in the optimal mechanism, our results also apply to the case where some bidders have higher budget that are never binding.

In the related literature, the paper closest to ours is Laffont and Robert (1996) who consider a similar environment with commonly known but equal budgets among the bidders. They derive an optimal mechanism which is a special case of our top auction. Their optimal mechanism is symmetric and does not allow to understand what the seller should do when the buyers have different budgets, and hence are asymmetric from the ex-ante point of view. Maskin (2000) studies efficient mechanism design in the same environment as Laffont and Robert (1996). A surprising result of our analysis is that the bidders' thresholds remain equal when budget differences are sufficiently small. So, our "top auction" provides a generalization of the optimal auction of Laffont and Robert (1996) to an environment with small budget asymmetry. However, a qualitatively different mechanism - "budget-handicap auction"- becomes optimal when budget differences are large.

Malakhov and Vohra (2008) derive optimal dominant strategy mechanism for two buyers with values distributed over a discrete support, one of whom faces no budget constraint and the other has a known fixed budget. Their mechanism is similar to the one that we derive in the extension of our example with two bidders one of whom has a small budget and the budget of the other is larger than the "monopoly" price with a single buyer. Pai and Vohra (2014) study an optimal mechanism with private budgets and values identically distributed across the bidders. In their work, the budgets and valuations have a finite support, with a continuous distribution considered in an extension. They provide a significant contribution to multidimensional mechanism design showing how one can work directly with reduced form auctions. An extension of their paper considers bidders with equal and public budgets.

Thus, an important difference between our paper and Laffont and Robert (1996) and Pai and Vohra (2014) is that we allow for any commonly known budget profile, while these authors consider bidders with equal budgets and solve a symmetric mechanism design problem. In a symmetric mechanism design one has to derive a single allocation profile (probabilities of trading and transfers) which is offered to each buyer. In contrast, in our asymmetric mechanism design problem all allocation profiles are different, and the mechanism designer has to ensure that they are consistent with each other.

Discrimination by the seller between ex-ante asymmetric bidders has been studied in the literature in the case when buyers have ex-ante different distributions of valuations. Myerson (1981) shows that the optimal auction should handicap "stronger" bidders whose valuation are more likely to be high. More recently Jehiel and Lamy (2015) have considered the optimality of such discrimination when the buyers have to make costly entry decisions. They have shown that such discrimination is suboptimal if costly entry precedes buyers learning their valuation. However, "incumbent" bidders who do not face entry costs should be handicapped. In our model, the bidders asymmetry comes from another source- the budget differences. When these differences are sufficiently large, an asymmetry of virtual valuations arises endogenously in our optimal mechanism and leads to handicapping of richer bidders, even though the bidders valuations are distributed identically.

In the earlier literature on auctions with budget constraints, Che and Gale (1998) compare the performance of first- and second-price auctions when the buyers have privately known budgets and values. They show that the first-price auction yields higher expected social surplus and expected revenue. Che and Gale (1996) show that the all-pay auction performs better than the first-price auction under common value and private budgets. Che and Gale (2000) explore optimal nonlinear pricing for a buyer with privately known value and budget. Zheng (2001) studies the first-price auction in which budget-constrained buyers can bid above their budgets. In case of a win such buyer can either use costly financing to cover the deficit, or default and lose her budget if her valuation for the good turns out to be less than her bid. Hafalir, Ravi and Sayedi (2012) focus on a Vickrey auction with budget-constrained bidders. In their framework, the bidders have different and essentially known budgets. Although their mechanism is not optimal, it is "close" to a Pareto efficient mechanism.

Borgs et. al (2005) and Dobzinski, Lavi and Nisan (2012) are concerned with dominant strategy mechanisms for allocating multiple goods. Both papers establish impossibility results under private budgets, the latter- for Pareto optimal allocation, the former- for allocation satisfying other properties that might be desirable. Borgs et. al (2005) then provide an auction that asymptotically (as maximal budgets becomes large) attains the same revenue as the posted price auction. Dobzinski, Lavi and Nisan (2012) demonstrate that with public budgets, a Pareto optimal allocation can be attained by using Ausubel's clinching auction. In contrast to Dobzinski, Lavi and Nisan (2012), Baisa (2015) demonstrates that clinching auction is a Pareto efficient mechanism under budget constraints when the bidders' beliefs satisfy full support assumption.

Importantly, Pareto optimality is inconsistent with the goal of revenue maximization pursued in this paper, and a revenue maximizing seller would not offer a Pareto optimal mechanism. In particular, handicapping a richer bidder, as in the budget-handicap auction, and allocating the good randomly between the bidders with values above the common threshold, as in the top auction, can not occur in a Pareto optimal mechanism.

Che, Gale and Kim (2013a) and (2013b) and Richter (2016) study revenue-maximizing and welfare-maximizing assignment of a divisible good to a continuum of budget-constrained agents. The nature of the problem studied by these authors is very different from that of our problem. In particular, as discussed in Richter (2016), his model can be reinterpreted as a single-agent problem in which budget and supply must be balanced on average, and transfers between types of this single agent are permitted.

The rest of the paper is organized as follows. Section 2 develops the model and preliminary results. Section 3 presents a two-bidder example. Section 3 contains the analysis. Section 4 presents the main results, including the optimal mechanisms and their qualitative properties. Section 5 contains additional examples, including the case of three bidders. Section 6 concludes. The proofs are relegated to the Appendix.

2 Model and Preliminaries

A seller with one unit of the good faces n bidders. Bidder $i \in \{1, ..., n\}$ has privately known value x_i for the good drawn from a common knowledge distribution $F(.)$, which possesses a continuous positive density function $f(.)$.[3] Without loss of generality, we assume that the support of $F(.)$ is $[0,1]$.

Bidder i has budget m_i which she can never exceed. The budgets are commonly known and will be assumed to be sufficiently small. A sufficient condition for all budget constraints to be binding in the optimal mechanism is $\max_i m_i \leq \arg\max p(1-F(p))$ i.e., the highest budget is below the price set by a seller facing a single buyer without a budget constraint. With multiple bidders, competition causes a bidder's budget constraint to be binding even if her budget exceeds this threshold. Furthermore, our results easily generalize to the situation when only some bidders have binding budget constraints.

We will impose a standard assumption on the distribution $F(.)$:

Assumption 1 *Increasing Hazard rate:*

$$\frac{f(x)}{1-F(x)} \quad \textit{is increasing in} \quad x \quad \textit{for all} \quad x \in [0,1] \tag{1}$$

In fact, a weaker assumption that $x - \frac{1-F(x)}{f(x)}$ is increasing is sufficient, and we make the increasing hazard rate assumption mainly for the sake of conformity with the literature.[4]

Bidder i with valuation x_i gets a payoff equal to $x_i q_i - t_i$ if she gets the good with probability q_i and makes a payment t_i to the seller. The seller has zero value for the good, so her payoff is simply the sum of the payments that she receives from the buyers, $\sum_{i=1,..,n} t_i$. All the bidders and the seller are risk-neutral and wish to maximize their expected payoffs.

The seller has all bargaining power and acts as a mechanism designer. Thus, we focus on the optimal mechanism maximizing the expected revenue of the seller.

By the Revelation principle (Myerson 1979) we can restrict attention to direct truthful mechanisms which specify the probabilities of trading $(Q_1(.), ..., Q_n(.))$ and the payments

[3] Although it is feasible to extend our analysis to the case where each bidder's valuation is drawn from a different probability distribution $F_i(.)$, we focus on the symmetric distribution case in order to highlight the consequences of the budget differences between the bidders.

[4] Pai and Vohra (2014) suggest that a stronger assumption that $f(x)$ is nonincreasing is necessary in the setting with budget constraints because bidder i's virtual value is $x - \frac{1-F(x)-\lambda_i}{f(x)}$ on the interval of x adjacent to zero, where λ_i is a Lagrange multiplier associated with i's budget constraint. However, as we show below, in the optimal mechanism $\lambda_i \leq 1-F(x)$ on the appropriate interval of x. Therefore, the monotonicity of $x - \frac{1-F(x)}{f(x)}$ guarantees the monotonicity of $x - \frac{1-F(x)-\lambda_i}{f(x)}$.

$(T_1(.), ..., T_n(.))$ for the bidders as the functions of their announced valuations $(\hat{x}_1, ..., \hat{x}_n)$. Specifically, $Q_i(\hat{x}_1, ..., \hat{x}_n)$ is the probability that the bidder i gets the good and $T_i(\hat{x}_1, ..., \hat{x}_n)$ is the transfer that she pays to the seller.

Further, $q_i(x_i) = \int_{x_{-i}\in[0,1]^{n-1}} Q_i(x_i, x_{-i}) \prod_{j\neq i} dF(x_j)$ and $t_i(x_i) = \int_{x_{-i}\in[0,1]^{n-1}} T_i(x_i, x_{-i}) \prod_{j\neq i} dF(x_j)$ are the expected probability that bidder i gets the good and her expected payment, respectively, when she announces type x_i and all other bidders announce their types truthfully.

The optimal mechanism $(Q_1(.), ..., Q_n(.), T_1(.), ..., T_n(.))$ solves the revenue maximization problem of the seller:

$$\max \sum_{i=1,...,n} \int_{(x_1,...,x_n)\in[0,1]^n} T_i(x_1, ..., x_n) \prod_{i=1,...,n} dF(x_i) \tag{2}$$

subject to the following:

(i) *interim incentive constraints:*

$$x_i q_i(x_i) - t_i(x_i) \geq x_i q_i(\hat{x}_i) - t_i(\hat{x}_i), \quad \text{for all } (x_i, \hat{x}_i) \in [0,1]^2 \text{ and all } i \in \{1, ..., n\}. \tag{3}$$

(ii) *individual rationality constraints:*

$$x_i q_i(x_i) - t_i(x_i) \geq 0 \quad \text{for all } i \quad \text{and} \quad x_i \in [0,1]. \tag{4}$$

(iii) *budget constraints:*

$$T_i(x_i, x_{-i}) \leq m_i \quad \text{for all } i,\ x_i \in [0,1],\ x_{-i} \in [0,1]^{n-1}. \tag{5}$$

(iv) *feasibility constraints*:

$$\sum_i Q_i(x_1, ..., x_n) \leq 1 \text{ and } Q_i(x_1, ..., x_n) \geq 0 \text{ for all } (x_1, ..., x_n) \in [0,1]^n. \tag{6}$$

3 Example

To illustrate our results we first present the optimal mechanism for two bidders, 1 and 2, with budgets m_1 and m_2, respectively, satisfying $m_1 \geq m_2$ without loss of generality.

If the budgets are sufficiently small[5] and close to each other then the optimal mechanism

[5]The condition $m_1 \leq \arg\max_p p(1 - F(p))$ is sufficient for both bidders' budgets to bind in the optimal mechanism. It says that bidder 1's budget is smaller than the seller's optimal price when she faces a single bidder. However, the following weaker condition is necessary and sufficient for both budget constraints to be binding when the conditions for the top auction are met: $m_1 < 1 - \int_{r':r'=\frac{1-F(r')}{f(r')}}^{1} F(x)dx$.

is a "top auction" defined by four parameters: reservation value r^t; threshold value $\bar{x}^t$; and expected probabilities of trading "at the top," $q_1(\bar{x}^t)$ and $q_2(\bar{x}^t)$ (see Theorem 3 for details).

In the "top auction," the budget constraint of either bidder is not binding when her value is below $\bar{x}^t$. Despite budget asymmetry, for bidder i with value in $[r^t, \bar{x}^t)$ the top auction looks exactly like a standard symmetric auction: she gets the good when her competitor has a lower value, although the reservation value r^t is lower than without budget constraints.

At $\bar{x}^t$ each bidder's budget constraint becomes binding. So bidder i with value in $[\bar{x}^t, 1]$ pays a transfer equal to her budget and gets the good with probability $q_i(\bar{x}^t)$. Naturally, a richer bidder has a higher probability of trading at the top i.e., $q_1(\bar{x}^t) > q_2(\bar{x}^t)$. In fact, both $q_1(x)$ and $q_2(x)$ jump upwards at $x = \bar{x}^t$, except in the borderline parameter case in which only $q_1(x)$ jumps to 1 and $q_2(x)$ remains continuous. Thus, the top auction discriminates between the buyers only in the upper range of values $[\bar{x}^t, 1]$.

Since the net payoff of a bidder with value $x_i \in [\bar{x}^t, 1]$ is equal to $q_i(\bar{x}^t)x_i - m_i = q_i(\bar{x}^t)(x_i - \bar{x}^t) + \int_{r^t}^{\bar{x}^t} F(x)dx$, a high-value bidder with a higher budget gets a higher payoff than the bidder with the same value but a lower budget.

The threshold value $\bar{x}^t$ is found from the aggregate budget constraint (see Theorem 3):

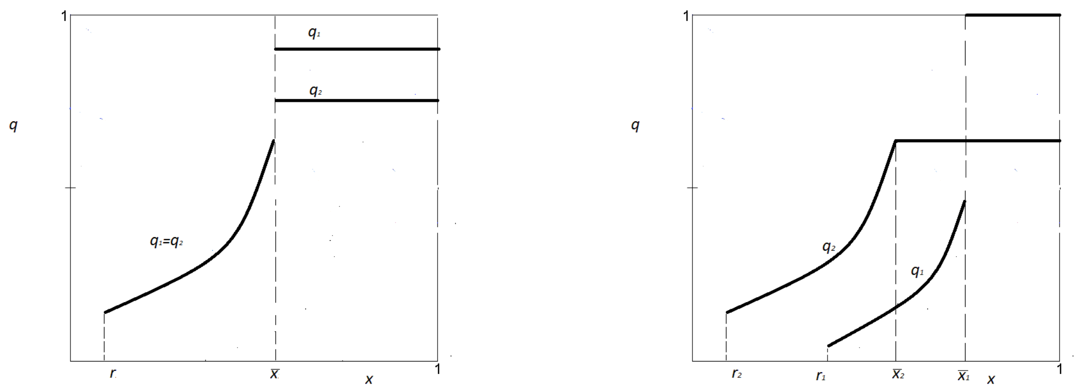
$$m_1 + m_2 = \bar{x}^t(1 + F(\bar{x}^t)) - 2\int_{r^t}^{\bar{x}_t} F(x)dx$$

where $(1 + F(\bar{x}^t))$ is the maximal feasible value of $q_1(\bar{x}^t) + q_2(\bar{x}^t)$, and r^t is the level at which a bidder's virtual value in this mechanism is zero. Precisely, $r^t = \frac{1-F(r^t)-\frac{(1-F(\bar{x}^t))^2}{1-F(\bar{x}^t)+\bar{x}^t f(\bar{x}^t)}}{f(r^t)}$.

The top auction is not always feasible. It has to satisfy $q_1(\bar{x}^t) \leq 1$ and $q_2(\bar{x}^t) \geq F(\bar{x}^t)$. These conditions together with binding budget constraints at $\bar{x}^t$ i.e., $m_i = q_i(\bar{x}^t)\bar{x}^t - \int_{r^t}^{\bar{x}^t} F(x)dx$, imply that $m_1 - m_2 \leq \bar{x}^t(1 - F(\bar{x}^t))$. If this inequality holds, the top auction is the optimal mechanism. If it fails, the top auction is infeasible. Instead, the threshold $\bar{x}_1$ at which the budget constraint of the richer bidder 1 becomes binding has to be greater than the corresponding threshold $\bar{x}_2$ of the poorer bidder 2.

This has a number of consequences. First, bidder 1's probability of trading jumps to 1 at $\bar{x}_1$, while bidder 2's probability of trading is continuous at $\bar{x}_2$ reaching its maximal value $F(\bar{x}_1)$ at this point. Significantly, the bidders no longer face a symmetric auction at lower values. Instead, richer bidder 1 is handicapped. She faces a higher reservation value i.e.,

Figure 1: Expected Probabilities of Trading with Two Bidders



(a) Top Auction

(b) Budget-Handicap Auction

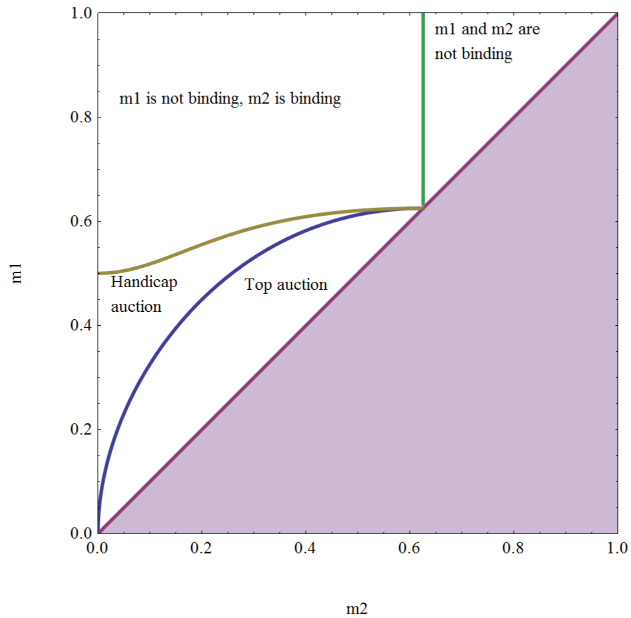
$r_1 > r_2$. Also, bidder 1 with value $x \in [r_1, \bar{x}_1)$ gets the good with a lower probability than bidder 2 with the same value. Because of this, we refer to this mechanism as a "budget-handicap" auction (Theorem 4). The handicapping of bidder 1 generates more competition for her from bidder 2, and allows the seller to extract more surplus from bidder 1.

Naturally, for some values of the budgets neither budget constraint or only bidder 2's budget constraint is binding. The condition for the former is simple: the poorer bidder 2 with valuation 1 must not be budget constrained in a standard symmetric auction i.e., $m_2 \geq 1 - \int_{r^s}^{1} F(x)dx$ with r^s satisfying $r^s - \frac{1-F(r^s)}{f(r^s)} = 0$.

Finally, only bidder 2's budget constraint is binding if m_2 fails the previous condition, while m_1 is sufficiently large i.e., $m_1 \geq 1 - \int_{r^s}^{1} q_1^h(x)dx$ where $q_1^h(x)$ is the handicapped probability of trading of bidder 1 which is less than $F(x)$ and depends on the budget of buyer 2. In particular, m_1 must be above the "monopoly" price $p^m = \arg\max_p p(1 - F(p))$. In this case, the optimal mechanism closely resembles the "budget-handicap" auction. The only difference is that bidder 1 with value above her optimally chosen threshold $\bar{x}_1$ gets the good with probability 1 but pays less than her budget.

The implementation of the "top-auction" and "budget-handicap auction" via an indirect mechanism that combines an all-pay auction with a lottery is discussed in Section 5 for the case of n bidders. Here it is worth noting that with two bidders, the implementation of

Figure 2: The Optimal Mechanism and Bidders' Budgets.



the budget-handicap auction includes only an all-pay auction for the poorer bidder 2. In contrast, richer bidder 1 is offered a choice between the all-pay-auction and a "buy-it-now" option: she can get the good for sure by paying his budget.

The expected probabilities of trading in the "top auction" and "budget-handicap auction" are depicted in Figure 1. Figure 2 summarizes how the nature of the optimal mechanism depends on the budgets m_1 and m_2 when the bidders' types are distributed uniformly.

4 Analysis

Our first result establishes the existence and uniqueness of the optimal mechanism.

Lemma 1 *There exists an optimal mechanism* $(Q_1(.), ..., Q_n(.), T_1(.), ..., T_n(.))$ *that solves the problem (2) subject to the constraints (3)-(6).*

Proof of Lemma 1: The objective of the maximization problem (2) is a continuous linear functional in the Hilbert space $L^2([0,1]^n)$. Its admissible set specified by constraints (3)-(6) is convex. Therefore, by Theorem 2.6.1 in Balakrishnan (1993) the solution to our problem exists. The uniqueness almost everywhere follows by standard arguments, in particular, the linearity of the objective and the convexity of the constraints. *Q.E.D.*

Let $U_i(x_i) \equiv q_i(x_i)x_i - t_i(x_i)$ be the net expected payoff of buyer i of type x_i in the optimal mechanism. The following result is standard and is left without proof:

Lemma 2 *A mechanism $(Q_1(.), ..., Q_n(.), T_1(.), ..., T_n(.))$ is incentive compatible and individually rational if any only if the expected trading probability $q_i(x_i)$ is nondecreasing in x_i for all i and $x_i \in [0, 1]$, and:*

$$U_i(x_i) = \int_0^{x_i} q_i(s)ds + c_i \;\; \textit{for some} \;\; c_i \in \mathbb{R}_+ \tag{7}$$

The individual rationality requires the constant c_i to be nonnegative. The optimality then implies that $c_i = 0$. Given this equality, we drop c_i altogether from the analysis.

Combining $U_i(x_i) = x_i q_i(x_i) - t_i(x_i)$ with (7) yields the following expression:

$$t_i(x_i) = x_i q_i(x_i) - \int_0^{x_i} q_i(s)ds \tag{8}$$

Consider now the budget constraints. First, we can replace the ex-post budget constraint in (5) with the interim one, $t_i(x_i) \le m_i$ for all i and x_i. Indeed, the interim budget constraints obviously hold when (5) holds. In the opposite directions, if $t_i(x_i) \le m_i$ for all i and x_i, then (5) holds if we set $T_i(x_i, x_{-i}) = t_i(x_i)$ for all i and x_i. Doing so does not affect the seller's objective, the incentive or individual rationality constraints, since these depend only on the expected transfers $t_i(.)$, but it can potentially relax the budget constraint in some states of the world since the maximal payment by bidder i weakly decreases.

Next, let us establish the following useful result:

Lemma 3 *Let $\bar{x}_i \in [0, 1]$ be defined as follows:*

$$\bar{x}_i = \sup\{x_i \in [0, 1] | t_i(x_i) < m_i\} \tag{9}$$

If $\bar{x}_i < 1$, then $t_i(x_i) = m_i$ for all $x_i \in [\bar{x}_i, 1]$

Proof of Lemma 3: Since by Lemma 2 $q_i(x_i)$ must be increasing in x_i, the expected transfer $t_i(x_i)$ must also be increasing in x_i, for otherwise the mechanism will not be incentive compatible. Therefore, if $t_i(x_i) = m_i$, then $t_i(x'_i) = m_i$ for all $x'_i \in [x_i, 1]$. *Q.E.D.*

The threshold values $\bar{x}_i$ play an important role as the key choice variables which ultimately determine the whole mechanism. Lemma 3 and equation (8) imply that whenever

$\bar{x}_i < 1$, the budget constraint $t_i(x_i) \le m_i$ can be replaced with the following two conditions:

$$m_i = \bar{x}_i q_i(\bar{x}_i) - \int_0^{\bar{x}_i} q_i(s)ds \tag{10}$$

$$q_i(x) = q_i(\bar{x}_i) \text{ for all } x_i \in [\bar{x}_i, 1] \tag{11}$$

Let us now substitute the expression (8) for the transfers into the objective (2). Then using (11) and integrating by parts yields the modified objective:

$$\sum_{i=1}^{n} \int_0^1 t_i(x_i)dF(x_i) = \sum_{i=1}^{n} \int_0^1 \left(q_i(x_i)x_i - \int_0^{x_i} q_i(x)dx \right) dF(x_i)$$
$$= \sum_{i=1}^{n} \int_0^{\bar{x}_i} q_i(x_i) \left(x_i - \frac{1 - F(x_i)}{f(x_i)} \right) dF(x_i) + \sum_{i=1}^{n} \int_{\bar{x}_i}^1 q_i(\bar{x}_i)\bar{x}_i dF(x_i) \tag{12}$$

By Lemma 2, the incentive constraints (3) and the individual rationality constraints (4) can now be replaced with the condition that $q_i(x_i)$ is increasing in x_i for all i. Following standard approach, we will consider a relaxed program omitting the latter condition. We will also replace condition (9) with the condition (10) that the budget constraint is binding for type $\bar{x}_i$. Since we have imposed constraint (11) on the objective explicitly, this will be sufficient to guarantee that the budget constraint is binding for all types in $[\bar{x}_i, 1]$.

After solving the relaxed program, we will check that our solution is such that $q_i(.)$ is increasing, strictly at $\bar{x}_i$ from the left. The latter guarantees that (9) holds i.e., $\bar{x}_i$ is the lowest type for whom the budget constraint is binding. This would imply that the solution to the relaxed problem does, in fact, solve the full unrelaxed problem. Finally, we will take care of the feasibility constraints (6) by imposing them directly on the probabilities of trading.

Imposing constraint (10) on the objective (12) yields the relaxed program Lagrangian:

$$\mathcal{L}(Q, \bar{x}, \lambda) = \sum_{i=1}^{n} \int_0^{\bar{x}_i} q_i(x_i) \left(x_i - \frac{1 - F(x_i)}{f(x_i)} \right) dF(x_i) + \int_{\bar{x}_i}^{1} q_i(\bar{x}_i)\bar{x}_i dF(x_i) - \lambda_i \left(q_i(\bar{x}_i)\bar{x}_i - \int_0^{\bar{x}_i} q_i(x)dx - m_i \right)$$
$$= \sum_{i=1}^{n} \left(\int_0^{\bar{x}_i} q_i(x_i) \left(x_i - \frac{1 - \lambda_i - F(x_i)}{f(x_i)} \right) dF(x_i) + \int_{\bar{x}_i}^{1} q_i(\bar{x}_i) \left(\bar{x}_i - \frac{\lambda_i \bar{x}_i}{1 - F(\bar{x}_i)} \right) dF(x_i) + \lambda_i m_i \right) \tag{13}$$

where λ_i is a Lagrange multiplier associated with bidder i's budget constraint (10).

Next, using $q_i(x_i) = \int_{x_{-i}\in[0,1]^{n-1}} Q(x_i, x_{-i}) \prod_{j\neq i} dF(x_j)$ and changing the order of summation and integration in (13) we can rewrite it as follows:

$$\mathcal{L}(Q, \bar{x}, \lambda) = \int_{(x_1,...,x_n)\in[0,1]^n} \sum_{i=1}^{n} Q_i(x_1, ..., x_n)\gamma_i(x_i) \prod_{i=1,...,n} dF(x_i) + \sum_{i=1}^{n} \lambda_i m_i. \qquad (14)$$

where $\gamma_i(x_i)$ is defined as follows for $i \in \{1, ..., n\}$:

$$\gamma_i(x_i) = \begin{cases} x_i - \frac{1-\lambda_i - F(x_i)}{f(x_i)}, & \text{if} \quad x_i < \overline{x}_i, \\ \overline{x}_i - \frac{\lambda_i \overline{x}_i}{1-F(\overline{x}_i)}, & \text{if} \quad x_i \geq \overline{x}_i. \end{cases} \qquad (15)$$

As one can see from (14), $\gamma_i(.)$ plays the role of the virtual value of player i. Recall that without budget constraints, bidder i's virtual value is $x_i - \frac{1-F(x_i)}{f(x_i)}$. So budget constraints cause the virtual value of type $x_i \in [0, \overline{x}_i)$ to increase by an amount proportional to the value of the Lagrange multiplier. Intuitively, this happens because all types above $\bar{x}_i$ pay their whole budget and hence the seller cannot extract more surplus from them. Therefore, increasing the probability with which lower types get the good depresses the seller's profits by less than without budget constraints. Further, all types in $[\overline{x}_i, 1]$ get the same allocation, and every type in this endogenous "atom" has the same virtual value, $\overline{x}_i - \frac{\lambda_i \overline{x}_i}{1-F(\overline{x}_i)}$.

Note that if the budget constraint of bidder i is non-binding, then $\overline{x}_i = 1$ and $\lambda_i = 0$, and so according to (15) we recover the standard formula for the virtual value. Hence, our analysis also applies to the case when only some bidders have binding budget constraints.

Inspection of (14) yields the following Lemma:

Lemma 4 *For any bidder $i \in \{1, 2, ..., n\}$ and any $(x_i, x_{-i}) \in [0, \bar{x}_i]^n$, the optimal $Q_i(.)$ is such that:*

1. $Q_i(x_i, x_{-i}) = 1$ *if* $\gamma_i(x_i) > \max\{0, \max_{j\neq i} \gamma_j(x_j)\}$;

2. $Q_i(x_i, x_{-i}) \in [0, 1]$ *if* $\gamma_i(x_i) = \max\{0, \max_{j\neq i} \gamma_j(x_j)\}$;

3. $Q_i(x_i, x_{-i}) = 0$ *if* $\gamma_i(x_i) < \max\{0, \max_{j\neq i} \gamma_j(x_j)\}$.

4. $\sum_{i=1}^{n} Q_i(x_1, ..., x_n) = 1$ *if* $\min_i \gamma_i(x_i) > 0$.

According to this Lemma, the profile of virtual values $(\gamma_1(x_1), ..., \gamma_n(x_n))$ determines the trading probabilities $(Q_i(x), ..., Q_n(x))$ uniquely except in the case of ties, when two or more bidders have the highest virtual value. The ties that have zero probability can be ignored, as the designer can resolve them arbitrarily without affecting her expected profits. In particular, this applies to ties that involve a bidder type $x_i \in [0, \bar{x}_i)$.

However, all bidder types in $[\bar{x}_i, 1]$ have the same virtual value $\gamma_i(\bar{x}_i)$ and essentially constitute an atom of probability $1 - F(\bar{x}_i)$. If $\gamma_i(\bar{x}_i) = \gamma_j(\bar{x}_j)$ for some $j \neq i$, then every bidder type in $[\bar{x}_i, 1]$ is tied with every bidder type in $[\bar{x}_j, 1]$. Such tie has a positive probability. As we will see below, under certain conditions there will, in fact, exist clusters of bidders with the same threshold $\bar{x}$ and the same virtual value $\gamma(\bar{x})$, even if they have unequal budgets.[6] Tie-breaking rules between bidders in a cluster are an important part of our optimal design.

Significantly, Lemma 4 implies that in the optimal mechanism $\sum_{i=1}^{n} Q_i(x_1, ..., x_n)\gamma_i(x_i) = \max\{0, \max_i \gamma_i(x_i)\}$ for all $x = (x_1, ..., x_n) \in [0, 1]^n$. Therefore, we can replace the Lagrangian (14) with the following one that depends only on the profiles $(\bar{x}_1, ..., \bar{x}_n)$ and $(\lambda_1, ..., \lambda_n)$:

$$\mathcal{L}(\bar{x}, \lambda) = \max_{Q:\, 0 \leq Q_i(x) \leq 1,\, \sum_i Q_i(x) \leq 1} \mathcal{L}(Q, \overline{x}, \lambda) = \int_{x \in [0,1]^n} \max\{0, \max_{i=1,...,n} \gamma_i(x_i)\} \prod_i dF(x_i) + \sum_{i=1}^{n} \lambda_i m_i. \tag{16}$$

Now we are in a position to prove the following important result. To do this, define:

$$\gamma_i^-(\bar{x}_i) \equiv \lim_{x_i \uparrow \bar{x}_i} \gamma_i(x_i) = \bar{x}_i - \frac{1 - \lambda_i - F(\bar{x}_i)}{f(\bar{x}_i)}. \tag{17}$$

Then we have:

Theorem 1 *1. For all $i \in \{1, ..., n\}$ s.t. $\bar{x}_i \leq \bar{x}_j$ for some $j \neq i$, $\gamma_i(x_i)$ is continuous at $x_i = \bar{x}_i$ or, equivalently, $\lambda_i = \frac{(1-F(\bar{x}_i))^2}{(1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i))}$.*

2. For bidder $\hat{i}$ such that $\bar{x}_{\hat{i}} > \bar{x}_j$ for all $j \neq \hat{i}$, we have: $\gamma_{\hat{i}}(\bar{x}_{\hat{i}}) > \gamma_{\hat{i}}^-(\bar{x}_{\hat{i}}) = \max_{j \neq \hat{i}} \gamma_j(\bar{x}_j)$ or, equivalently,

$$\lambda_{\hat{i}} = 1 - F(\overline{x}_{\hat{i}}) - f(\overline{x}_{\hat{i}}) \left(\overline{x}_{\hat{i}} - \max_{j \neq \hat{i}} \gamma_j(\bar{x}_j) \right).$$

[6] A cluster of bidders at threshold $\bar{x}^C$ is defined as $C(\bar{x}^C) \equiv \{i | \bar{x}_i = \bar{x}^C\}$.

Although the proof of Theorem 1 is fairly intricate, it relies on an intuitive observation: if some bidder i's virtual value is not continuous at her threshold $\bar{x}_i$, then the seller can attain a higher payoff by modifying $\bar{x}_i$ slightly.

Importantly, this Theorem shows that the profile $(\bar{x}_1, ..., \bar{x}_n)$ uniquely determines the profile $(\lambda_1, ..., \lambda_n)$, and so the number of choice variable in our problem can be reduced from $2n$ to n. Significantly, the relationship between the profiles $(\bar{x}_1, ..., \bar{x}_n)$ and $(\lambda_1, ..., \lambda_n)$ is 1-to-1: we will confirm below that the profile $(\lambda_1, ..., \lambda_n)$ also uniquely determines $(\bar{x}_1, ..., \bar{x}_n)$. Using this, we will solve our problem by choosing the optimal profile $(\lambda_1, ..., \lambda_n)$.

Theorem 1 is consistent both with binding and non-binding budget constraints of any player i. Particularly, if the budget constraint of bidder i is non-binding, then $\lambda_i = 0$. In this case, we either have $\bar{x}_i = 1$ or, if the only bidder whose budget constraint is non-binding is $\hat{i}$, then $\bar{x}_{\hat{i}} - \frac{1-F(\bar{x}_{\hat{i}})}{f(\bar{x}_{\hat{i}})} = \max_{j \neq \hat{i}} \frac{\bar{x}_j^2 f(\bar{x}_j)}{1-F(\bar{x}_j)+\bar{x}_j f(\bar{x}_j)}$, and bidder $\hat{i}$ of type above $\bar{x}_{\hat{i}}$ gets the good with probability 1 and pays a transfer which is below her budget.

Using Theorem 1 we can show the following:

Lemma 5 *The solution to the relaxed problem (16) is such that $\gamma_i(x_i)$ and $q_i(x_i)$ are strictly increasing on $[0, \bar{x}_i]$ for all i.*

Therefore, the solution to the relaxed program also solves the full unrelaxed program.

We can now establish the following intuitive relationship between the budgets and thresholds:

Lemma 6 *If $m_i > m_j$ for some $i, j \in \{1, ..., n\}$, then $\bar{x}_i \geq \bar{x}_j$.*

An immediate implication of this Lemma is that bidder $\hat{i}$ with the highest threshold $\bar{x}_{\hat{i}}$ is, in fact, the highest-budget bidder 1.

Finally, let us provide explicit expressions for the expected trading probabilities $q_i(x_i) = \int_{x_{-i} \in [0,1]^{n-1}} Q_i(x_i, x_{-i}) \prod_{j \neq i} dF(x_j)$ and the "reservation values" $r_i = \inf\{x_i \in [0,1] | q_i(x_i) > 0\}$. This characterization is provided in two Lemmas below and relies on Lemmas 4 and 5:

Lemma 7 *The expected probability of trading, $q_i(x_i)$, satisfies:*

$$\int_{x_{-i} \in [0,1]^{n-1} : \gamma_i(x_i) > \max\{0, \max_{j \neq i} \gamma_j(x_j)\}} \prod_{j \neq i} dF(x_j) \leq q_i(x_i) \leq \int_{x_{-i} \in [0,1]^{n-1} : \gamma_i(x_i) \geq \max\{0, \max_{j \neq i} \gamma_j(x_j)\}} \prod_{j \neq i} dF(x_j) \tag{18}$$

Both inequalities in (18) hold as equalities for almost all $x_i \in [0, \bar{x}_i)$ and for $x_i = \bar{x}_i$ if $\bar{x}_i \neq \bar{x}_j$ for all j, $j \neq i$. Hence, such $q_i(x_i)$ are uniquely defined by the vector of the thresholds $(\bar{x}_1, ..., \bar{x}_n)$.

Lemma 8 *The reservation value r_i for $i \in \{2, ..., n\}$ and $i = 1$ when $\bar{x}_1 = \bar{x}_2$ is uniquely determined by $\bar{x}_i$ and is increasing in it. It satisfies:*

$$r_i = \frac{1 - F(r_i) - \frac{(1-F(\bar{x}_i))^2}{1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i)}}{f(r_i)}. \tag{19}$$

When $\bar{x}_1 > \bar{x}_2$, the reservation value r_1 is uniquely determined by $\bar{x}_1$ and $\bar{x}_2$ and satisfies:

$$r_1 - \frac{F(\bar{x}_1) + f(\bar{x}_1)\left(\bar{x}_1 - \bar{x}_2 + \frac{\bar{x}_2(1-F(\bar{x}_2))}{1-F(\bar{x}_2)+\bar{x}_2 f(\bar{x}_2)}\right) - F(r_1)}{f(r_1)} = 0. \tag{20}$$

Lemma 8 implies that the bidders' reservation values in our mechanism are lower than the reservation value r in the standard case without budget constraints, which satisfies $r - \frac{1-F(r)}{f(r)} = 0$. This happens because under budget constraints the tradeoff between higher efficiency at lower values and leaving greater surplus to the bidders with higher values shifts towards higher efficiency, since the bidders with values above their respective thresholds pay their whole budgets and the seller cannot extract more surplus from them.

To complete the derivation of the optimal mechanism and solve (16) we will use the Lagrangian duality theory (see e.g. Boyd and Vandenberghe (2009) and Bertsekas (2001)). To this end, note that Theorem 1 implies that the profile of Lagrange multipliers $(\lambda_1, ..., \lambda_n)$ uniquely determines the profile $(\bar{x}_1, ..., \bar{x}_n)$. Indeed, let $\bar{x}_i(\lambda_i)$ be the solution to $\lambda_i = \frac{(1-F(\bar{x}_i))^2}{(1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i))}$ for $i \geq 2$, which is well-defined since the right-hand side of this expression is monotone decreasing in $\bar{x}_i$. Also, let $\bar{x}_1(\lambda_1, \lambda_2)$ be the solution for $\bar{x}_1$ of the equation in part (2) of Theorem 1. To see that $\bar{x}_1(\lambda_1, \lambda_2)$ is well-defined rewrite this equation as follows:

$$\bar{x}_1 - \frac{1 - F(\bar{x}_1) - \lambda_1}{f(\bar{x}_1)} = \frac{(\bar{x}_2(\lambda_2))^2 f(\bar{x}_2(\lambda_2))}{1 - F(\bar{x}_2(\lambda_2)) + \bar{x}_2(\lambda_2) f(\bar{x}_2(\lambda_2))}$$

The left-hand side of this equation is increasing in $\bar{x}_1$ (by increasing hazard rate property and because $\lambda_1 \leq 1 - F(\bar{x}_1)$) and increasing in λ_1, while its right-hand side is decreasing in λ_2. Therefore, $\bar{x}_1(\lambda_1, \lambda_2)$ is well-defined and is decreasing in both λ_1 and λ_2.

Next, let $g(\lambda)$ be Lagrange dual function (Boyd and Vandenberghe (2009), p. 215) i.e.:

$$g(\lambda) \equiv \mathcal{L}(\lambda, \bar{x}(\lambda)) = \max_{\bar{x}} \mathcal{L}(\lambda, \bar{x}) = max_{\bar{x}} \int_{x\in[0,1]^n} \max\{0, \max_{i=1,...,n} \gamma_i(x_i)\} \prod_i dF(x_i) + \sum_{i=1}^{n} \lambda_i m_i. \tag{21}$$

By Danskin's Theorem (Bertsekas (2001), Ch. 1, p. 131), $g(\lambda)$ is convex and therefore has a unique minimum characterized by its first-order conditions. Importantly, the next Lemma establishes the strong duality property for our problem implying that its solution can be obtained by minimizing $g(\lambda)$.

Lemma 9 *The maximization problem (16)* [7] *has strong duality property i.e.,*

$$\max_{\bar{x}} \min_{\lambda} \mathcal{L}(\bar{x}, \lambda) = \min_{\lambda} \max_{\bar{x}} \mathcal{L}(\bar{x}, \lambda)$$

5 Main Results

Using Lemma 9 we can now derive the solution to our problem by minimizing the Lagrange dual function $g(\lambda)$. The result is provided in the next Theorem. To state it, let us introduce the following notation. For any set $J \subseteq \{1, ..., n\}$ s.t. $i \notin J$, let $Prob.[\gamma_i(x_i) > \max_{j\in J} \gamma_j] = \prod_{j\in J} \int_{x_j\in[0,1]:\, \gamma_i(x_i)>\gamma_j(x_j)} dF(x_j)$. Then we have:

Theorem 2 *The optimal profile of threshold values $(\bar{x}_1, ..., \bar{x}_n)$ is unique and is characterized by the following necessary and sufficient conditions, in which $q_i(x)$ for all i and $x \in [0, \bar{x}_i)$ and $q_i(\bar{x}_i)$ for i s.t. $\bar{x}_i \neq \bar{x}_j$, $j \neq i$, are uniquely defined by (18) in Lemma 7:*

If i is such that $\bar{x}_i \neq \bar{x}_j$, $j \neq i$, then i's budget constraint must hold, i.e.:

$$m_i = \bar{x}_i q_i(\bar{x}_i) - \int_0^{\bar{x}_i} q_i(s) ds \tag{22}$$

[7] It is well know that the primal problem of maximizing (16), $\max_{\bar{x}} \mathcal{L}(\bar{x}, \lambda)$, is equivalent to the following one: $\max_{\bar{x}} \min_{\lambda} \mathcal{L}(\bar{x}, \lambda)$.

If bidders $k_1, ..., k_l$ form a cluster with threshold $\bar{x}^c$ i.e., $\bar{x}_{k_1} = ... = \bar{x}_{k_l} = \bar{x}^c \neq \bar{x}_j$ for any $j \notin \{k_1, ..., k_l\}$, then:[8,9]

$$\sum_{h \in \{1,...,l\}} m_{k_h} = \bar{x}^c \frac{1 - F(\bar{x}^c)^l}{1 - F(\bar{x}^c)} Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j \notin \{k_1,...,k_l\}} \gamma_j] - l \int_0^{\bar{x}^c} q_{k_1}(s)ds \tag{23}$$

Also, for all $r \in \{2, ..., l-1\}$ we have:

$$\frac{m_{k_1} + ... + m_{k_r}}{r} - \frac{m_{k_{r+1}} + ... + m_{k_l}}{l - r} \leq$$
$$\bar{x}^c \left(\frac{1 - F(\bar{x}^c)^r}{r(1 - F(\bar{x}^c))} - F(\bar{x}^c)^r \frac{1 - F(\bar{x}^c)^{l-r}}{(l-r)(1 - F(\bar{x}^c))} \right) Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j \notin \{k_1,...,k_l\}} \gamma_j] \tag{24}$$

It is important to understand the significance of Theorem 2. Conditions (22)-(24) are necessary and sufficient for the characterization of the minimum of the dual Lagrange function $g(\lambda) \equiv \mathcal{L}(\lambda, \bar{x}^*(\lambda))$. Since $g(\lambda)$ is convex, the solution to the system (22)-(24) is unique and constitutes the solution to our optimal mechanism design problem.

Condition (22) says that in the optimal mechanism the only necessary and sufficient condition for bidder i who does not belong to any cluster (i.e. $\bar{x}_i \neq \bar{x}_j$ for all $i \neq j$) is that her budget constraint is binding at her threshold value $\bar{x}_i$.

Condition (23) is the aggregate budget constraint for the bidders in a cluster with threshold $\bar{x}^c$. The probability that one of them gets the good, $\sum_{r=1,...,l} q_{k_r}(\bar{x}^c)$, is equal to $\frac{1-F(\bar{x}^c)^l}{1-F(\bar{x}^c)} Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j \notin \{k_1,...,k_l\}} \gamma_j]$. This can be shown by summing individual budget constraints (10) of the bidders in the cluster and comparing the result to (23).

Condition (24) is the feasibility condition for the existence of a cluster with threshold $\bar{x}^c$. Although it may appear non-transparent, it has a clear and intuitive economic interpretation. Its left-hand side is the difference between the average budget of the richest r bidders and the average budget of the poorest $l - r$ bidders in the cluster. For the cluster to exist, this

[8]Note that without loss of generality we may assume here that indexes $k_1, ..., k_l$ are ordered according to the budgets i.e., $k_1 < k_2 ... < k_{l-1} < k_l$ and $m_{k_1} \geq m_{k_2} ... \geq m_{k_{l-1}} \geq m_{k_l}$. This is so because when (24) holds for this ordering, it also holds for any alternative ordering.

[9]Note that the integrand in the last term of (23) is $q_{k_1}(.)$. Yet, every bidder in the cluster $\{k_1, ..., k_l\}$, to which (23) pertains, has the same threshold $\bar{x}^c$ and hence, by Theorem 1, the same λ and γ, we have $q_{k_1}(x) = ... q_{k_l}(x)$ for all x in the range $[0, \bar{x}^c]$ of this integral.

difference cannot be too large, for otherwise it would be impossible to satisfy the necessary condition (10) that budget constraints of all bidders in the cluster are binding when they have threshold values $\bar{x}^c$. Precisely, this difference cannot exceed the largest possible difference between the average transfers paid by these two groups. The latter difference is equal to the maximal difference between the average expected gross surpluses of these two groups (because their net surpluses $\int_{r^c}^{\bar{x}^c} q_{k_1}(s)ds$ are the same) which is the right-hand side of (24). Indeed, the maximal average gross surplus of the richest r bidders is equal to the threshold value $\bar{x}^c$ times the maximal average probability of trading in this group. The latter is a product of the probability $Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j \notin \{k_1,...,k_l\}} \gamma_j]$ that no bidder outside the cluster has a virtual value exceeding the virtual value of a cluster member of type $\bar{x}^c$, $\gamma_{k_1}(\bar{x}^c)$, and the average probability that at least one among r bidders has a type of at least $\bar{x}^c$, $\frac{1-F(\bar{x}^c)^r}{r(1-F(\bar{x}^c))}$. Similarly, the minimal average gross surplus of the poorest $l-r$ bidders is equal to the threshold value $\bar{x}^c$ times the minimal average probability of trading for that group. The latter probability is a product of $Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j \notin \{k_1,...,k_l\}} \gamma_j]$ and the average probability that at least one among $l-r$ bidders has a type of at least $\bar{x}^c$ and the other r bidders in the cluster have types below $\bar{x}^c$, $F(\bar{x}^c)^r \frac{1-F(\bar{x}^c)^{l-r}}{(l-r)(1-F(\bar{x}^c))}$.

So when conditions (23) and (24) hold, then the vector of trading probabilities "at the top" $(q_{k_1}(\bar{x}^c), ..., q_{k_l}(\bar{x}^c))$ for the bidders in the cluster $C(x^c)$ defined by the budget constraint $m_{k_j} = \bar{x}^c q_{k_j}(\bar{x}^c) - \int_0^{\bar{x}^c} q_{k_j}(s)ds$ is feasible.

Let us now confirm formally our intuitive explanation of conditions (23) and (24) as establishing the feasibility of the vector $(q_{k_1}(\bar{x}^c), ..., q_{k_l}(\bar{x}^c))$. Indeed, by Theorem 3 in Border (2007) a feasible vector $(q_{k_1}(\bar{x}^c), ..., q_{k_l}(\bar{x}^c))$ has to satisfy two conditions. The first one is the upper bound condition:

$$\sum_{j=1,...,h} q_{k_j}(\bar{x}^c) \leq \frac{1-F(\bar{x}^c)^h}{1-F(\bar{x}^c)} Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{i \notin \{k_1,...,k_l\}} \gamma_i] \quad \text{for all } \ h \in \{1, ..., l\} \qquad (25)$$

This condition requires that the probability of assigning the good to any subset of bidders from the cluster $C(\bar{x}^c)$ with values above the threshold $\bar{x}^c$ does not exceed the probability that a bidder from this subset has value in $[\bar{x}^c, 1]$ and the bidders outside the cluster have lower virtual values than $\gamma_{k_j}(\bar{x}^c)$ for $k \in \{1, ..., l\}$. Although this condition must hold for every subset of size $h \in \{1, ..., l\}$ of bidders in a cluster, it is sufficient to check that it holds for the subset including h richest bidders $k_1, ..., k_h$ since these bidders have higher trading probabilities at the threshold i.e., $q_{k_1}(\bar{x}^c) \geq ... \geq q_{k_l}(\bar{x}^c)$.

The second feasibility condition involves the lower bound on the probability of assigning the good to any subset of bidders in a cluster. It requires that the good should be assigned to a bidder from this subset if some bidder in it has value in $[\bar{x}^c, 1]$, the rest of the bidders in the cluster have values below $\bar{x}^c$, and the bidders outside the cluster have lower virtual values than $\gamma_{k_j}(\bar{x}^c)$. It is sufficient that this condition hold for the subsets of every size composed of the bidders with the lowest budgets because they have lower probabilities of trading $q_{k_j}(\bar{x}^c)$ "at the top." Thus, we must have:

$$\sum_{j=h,...,l} q_{k_j}(\bar{x}^c) \geq F(\bar{x}^c)^{h-1} \frac{1 - F(\bar{x}^c)^{l-h+1}}{1 - F(\bar{x}^c)} Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j \notin \{k_1,...,k_l\}} \gamma_j] \quad \text{for all } \ h \in \{1, ..., l\} \tag{26}$$

The proof of the Theorem 2 shows that the feasibility conditions (25) and (26) are equivalent to the first-order conditions for minimizing the Lagrange dual function $g(\lambda)$, (51) and (52) in the proof of the Theorem, and the latter, in turn, are equivalent to the conditions (23)-(24) in the statement of Theorem 2. Hence, the optimality conditions of Theorem 2 are equivalent to the feasibility conditions for the mechanism.

This equivalence of the optimality and the feasibility conditions for the solution to the dual problem $\min_\lambda g(\lambda)$ combined with the uniqueness of its solution imply that there is a unique feasible mechanism satisfying conditions (22)-(24) which therefore is optimal. Combining this conclusion with the results of Theorem 1 and Lemma 7, yields:

Corollary 1 *There is a unique profile of threshold values $(\bar{x}_1, ..., \bar{x}_n)$ that satisfies the conditions of Theorem 1, condition (18) in Lemma 7, and (22)-(24) in Theorem 2. This profile is a unique solution to the optimal mechanism design problem.*

Corollary 1 summarizes the set of necessary and sufficient conditions on the optimal profile of threshold values $(\bar{x}_1, ..., \bar{x}_n)$. In the next subsection, we will use these results to characterize two classes of optimal mechanisms.

5.1 Top and Budget-Handicap Auctions

In this section, we focus on the qualitative properties of the optimal mechanism and show how these properties depend on the profile of budgets.

Qualitatively, we will distinguish between two kinds of optimal mechanisms. A mechanism of the first kind is called a "top auction." In a top auction all thresholds are equal i.e., $\bar{x}_1 = ... = \bar{x}_n = \bar{x}^t$, so all bidders belong to a single cluster. Therefore, all bidders with valuations below $\bar{x}^t$ are treated symmetrically: every bidder with valuation $x \in [r^t, \bar{x}^t)$ pays the same transfer and gets the good when she has the highest valuation. But, because the bidders have unequal budgets, the seller discriminates between them "at the top:" a richer high valuation bidder gets the good with a higher probability and pays a higher transfer than a poorer high valuation bidder.

The mechanisms of the second kind are called "budget-handicap auctions." In a "budget-handicap auction" the mechanism designer sets different thresholds for different bidders or groups of bidders. There may exist clusters of bidders with the same threshold, but not all bidders belong to the same cluster. In this mechanism, there are two types of price discrimination. First, a richer bidder with a value above her threshold has a higher probability of trading than a poorer bidder with a value above her respective threshold. This type of price discrimination applies to any two bidders with different budgets, irrespectively of whether they belong to the same cluster or different clusters.

The second type of price discrimination works in the opposite direction. A poorer bidder with a lower value (below her threshold) has a higher probability of trading and pays a higher transfer than a richer bidder with the same value. So richer bidders are handicapped, and poorer bidders are given an advantage at lower valuations via a lower reserve price and a higher probability of trading. This motivates the use of the term "budget-handicap."

Which mechanism is offered by the designer - a top auction or a budget-handicap auction- ultimately depends on the budget profile. The designer offers a top auction whenever it is feasible, namely, when the budget differences across buyers are not too large. However, when these differences are large, price-discrimination only at the "top" is no longer feasible, as all budget constraints cannot be made binding at the same threshold. So, different thresholds have to be set across bidders, and the seller has to handicap richer bidders at lower valuations.

We will start our characterization of the optimal mechanism with the "top auction." First, let us define $\bar{x}^t$ as the unique solution to the following equation:

$$\sum_{i=1,\ldots,n} m_i = \bar{x}^t \frac{1-F(\bar{x}^t)^n}{1-F(\bar{x}^t)} - n \int_{r_t : r_t = \frac{1-F(r_t) - \frac{(1-F(\bar{x}^t))^2}{1-F(\bar{x}^t)+\bar{x}^t f(\bar{x}^t)}}{f(r_t)}.}^{\bar{x}^t} F(s)^{n-1} ds \tag{27}$$

Condition (27) is the equivalent of (23) for the case of top auction where all bidders belong to a single cluster. The solution to (27) is unique because its right-hand side is increasing in x^t,[10] is equal to zero when $x^t = 0$, and exceeds $\sum_i m_i$ when $x^t = 1$, since by assumption $m_1 \leq 1 - \int_{r:r-\frac{1-F(r)}{f(r)}=0}^{1} F^{n-1}(x)dx$.

Definition 1 *A "top auction" for n bidders with budgets $m_1, \ldots m_n$, with $m_i \geq m_{i+1}$ for all $i = 1, .., n-1$, is a mechanism with a common threshold $\bar{x}^t = \bar{x}_1 = \ldots \bar{x}_n$ uniquely solving (27), reservation values $r_1 = \ldots = r_n = r_t$ defined by $r_t = \frac{1-F(r_t) - \frac{(1-F(\bar{x}^t))^2}{1-F(\bar{x}^t)+\bar{x}^t f(\bar{x}^t)}}{f(r_t)}$, and trading probabilities $q_i(x_i) = F(x_i)^{n-1}$ for all $x_i \in [r, \bar{x}^t)$ and $q_i(\bar{x}^t)$ satisfying:*

$$m_i = \bar{x}^t q_i(\bar{x}^t) - \int_{r_t}^{\bar{x}^t} F(s)^{n-1} ds \tag{28}$$

$$\sum_{i=1,\ldots,n} q_i(\bar{x}^t) = \frac{1-F(\bar{x}^t)^n}{1-F(\bar{x}^t)} \tag{29}$$

Condition (29) says that with probability 1 the good is given to a bidder with value of at least $\bar{x}^t$ if there is at least one such bidder. Also (28) and (29) are consistent with the definition of $\bar{x}^t$ in (27): summing up (28) over i and substituting (29) into this sum yields (27).

Our next result, which is a direct consequence of Theorem 2, shows that the "top auction" is optimal whenever it is feasible i.e., whenever the remaining feasibility condition (24) in Theorem 2 adapted to the top auction is satisfied.

Theorem 3 *Suppose that for a profile of bidders with budgets $m_1, \ldots, m_n$, with $m_i \geq m_{i+1}$ for all $i = 1, \ldots, n-1$, the threshold $\bar{x}^t$ uniquely solving (27) is such that $\bar{x}^t < 1$.*

[10]Indeed, its derivative is equal to $\frac{1-F(\bar{x}^t)^n}{1-F(\bar{x}^t)} + \frac{xf(\bar{x}^t)}{(1-F(\bar{x}^t))^2}(1+(n-1)F(\bar{x}^t)^n - nF(\bar{x}^t)^{n-1}) - nF(\bar{x}^t)^{n-1} + nF(r(\bar{x}^t))^{n-1}\frac{dr(\bar{x}^t)}{d\bar{x}^t}$. It is easy to ascertain that this expression is positive, in particular, because $\frac{dr(\bar{x}^t)}{d\bar{x}^t} > 0$.

The unique optimal mechanism is a "top auction" with a common threshold $\bar{x}^t$ if and only if for every $k = 1, 2, ..., n-1$ we have:

$$\frac{m_1 + ... + m_k}{k} - \frac{m_{k+1} + ... + m_n}{n-k} \leq \bar{x}^t \left(\frac{1 - F(\bar{x}^t)^k}{k(1 - F(\bar{x}^t))} - F(\bar{x}^t)^k \frac{1 - F(\bar{x}^t)^{n-k}}{(n-k)(1 - F(\bar{x}^t))} \right) \tag{30}$$

As the discussion following Theorem 2 points out, condition (30) (equivalently, condition (24) in Theorem 2), says that the difference between the average budget of the richest k bidders and the average budget of the poorest $n-k$ bidders does not exceed the maximal difference between the average expected surpluses of these two groups. This allows to allocate the good "at the top" in such a way that all budget constraints (10) hold at the threshold $\bar{x}^t$.

The top auction allocates the good efficiently when all buyers' valuations lie in $[r_t, \bar{x}^t]$, where r_t is the reservation value which is below the reservation value in the optimal auction without budget constraints. The additional inefficiency compared to the standard optimal auction is at the "top:" when several buyers have valuations above $\bar{x}^t$, the good is allocated randomly among them, with probabilities increasing in their budgets. So a bidder with a lower value in $[\bar{x}^t, 1]$ may end up getting the good even if another bidder has a higher value.

The following Corollary of Theorem 3 shows that the seller's expected revenue in the top auction depends only on the aggregate budget $\sum_i m_i$, and not on the distribution of the budgets across the bidders:

Corollary 2 *Suppose that the top auction is the optimal mechanism under budget profiles $(m_1, ..., m_n)$ and $(m'_1, ..., m'_n)$ such that $\sum_i m_i = \sum_i m'_i$. Then the optimal threshold $\bar{x}^t$ and the expected seller's revenue is the same in both cases.*

As an application of this Corollary, suppose first that all bidders have the same budgets. So the seller offers a top auction which in this case coincides with the optimal mechanism of Laffont and Robert (1996). Then the nature of the optimal mechanism and its profitability for the seller do not change after a sufficiently small budget redistribution among the bidders that does not violate (30).

The top auction can be implemented via an indirect mechanism which combines an all-pay auction with a lottery. Specifically, each bidder is offered a choice between the former and the latter. If bidder i chooses the lottery, she pays m_i for a "lottery ticket" which gets her the good with probability $q_i(\bar{x}^t)$. If i chooses the all-pay auction she submits a bid b_i and gets the good if she is the highest bidder, her bid is above the "reserve price," and no bidder has chosen to take part in the lottery. The reserve price in the all-pay auction is equal to $t_i(r^t) = r^t F^{n-1}(r^t)$. This mechanism implements the same allocation as the top-auction with optimal threshold $\bar{x}^t$. Indeed, it is easy to see that the optimal strategy of bidder i is to buy the lottery ticket if $x_i \in [\bar{x}^t, 1]$; to bid $b_i = t_i(x_i) = x_i F^{n-1}(x_i) - \int_{r^t}^{x_i} F^{n-1}(s)ds$ if $x_i \in [r^t, \bar{x}^t]$; and not to participate if $x_i < r^t$. Note that this mechanism is envy-free since any two bidders i and j get equal payoffs if $x_i = x_j \in [0, \bar{x}^t]$, while richer bidder i gets a higher payoff than poorer bidder j when $x_i = x_j > \bar{x}^t$, but j cannot afford i's lottery ticket which costs m_i.

When the feasibility condition (30) fails, the seller has to use additional tools to discriminate between the bidders and, in particular, set different thresholds for them. Naturally, lower-budget bidders have lower thresholds (Lemma 6), although there may still exist some clusters of bidders sharing the same threshold. Richer bidders with valuations above their higher thresholds get the good with a higher probability and pay more than poorer bidders with valuations above their lower thresholds.

Importantly, there is another type of "price discrimination" in this second kind of mechanism, which we call "budget handicap auction:" a poorer bidder with a low value has a higher probability of trading and pays a higher transfer than a richer bidder with the same value. This handicapping of higher-budget bidders creates a stronger competition for them from lower-budget bidders, and extracts higher payments from the former when they have high values. It also unavoidably increases inefficiency. Formally, we have:

Theorem 4 *Suppose that (30) fails for some k. Then the optimal auction is a "budget handicap auction" which is uniquely defined by a vector of thresholds $(\bar{x}_1, ..., \bar{x}_n)$ s.t. $\bar{x}_i \geq \bar{x}_{i+1}$ for all $i \in \{1, .., n-1\}$, with strict inequality for at least some i.*

In budget handicap auction, if $\bar{x}_i > \bar{x}_j$ then $r_i > r_j$ and $q_i(x) < q_j(x)$ for all $x \in [r_j, \bar{x}_j]$.

If bidders $k_1, ...k_l$, $k_1 \leq k_2 ... \leq k_l$ form a cluster with threshold $\bar{x}^c$, then these bidders have *the closest budgets i.e., there exists* $j \in \{0, n-l\}$ *such that* $k_h = j + h$ *for all* $h \in \{1, ..., l\}$.

By Corollary 1, the vector of thresholds $(\bar{x}_1, ..., \bar{x}_n)$ is uniquely defined by conditions (22)-(24) in Theorem 2, the conditions of Theorem 1, and condition (18) in Lemma 7.

Similarly to the top auction, the cluster configuration in the budget-handicap auction and the seller's revenue remain robust to certain sufficiently small redistributions of the budgets:

Corollary 3 *Suppose that under budget profile* $(m_1, ..., m_n)$ *the optimal mechanism is a budget-handicap auction with thresholds* $(\bar{x}_1, ..., \bar{x}_n)$. *Consider a budget profile* $(m'_1, ..., m'_n)$ *such that* $|m_i - m'_i|$ *is sufficiently small for all* i *and the aggregate budget of any set of bidders that have a common threshold under* $(m_1, ..., m_n)$ *is the same under both budget profiles.*

Then the optimal mechanism under $(m'_1, ..., m'_n)$ *is a budget-handicap auction with the same profile of thresholds* $(\bar{x}_1, ..., \bar{x}_n)$ *and the same expected seller's revenue as under profile* $(m_1, ..., m_n)$.

Implementation of a budget-handicap auction via an indirect bidding mechanism is similar to that for the top-auction. As in the latter, bidder i is offered a choice between participating in an all-pay auction (with a handicap) and buying a lottery ticket that costs m_i and wins the good with probability $q_i(\bar{x}_i)$. Bidder i chooses the lottery if her valuation exceeds $\bar{x}_i$ and submits a bid in the auction otherwise. The all-pay auction is not symmetric as in the top-auction case, since richer bidders now have to be handicapped. Specifically, bidder i with threshold $\bar{x}_i$ participating in this auction gets the good when her bid: (i) exceeds the bid of any richer bidder j with a higher threshold $\bar{x}_j$ lowered by a certain margin; (ii) exceeds the bid of a poorer bidder h with a lower threshold $\bar{x}_h$ by a certain margin. These margins depend both on the bidders' thresholds and type distribution.[11]

The most challenging part in computing the optimal "budget handicap" auction is determining which groups of bidders form clusters with common thresholds. Theorem 4 simplifies

[11]The mapping of bids into the allocation of the good is defined via the formula for the transfers $t_i(x_i) = q_i(x_i)x_i - \int_{r_i}^{x_i} q_i(s)ds$. In the optimal budget-handicap mechanism $t_i(x_i)$ is strictly increasing in x_i on $[r_i, \bar{x}_i)$ and therefore b_i uniquely defines x_i on this interval via $b_i = t_i(x_i)$. Thus, when i submits bid $b_i = t_i(x_i)$ and bidder $j \neq i$ submit a bid $b_j = t_j(x_j)$, bidder i gets the good whenever $\gamma_i(x_i) \geq \max_{j \neq i} \gamma_j(x_j)$ which occurs with probability $q_i(x_i)$ from i's point of view.

this task by showing that any cluster contains only "adjacent" bidders with the smallest budget differences. So the number of possible cluster configurations is $2^n - 1$, and potentially one may have to go over all of them to compute the solution. Our results provide a tractable method to check whether a particular cluster configuration is optimal. For example, the optimal mechanism is a budget-handicap auction without any clusters if the following system of n equations has a solution $(\bar{x}_1, ..., \bar{x}_n)$ satisfying $\bar{x}_i > \bar{x}_{i+1}$ for all $i \in \{1, ..., n-1\}$:

$$m_1 = \bar{x}_1 - \int_0^{\bar{x}_1} \int_{x_{-1}:\gamma_1(x_1) > \max\{0, \max_{j\neq 1} \gamma_j(x_j)\}} \prod_{j\neq 1} dF(x_j) dx_1$$
$$m_i = \bar{x}_i \int_{x_{-i}:\gamma_i(\bar{x}_i) > \max\{0, \max_{j\neq i} \gamma_j(x_j)\}} \prod_{j\neq i} dF(x_j) dx_1 - \int_0^{\bar{x}_i} \int_{x_{-i}:\gamma_i(x_i) > \max\{0, \max_{j\neq i} \gamma_j(x_j)\}} \prod_{j\neq i} dF(x_j) dx_i \tag{31}$$

Similarly, we can write down necessary and sufficient conditions for the optimality of any other cluster configuration. In the next section we will consider an example with three bidders and exhibit conditions for optimality of various cluster configurations in that case.

In the remainder of this section, we will focus on the properties of the seller's expected payoff function, in particular, we will show that it is (weakly) concave in the bidders' budgets and explore the implications of this. Recall that by Lemma 9 (strong duality), the seller's expected profit in the optimal mechanism is given by the minimum of the dual Lagrange function $g(\lambda)$ in (21) which can be written as a function of the vector of budgets $(m_1, ..., m_n)$ in the following way:

$$\pi(m_1, ..., m_n) = \min_{\lambda} \left\{ \int_{x\in[0,1]^n} \max\{0, \max_{i=1,...,n} \gamma_i(x_i, \lambda)\} dF(x) + \sum_{i=1}^{n} \lambda_i m_i \right\}. \tag{32}$$

Since $\pi(m_1, ..., m_n)$ is a pointwise minimum in λ of a function affine in $(m_1, ..m_n)$, it is concave in the vector $(m_1, ..., m_n)$.[12]

Concavity of $\pi(.)$ has the following consequences for the seller's revenue:

Lemma 10 *Suppose that the aggregate budget of all bidders is fixed i.e.* $\sum_i m_i = M$[13]

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[12]Note that this is true even if some bidder i's budget constraint is not binding. In this case $\lambda_i = 0$ and $\pi(m_1, ..., m_n)$ does not depend on m_i.

[13]To make this result non-trivial M has to be sufficiently small. In particular, we will assume that $M \le np^m$ where p^m is a monopoly price i.e., $p^m = \arg\max_p F(p)(1-p)$.

Then the seller gets the maximal payoff in the optimal mechanism when all bidders' budgets are equal i.e., $m_i = \frac{M}{n}$ for all $i = 1, ..., n$.

Moreover, consider two budget profiles $(m_1, ..., m_n)$ and $(m_1', ..., m_n')$, ordered from the highest to the lowest, and suppose that $\sum_{j=1}^n m_j = \sum_{j=1}^n m_j'$ and $\sum_{j=i}^n m_j \leq \sum_{j=i}^n m_j'$ for all $i \in \{2, ..., n\}$. Then $\pi(m_1, ..., m_n) \leq \pi(m_1', ..., m_n')$.

Intuitively, the second part of the Lemma says that if budget profile $(m_1, ..., m_n)$ is a mean preserving spread of the profile $(m_1', ..., m_n')$, in the sense of Rothschild and Stiglitz (1970), then the seller's expected revenue is greater under the latter than under the former. However, Corollaries and 2 and 3 show that the inequality $\pi(m_1, ..., m_n) \leq \pi(m_1', ..., m_n')$ is strict only if the difference between the two budget profiles is sufficiently large that the sets of thresholds under the two budget profiles are different. If these two budget profiles are sufficiently close, the thresholds and the seller's revenue are the same in both cases.

6 Examples with the Uniform Type Distribution.

6.1 Two Bidders.

We have described the qualitative properties of the optimal mechanism for two bidders in section 3. Below we compute its exact parameters under uniform type distribution on $[0, 1]$.

Starting with the top auction, equation (27) defining the threshold value in it becomes:

$$m_1 + m_2 = \bar{x}^t + \left(\bar{x}^t\right)^2 - \left(\bar{x}^t\right)^3 + \frac{\left(\bar{x}^t\right)^4}{4}$$

Condition (30) for the feasibility of the top auction simplifies to $m_1 - m_2 \leq \bar{x}^t(1 - \bar{x}^t)$. If this holds, then $\gamma_1(x) = \gamma_2(x) = 2x - 2\bar{x}^t + (\bar{x}^t)^2$ for $x \leq \bar{x}^t$, and by Lemma 7 $q_i(x_i) = 0$ if $x_i < \bar{x}^t - \frac{\left(\bar{x}^t\right)^2}{2}$; $q_i(x_i) = x_i$ if $x_i \in [\bar{x}^t - \frac{\left(\bar{x}^t\right)^2}{2}, \bar{x}^t)$; $q_i(x_i) = \frac{1+\bar{x}^t}{2} + \frac{m_i - m_j}{2}$ if $x_i \geq \bar{x}^t$.

So, $q_1(x)$ and $q_2(x)$ jump upwards at $x = \bar{x}^t$, except in the borderline case $m_1 - m_2 = \bar{x}^t(1 - \bar{x}^t)$ where $q_1(x)$ jumps to 1 at $\bar{x}^t$, and $q_2(x)$ is continuous at $\bar{x}^t$, with $q_2(\bar{x}^t) = F(\bar{x}^t) = \bar{x}^t$.

If $m_1 - m_2 > \bar{x}^t(1 - \bar{x}^t)$, then the optimal mechanism is a "budget-handicap auction" with thresholds $\bar{x}_1$ and $\bar{x}_2$ such that $\bar{x}_1 > \bar{x}_2$. By Theorem 1, $\gamma_1(x_1) = x_1 - \bar{x}_1 + \frac{\bar{x}_2^2}{2}$ for $x_1 \in [0, \bar{x}_1)$, $\gamma_2(x_2) = x_2 - \bar{x}_2 + \frac{\bar{x}_2^2}{2}$ for $x_2 \in [0, \bar{x}_2)$, $\gamma_2(\bar{x}_2) = \gamma_2^-(\bar{x}_2) = \gamma_1^-(\bar{x}_1) = \frac{\bar{x}_2^2}{2} < \gamma_1(\bar{x}_1)$. Also, by Lemma 8 $r_1 = \bar{x}_1 - \frac{\bar{x}_2^2}{2}$, $r_2 = \bar{x}_2 - \frac{\bar{x}_2^2}{2}$.

By Theorem 2 the thresholds $\bar{x}_1$ and $\bar{x}_2$ solve the following equations:

$$m_1 = \bar{x}_1 - \int_{r_1}^{\bar{x}_1} \int_{\gamma(x_1)>\gamma(x_2)} dx_2 dx_1 = \bar{x}_1 - \frac{\bar{x}_2^3}{2} + \frac{\bar{x}_2^4}{8} \tag{33}$$

$$m_2 = \bar{x}_2 F(\bar{x}_1) - \int_{r_2}^{\bar{x}_2} \int_{\gamma(x_2)>\gamma(x_1)} dx_1 dx_2 = \bar{x}_1\bar{x}_2 - \bar{x}_1\frac{\bar{x}_2^2}{2} + \frac{\bar{x}_2^4}{8} \tag{34}$$

Also by Lemma 7, $q_1(\bar{x}_1) = 1$, $q_2(\bar{x}_2) = F(\bar{x}_1) = x_1$, and for $x_i \in [\bar{x}_i - \frac{\bar{x}_2^2}{2}, \bar{x}_i)$, $i \in \{1,2\}$:

$$q_i(x_i) = \int_{\gamma_i(x_i)>\gamma_j(s)} ds = \int_{x_i - \bar{x}_i > s - \bar{x}_j} ds = x_i - \bar{x}_i + \bar{x}_j \text{ for } i,j \in \{1,2\}\ i \neq j.$$

So $q_1(x_1)$ increases continuously on $[\bar{x}_1 - \frac{\bar{x}_2^2}{2}, \bar{x}_1)$ and jumps at $\bar{x}_1$ from $\bar{x}_2$ to 1, while $q_2(x_2)$ increases continuously on $[\bar{x}_2 - \frac{\bar{x}_2^2}{2}, \bar{x}_2]$ to its maximum $\bar{x}_1$. Note that $q_1(x) - q_2(x) = 2(\bar{x}_2 - \bar{x}_1) < 0$ for $x \in [\bar{x}_1 - \frac{\bar{x}_2^2}{2}, \bar{x}_2]$, as buyer 1 is handicapped in the intermediate range of values.

To conclude with this example, let us consider how the budget-handicap auction changes when bidder 1's budget increases and becomes non-binding, while the budget of bidder 2 remains small. By Theorem 1, $\lambda_2 = (1-\bar{x}_2)^2$ and $\lambda_1 = 1 - 2\bar{x}_1 + \bar{x}_2^2$. Thus, $\lambda_1 > 0$ and hence bidder 1's budget constraint is binding if and only if the solutions to (33) and (34) are such that $\bar{x}_1 < \frac{1+\bar{x}_2^2}{2}$. Substituting $\bar{x}_1 = \frac{1+\bar{x}_2^2}{2}$ into (33) yields the necessary condition for binding budget constraint of bidder 1: $m_1 \leq \frac{1+\bar{x}_2^2}{2} - \frac{\bar{x}_2^3}{2} + \frac{\bar{x}_2^4}{8}$, which holds when $m_1 \leq \frac{1}{2}$.

Now suppose that the solution to (33) and (34) is such that $\bar{x}_1 > \frac{1+\bar{x}_2^2}{2}$ which holds in the upper-left triangle of Figure 2. In this case, bidder 1's budget constraint is never binding and the Lagrange multipliers are $\lambda_1 = 0$ and $\lambda_2 = (1-\bar{x}_2)^2$. Then the thresholds have to be adjusted. Particularly, $\lambda_1 = 0$ implies that optimal thresholds satisfy $\bar{x}_1 = \frac{1+\bar{x}_2^2}{2}$, substituting which into (34) we obtain that $\bar{x}_2$ is now uniquely defined by $m_2 = \frac{\bar{x}_2+\bar{x}_2^3}{2} - \frac{\bar{x}_2^2}{4} - \frac{\bar{x}_2^4}{8}$.

Consequently, $\gamma_1(x_1) = 2x_1 - 1$, $\gamma_2(x_2) = 2x_2 - 2\bar{x}_2 + \bar{x}_2^2$, and so by Lemma 7 $q_1(x_1) = 0$ if $x_1 < \frac{1}{2}$; $q_1(x_1) = x_1 + \bar{x}_2 - \frac{1+\bar{x}_2^2}{2}$ if $x_1 \in \left[\frac{1}{2}, \frac{1+\bar{x}_2^2}{2}\right)$; and $q_1(x_1) = 1$ if $x_1 \geq \frac{1+\bar{x}_2^2}{2}$. Also, $q_2(x_2) = 0$ if $x_2 < \bar{x}_2 - \frac{\bar{x}_2^2}{2}$; $q_2(x_2) = x_2 - \bar{x}_2 + \frac{1+\bar{x}_2^2}{2}$ if $x_2 \in \left[\bar{x}_2 - \frac{\bar{x}_2^2}{2}, \bar{x}_2\right)$; and $q_2(\bar{x}_2) = \frac{1+\bar{x}_2^2}{2}$ if $x_2 \geq \bar{x}_2$. Bidder 1 is handicapped, as $q_1(x) - q_2(x) = -(1-\bar{x}_2)^2 < 0$ when $x \in [\frac{1}{2}, \frac{1+\bar{x}_2^2}{2})$.

Finally, substituting $\bar{x}_1 = \frac{1+\bar{x}_2^2}{2}$ into (33) yields the maximal transfer paid by bidder 1, when her value exceeds $\frac{1+\bar{x}_2^2}{2}$, $\frac{1+\bar{x}_2^2}{2} - \frac{\bar{x}_2^3}{2} + \frac{\bar{x}_2^4}{8} < m_1$. The last inequality holds because we are in the case where the solution to the system of (33) and (34) satisfies $\bar{x}_1 > \frac{1+\bar{x}_2^2}{2}$.

6.2 Three-Bidder Mechanism Under the Uniform Distribution

The optimal mechanism with three bidders can be of four kinds:

- "top-auction:" $\bar{x}_1 = \bar{x}_2 = \bar{x}_3 = \bar{x}^t$;
- "budget-handicap auctions" with:
  - "top cluster:" $\bar{x}_1 = \bar{x}_2 > \bar{x}_3$.
  - "lower cluster:" $\bar{x}_1 > \bar{x}_2 = \bar{x}_3$.
  - "no clusters:" $\bar{x}_1 > \bar{x}_2 > \bar{x}_3$.

In this section we provide the conditions on the budgets for the optimality of these mechanisms. To save space, the details of the derivations are omitted. An interested reader is referred to the online Appendix available at $http://www.severinov.com/bca\_uniform$.

Interestingly, each of these mechanisms is optimal for a set of budgets of a positive measure, as illustrated in Figures 3-6. These four sets of budgets possess a common boundary depicted in Figure 8 where all of them meet. Also, Figure 7 depicts budget combinations for which all budget constraints are binding.

Top Auction. In the top auction, the reservation value is given by $r_i = \bar{x}^t - \frac{(\bar{x}^t)^2}{2}$. Also, $q_i(x) = x^2$ for all $x \in \left[\bar{x}^t - \frac{(\bar{x}^t)^2}{2}, \bar{x}^t\right)$, and $q_i(\bar{x}^t)$ is set to satisfy the budget constraint of bidder i for $i \in \{1, 2, 3\}$. Then conditions (27) and (30) simplify to:

$$\sum_{i=1}^{3} m_i = \bar{x}^t(1+\bar{x}^t) + \left(\bar{x}^t - \frac{(\bar{x}^t)^2}{2}\right)^3$$
$$m_1 - \frac{m_2+m_3}{2} \leq \bar{x}^t\left(1 - \bar{x}^t\frac{1+\bar{x}^t}{2}\right)$$
$$m_1 - m_3 \leq \bar{x}^t\left(1-\left(\bar{x}^t\right)^2\right) \tag{35}$$

Top auction is an optimal mechanism when the system (35) has a solution $\bar{x}^t$. Figure 3 depicts the set of budgets for which this is the case. If the system (35) does not have a solution, then the optimal mechanism is a "budget-handicap auction."

Budget-Handicap Auction with Top cluster. Since $\bar{x}_1 = \bar{x}_2$ in the top cluster, we will simplify the notation and let $\bar{x}_1$ denote the threshold of bidders 1 and 2 in the rest of this subsection, and let $\bar{x}_3$ denote the threshold of bidder 3, with $\bar{x}_1 > \bar{x}_3$. The budget-handicap auction with a top cluster is optimal if the following system -which correspond to conditions (22)-(24) in Theorem 2- has a solution:

$$
\begin{aligned}
m_3 &= -\frac{\bar{x}_3^6}{24} + \frac{\bar{x}_3^5}{4} + \bar{x}_3^3\left(1 - \frac{\bar{x}_3}{4}\right)\left(\bar{x}_1 - \frac{\bar{x}_1^2}{2}\right) + \left(\bar{x}_3 - \frac{\bar{x}_3^2}{2}\right)\left(\bar{x}_1 - \frac{\bar{x}_1^2}{2}\right)^2 \\
m_1 + m_2 &= \bar{x}_1(1 + \bar{x}_1) + \frac{\bar{x}_3^4}{4}\left(1 - \bar{x}_3 + \frac{\bar{x}_3^2}{6}\right) - \bar{x}_1^3\left(1 - \frac{\bar{x}_1}{4}\right) + \left(\bar{x}_1 - \frac{\bar{x}_1^2}{2}\right)\bar{x}_3^2\left(1 - \frac{\bar{x}_3}{2}\right)^2 \\
m_1 - m_2 &\leq \bar{x}_1(1 - \bar{x}_1)
\end{aligned}
$$

The set of budgets under which the above system is compatible is depicted in Figure 4.

Lower cluster. Next, consider the case of the “lower cluster” with $\bar{x}_1 > \bar{x}_2 = \bar{x}_3$. We let $\bar{x}_2$ denote the threshold of bidders 2 and 3 and drop $\bar{x}_3$ from the notation. By Theorem 2, condition (22) must hold for bidder 1 and conditions (23) and (24) must hold for bidders 2 and 3. Under uniform distribution these conditions can be written as follows:

$$
m_1 = \bar{x}_1 - \frac{\bar{x}_2^4}{2}\left(1 - \frac{\bar{x}_2}{2} + \frac{\bar{x}_2^2}{12}\right) \tag{36}
$$

$$
m_2 + m_3 = \bar{x}_1\bar{x}_2(1 + \bar{x}_2) + \frac{\bar{x}_2^5}{4}\left(1 - \frac{\bar{x}_2}{3}\right) - \bar{x}_2^3\bar{x}_1\left(1 - \frac{\bar{x}_2}{4}\right) \tag{37}
$$

$$
m_2 - m_3 \leq \bar{x}_1\bar{x}_2(1 - \bar{x}_2) \tag{38}
$$

Equations (36) and (37) implicitly define $\bar{x}_1$ and $\bar{x}_2$. If the solution satisfies (38), the optimal mechanism is the handicap auction with the lower cluster and thresholds $\bar{x}_1$ and $\bar{x}_2 = \bar{x}_3$. The set of budgets for which this is true is depicted in Figure 5.

No Clusters. Finally, consider the case with no clusters i.e., $\bar{x}_1 > \bar{x}_2 > \bar{x}_3$. By Theorem 2, the necessary and sufficient conditions characterizing the optimal thresholds in this case are the budget constraints (22): $m_i = \bar{x}_i q_i(\bar{x}_i) - \int_{r_i}^{\bar{x}_i} q_i(s)ds$ for $i = 1, 2, 3$. If the solution to this system exists and satisfies $1 \geq \bar{x}_1 > \bar{x}_2 > \bar{x}_3 \geq 0$, then there are no clusters in the optimal mechanism. We can rewrite the system of budget constraints as follows:

$$m_1 = \bar{x}_1 + \frac{\bar{x}_3^4}{8}\left(1 - \bar{x}_3 + \frac{\bar{x}_3^2}{6}\right) - \frac{\bar{x}_2^3}{2}\left(1 - \frac{\bar{x}_2}{4}\right) + \left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)\frac{\bar{x}_3^2}{2}\left(1 - \frac{\bar{x}_3}{2}\right)^2$$

$$m_1 - m_2 = \bar{x}_1(1 - \bar{x}_2) + \frac{\bar{x}_1 - \bar{x}_2}{2}\left(\bar{x}_2^2 - \left(\bar{x}_3 - \frac{\bar{x}_3^2}{2}\right)^2\right).$$

$$m_2 - m_3 =$$

$$\bar{x}_1\bar{x}_2 + (\bar{x}_2\bar{x}_3 - \bar{x}_1(1 - \bar{x}_3))\frac{\bar{x}_2^2 - \bar{x}_3^2}{2} + \left(\frac{1}{2} - \bar{x}_3\right)\left(\frac{\bar{x}_2^2}{2} - \frac{\bar{x}_3^2}{2}\right)^2 + \frac{\bar{x}_3^2}{2}\left(\bar{x}_2 - \frac{\bar{x}_2^2}{2} - \bar{x}_3 + \frac{\bar{x}_3^2}{2}\right)\left(\bar{x}_1 + \frac{\bar{x}_3^2}{4} - \frac{\bar{x}_2^2}{2}\right) \quad (39)$$

When the solution to the system (39) satisfies $\bar{x}_1 > \bar{x}_2 > \bar{x}_3$, these thresholds define an optimal mechanism with no clusters. The corresponding set of budgets is depicted in Figure 5.

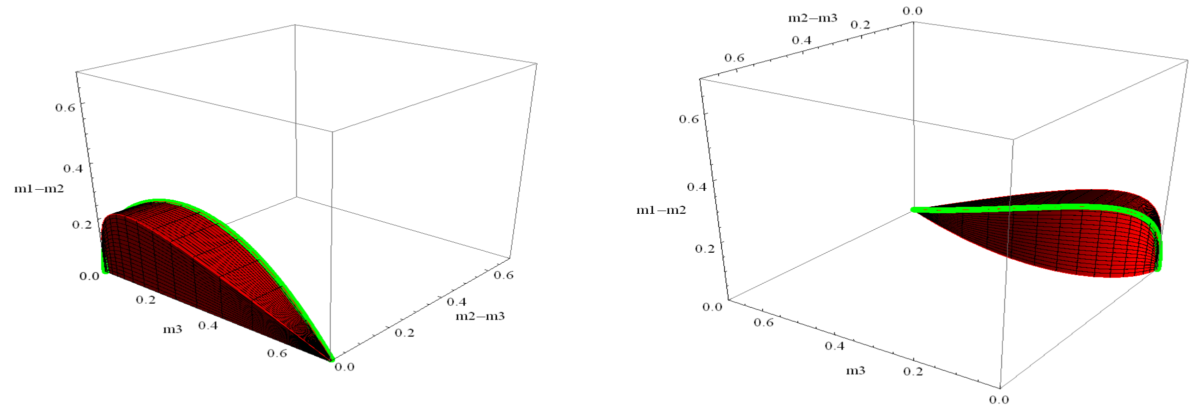
7 Conclusions

In this paper, we have derived an optimal mechanism for a seller facing bidders who have commonly known and unequal budgets. We have shown that when the differences between the budgets are not too large, the seller uses a "top auction" mechanism in which all bidders are treated symmetrically when their valuation do not exceed a certain threshold valuation. At that threshold all budget constraints become binding, and the richer bidders are given the good with a higher probability.

When the differences between the budgets are sufficiently large, then the seller uses a "budget-handicap" auction in which the valuation thresholds at which budget constraints become binding differ across the bidders. Budget-handicap auction also discriminates between the bidders at low valuations favoring low-budget bidders, who have higher probabilities of trading at low valuations and lower reserve prices. The seller does so to create a stronger competition for higher-budget bidders and extract more surplus from them. The latter result can be interpreted as providing justification for favoring smaller or minority-owned businesses in public procurement and other allocation mechanisms, such as spectrum auctions.

Our mechanisms have the nature of an all-pay auction, since a bidder always pays her bid. It would be interesting to consider a modification of our set-up and consider mechanisms in which a bidder pays only when (s)he gets the good. We leave this issue for future research.

Figure 3: Region of Optimality of The Top Auction



(a) View 1 (b) View 2

Another interesting qualitative property of the optimal mechanism emerges from our analysis of the two bidder case. There, we show that when one bidder has a significantly larger budget than the other, a mechanism with "buy-it-now" features is optimal. Precisely, a rich bidder is given an option to either participate in an auction, where she competes with the not-so-rich bidder, or to purchase the good immediately at a higher price. Generalizing this result to a more general set-up with many bidders is another extension which we leave for future research.

8 Appendix

Proof of Theorem 1: The proof of the Theorem relies on the following Lemma:

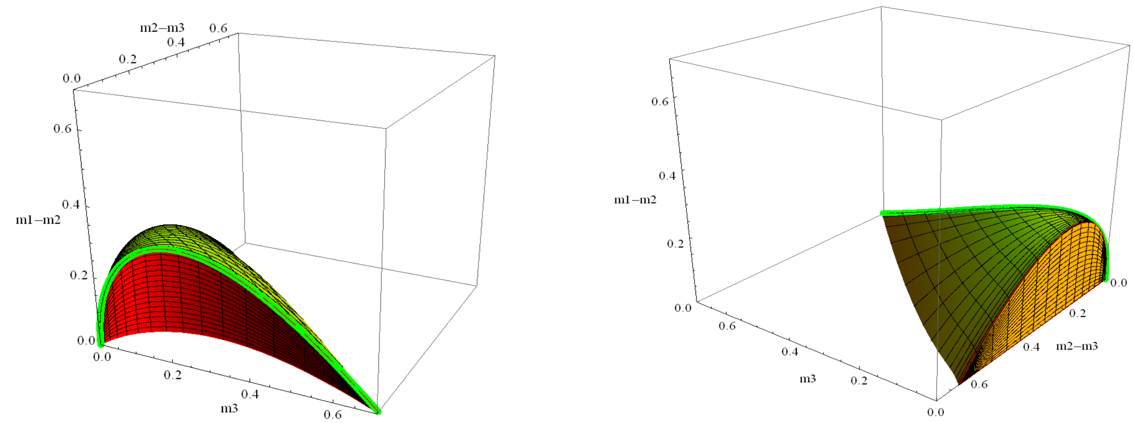
Lemma 11 *The following cannot hold in the optimal mechanism for any* $i \in \{1, ..., n\}$*:*

(a) $\gamma_i^-(\bar{x}_i) > \gamma_i(\bar{x}_i)$ *and the set* $A_i(\gamma_i^-(\bar{x}_i), \gamma_i(\bar{x}_i)) = \{x_{-i} \in [0,1]^{n-1} : \gamma_i(\bar{x}_i) \leq \max_{j \neq i} \gamma_j(x_j) < \gamma_i^-(\bar{x}_i)\}$ *has a positive measure:.*

(b) $\gamma_i(\bar{x}_i) > \gamma_i^-(\bar{x}_i)$ *and the set* $B_i(\gamma_i^-(\bar{x}_i), \gamma_i(\bar{x}_i)) = \{x_{-i} \in [0,1]^{n-1} : \gamma_i^-(\bar{x}_i) < \max_{j \neq i} \gamma_j(x_j) \leq \gamma_i(\bar{x}_i)\}$ *has a positive measure.*

Proof of Lemma 11:

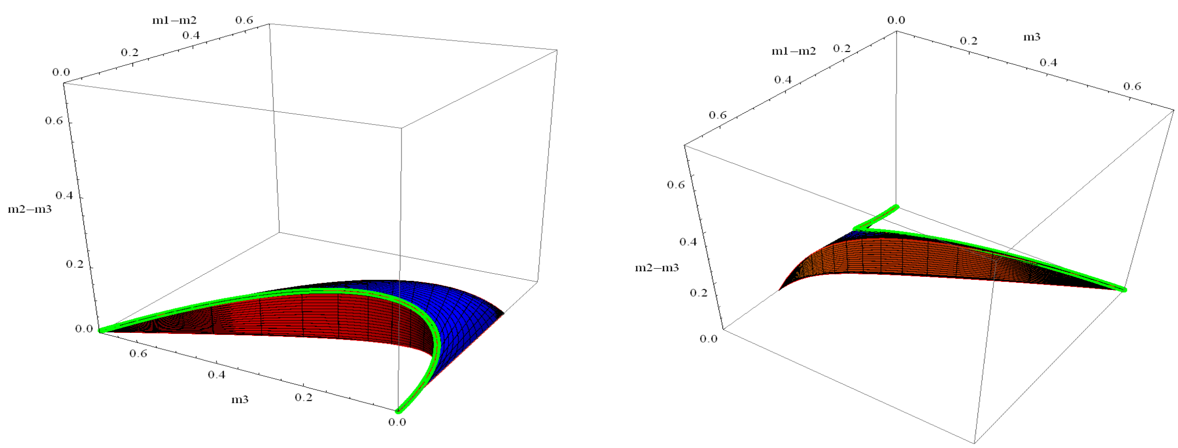
Figure 4: Region of Optimality of the Budget Handicap Auction with Top Cluster



(a) View 1

(b) View 2

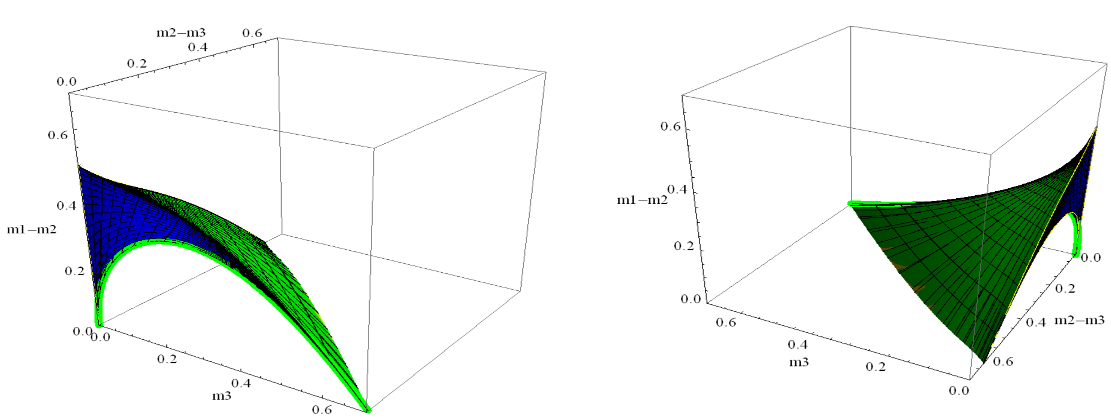
Figure 5: Region of Optimality of the Budget Handicap Auction with Lower Cluster



(a) View 1

(b) View 2

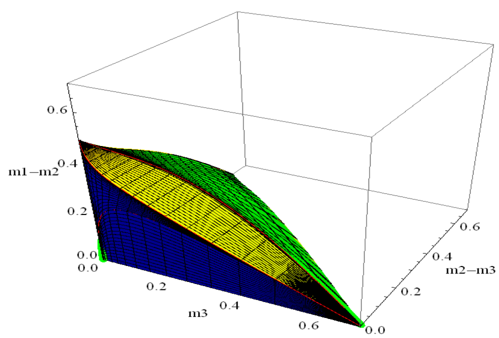
Figure 6: Region of Optimality of the Budget Handicap Auction with No Clusters



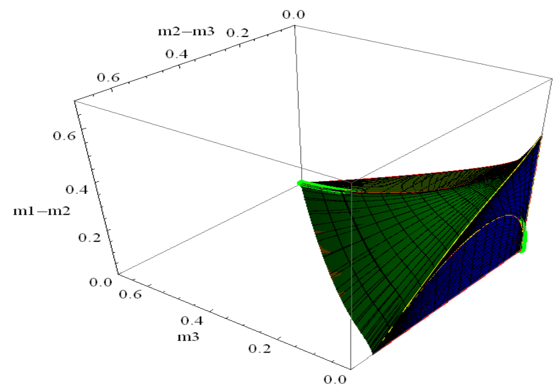
(a) View 1

(b) View 2

Figure 7: Region where Both Budget Constraints are Binding.

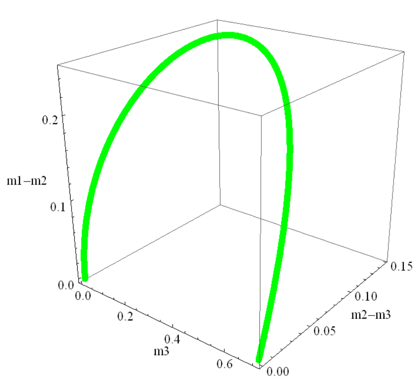


(a) View 1



(b) View 2

Figure 8: The Common Boundary between All Regions of Optimality



In order to prove the Lemma, we need to characterize the first-order conditions of the optimization problem (16) with respect to $\bar{x}_i$. Although (16) may not possess a derivative with respect to $\bar{x}_i$ because its second term contains a max operator, it does however possess left- and right- derivatives which we denote by $\frac{\partial_- \mathcal{L}}{\partial \bar{x}_i}$ and $\frac{\partial_+ \mathcal{L}}{\partial \bar{x}_i}$, respectively.

Then at the optimum the following must hold. If $\bar{x}_i \in (0,1)$, then $\frac{\partial_+ \mathcal{L}}{\partial \bar{x}_i} \geq 0$ and $\frac{\partial_- \mathcal{L}}{\partial \bar{x}_i} \leq 0$. If $\bar{x}_i = 1$, then $\frac{\partial_- \mathcal{L}}{\partial \bar{x}_i} \geq 0$. Note that $\bar{x}_i = 0$ cannot be optimal because in this case $t_i(x_i) = q_i(x_i) = 0$ for all $x_i \in [0,1]$.

Differentiating (16) and using the notation $\gamma_i^-(\bar{x}_i) = \bar{x}_i - \frac{1-\lambda_i - F(\bar{x}_i)}{f(\bar{x}_i)}$ from (17) yields:

$$\frac{\partial_+ \mathcal{L}}{\partial \bar{x}_i} = f(\bar{x}_i) \int_{x_{-i} \in [0,1]^{n-1}} \left(\max\{0, \gamma_i^-(\bar{x}_i), \max_{j \neq i} \gamma_j(x_j)\} - \max\{0, \gamma_i(\bar{x}_i), \max_{j \neq i} \gamma_j(x_j)\} \right) dF(x_{-i}) + \int_{x \in [0,1]^n} \frac{\partial_+ \max\{0, \max_{j=1,\ldots,n} \gamma_j(x_j)\}}{\partial \bar{x}_i} dF(x) \tag{40}$$

The first term in (40) appears because the range of integration in (16) over x_i includes the point $\bar{x}_i$ at which the integrand may be discontinuous. The second term comes from differentiating the integrand of $\mathcal{L}$.

Let us now focus on part (a) of the Lemma where $\gamma_i(\bar{x}_i) - \gamma_i^-(\bar{x}_i) > 0$ or, equivalently, $\lambda_i > \frac{(1-F(\bar{x}_i))^2}{1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i)}$. The latter implies that $\gamma_i^-(\bar{x}_i) > 0$.

Then the first term in (40) can be rewritten as follows:

$$f(\bar{x}_i) \int_{x_{-i} \in [0,1]^{n-1} : \max_{j \neq i} \gamma_j(x_j) < \gamma_i(\bar{x}_i)} \gamma_i^-(\bar{x}_i) - \gamma_i(\bar{x}_i) dF(x_{-i}) + f(\bar{x}_i) \int_{x_{-i} \in [0,1]^{n-1} : \gamma_i(\bar{x}_i) \leq \max_{j \neq i} \gamma_j(x_j) < \gamma_i^-(\bar{x}_i)} \left(\gamma_i^-(\bar{x}_i) - \max_{j \neq i} \gamma_j(x_j) \right) dF(x_{-i}) \tag{41}$$

If $\gamma_i(\bar{x}_i) < 0$, then the first term in (40) is strictly positive because in this case the set $A_i(\gamma_i^-(\bar{x}_i), \gamma_i(\bar{x}_i)) = \{x_{-i} \in [0,1]^{n-1} : \gamma_i(\bar{x}_i) \leq \max_{j \neq i} \gamma_j(x_j) < \gamma_i^-(\bar{x}_i)\}$ has a positive measure, and the second term is zero. So $\frac{\partial_+ \mathcal{L}}{\partial \bar{x}_i} > 0$, which cannot hold at the optimum.

Now, consider the case $\gamma_i^-(\bar{x}_i) > \gamma_i(\bar{x}_i) \geq 0$. Let us simplify the second term of (40). Note that (15) implies that $\frac{\partial \gamma_j(x_j)}{\partial \bar{x}_i} = 0$ for $j \neq i$, and

$$\frac{\partial_+ \gamma_i(x_i)}{\partial \overline{x}_i} = \begin{cases} 0, & \text{if } x_i < \overline{x}_i, \\ 1 - \frac{\lambda_i}{(1-F(\overline{x}_i))^2} \left(1 - F(\overline{x}_i) + \overline{x}_i f(\overline{x}_i)\right) = \frac{f(\bar{x}_i)}{1-F(\bar{x}_i)} \left(\gamma_i(\bar{x}_i) - \gamma_i^-(\bar{x}_i)\right), & \text{if } x_i \geq \bar{x}_i, \end{cases} \tag{42}$$

Then by (42) we have $\frac{\partial_+ \gamma_i(x_i)}{\partial \bar{x}_i} < 0$ for $x_i \geq \overline{x}_i$. Therefore, the second term in (40) equals:

$$\int_{x:\, x_i \in [\bar{x}_i,1]: \max_{j\neq i} \gamma_j(x_j) < \gamma_i(\bar{x}_i)} \frac{\partial_+ \gamma_i(x_i)}{\partial \bar{x}_i} dF(x) = \quad f(\bar{x}_i) \int_{x_{-i} \in [0,1]^{n-1}: \max_{j\neq i} \gamma_j(x_j) < \gamma_i(\bar{x}_i)} \gamma_i(\bar{x}_i) - \gamma_i^-(\bar{x}_i) dF(x_{-i}) \tag{43}$$

Summing (41) and (43) yields:

$$\frac{\partial_+ \mathcal{L}}{\partial \bar{x}_i} = f(\bar{x}_i) \int_{x_{-i} \in [0,1]^{n-1}: \gamma_i(\bar{x}_i) \leq \max_{j\neq i} \gamma_j(x_j) < \gamma_i^-(\bar{x}_i)} \left(\gamma_i^-(\bar{x}_i) - \max_{j\neq i} \gamma_j(x_j) \right) dF(x_{-i}) \tag{44}$$

By inspection, (44) is strictly positive - which cannot hold at the optimum- when the set $A_i(\gamma_i^-(\bar{x}_i), \gamma_i(\bar{x}_i)) = \{x_{-i} \in [0,1]^{n-1} : \gamma_i(\bar{x}_i) \leq \max_{j\neq i} \gamma_j(x_j) < \gamma_i^-(\bar{x}_i)\}$ has a positive measure. This completed the proof of Part (a) of the Lemma.

Now, let us consider part (b) of the Lemma. So suppose that $\gamma_i(\bar{x}_i) > \gamma_i^-(\bar{x}_i)$ or, equivalently, $\lambda_i < \frac{(1-F(\bar{x}_i))^2}{1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i)}$. Then $\gamma_i(\overline{x}_i) > 0$, and the first term in (40) is equal to:

$$f(\bar{x}_i) \int_{x_{-i} \in [0,1]^{n-1}: \max_{j\neq i} \gamma_j(x_j) \leq \gamma_i(\bar{x}_i)} \left(\max\{0, \gamma_i^-(\bar{x}_i), \max_{j\neq i} \gamma_j(x_j)\} - \gamma_i(\bar{x}_i) \right) dF(x_{-i}) \tag{45}$$

From (42) it follows that in Part (b) of the Lemma $\frac{\partial_+ \gamma_i(x_i)}{\partial \bar{x}_i} > 0$ if $x_i \geq \bar{x}_i$ and $\frac{\partial_+ \gamma_i(\bar{x}_i)}{\partial \bar{x}_i} = 0$ if $x_i < \bar{x}_i$. Recall that in Part (a) we had $\frac{\partial_+ \gamma_i(x_i)}{\partial \bar{x}_i} < 0$ if $x_i \geq \bar{x}_i$ and $\frac{\partial_+ \gamma_i(\bar{x}_i)}{\partial \bar{x}_i} = 0$ if $x_i < \bar{x}_i$. Therefore, since we also have $\gamma_i(\overline{x}_i) > 0$ in Part (b), the second term of (40) in Part (b) differs from (43) only in the range of integration: in Part (b) this range is $\{x_{-i} \in [0,1]^{n-1} : \max_{j\neq i} \gamma_j(x_j) \leq \gamma_i(\bar{x}_i)\}$, not $\{x_{-i} \in [0,1]^{n-1} : \max_{j\neq i} \gamma_j(x_j) < \gamma_i(\bar{x}_i)\}$ as in Case (a). So, the second term of (40) in Part (b) becomes:

$$f(\bar{x}_i) \int_{x_{-i} \in [0,1]^{n-1}: \max_{j\neq i} \gamma_j(x_j) \leq \gamma_i(\bar{x}_i)} \gamma_i(\bar{x}_i) - \gamma_i^-(\bar{x}_i) dF(x_{-i}) \tag{46}$$

Combining (45) and (46) yields for Part (b):

$$\frac{\partial_+ \mathcal{L}}{\partial \bar{x}_i} = \quad f(\bar{x}_i) \int_{x_{-i} \in [0,1]^{n-1}: \max_{j\neq i} \gamma_j(x_j) \leq \gamma_i(\bar{x}_i)} \max\{0, \gamma_i^-(\bar{x}_i), \max_{j\neq i} \gamma_j(x_j)\} - \gamma_i^-(\bar{x}_i) dF(x_{-i}) \tag{47}$$

By inspection of (47), $\frac{\partial_+ \mathcal{L}}{\partial \bar{x}_i} > 0$ if the set $B_i(\gamma_i^-(\bar{x}_i), \gamma_i(\bar{x}_i)) = \{x_{-i} \in [0,1]^{n-1} : \gamma_i^-(\bar{x}_i) < \max_{j\neq i} \gamma_j(x_j) \leq \gamma_i(\bar{x}_i)\}$ has a positive measure. This completes the proof of the Lemma. *Q.E.D.*

Now we can complete the proof of the Theorem by relying on Lemma 11.

Let us start by establishing part (1) of the Theorem. So, suppose that there exists $h \neq i$ such that $\bar{x}_h \geq \bar{x}_i$. Let us show that $\gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i)$.

First, suppose that $\gamma_i^-(\bar{x}_i) < \gamma_h^-(\bar{x}_h)$. Then we must have $\gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i)$, because $\gamma_i^-(\bar{x}_i) > \gamma_i(\bar{x}_i)$ cannot hold by part (a) of Lemma 11 and $\gamma_i^-(\bar{x}_i) < \gamma_i(\bar{x}_i)$ cannot hold by part (b) of this Lemma.

Next, suppose that $\gamma_i^-(\bar{x}_i) > \gamma_h^-(\bar{x}_h)$. Then, again, Lemma 11 implies that we must have $\gamma_h^-(\bar{x}_h) = \gamma_h(\bar{x}_h) = \bar{x}_h - \frac{\bar{x}_h(1-F(\bar{x}_h))}{1-F(\bar{x}_h)+\bar{x}_h f(\bar{x}_h)}$ or, equivalently, $\lambda_h = \frac{(1-F(\bar{x}_h))^2}{(1-F(\bar{x}_h)+\bar{x}_h f(\bar{x}_h))}$. The inequalities $\bar{x}_h \geq \bar{x}_i$, and $\gamma_i^-(\bar{x}_i) > \gamma_h^-(\bar{x}_h)$ together imply that $\lambda_i > \frac{(1-F(\bar{x}_i))^2}{(1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i))}$. Using the latter inequality and $\bar{x}_h > \bar{x}_i$ we obtain that

$$\gamma_i(\bar{x}_i) < \bar{x}_i - \frac{\bar{x}_i(1-F(\bar{x}_i))}{1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i)} \leq \bar{x}_h - \frac{\bar{x}_h(1-F(\bar{x}_h))}{1-F(\bar{x}_h)+\bar{x}_h f(\bar{x}_h)} = \gamma_h^-(\bar{x}_h) = \gamma_h(\bar{x}_h) < \gamma_i^-(\bar{x}_i).$$

But such configuration cannot be optimal by part (a) of Lemma 11.

It remains to consider the case $\gamma_i^-(\bar{x}_i) = \gamma_h^-(\bar{x}_h)$. From Lemma 11 it follows that $\min\{\gamma_i(\bar{x}_i), \gamma_h(\bar{x}_h)\} = \gamma_i^-(\bar{x}_i) = \gamma_h^-(\bar{x}_h)$. So, to show that $\gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i)$, we only need to rule out $\gamma_i(\bar{x}_i) > \gamma_h(\bar{x}_h) = \gamma_i^-(\bar{x}_i) = \gamma_h^-(\bar{x}_h)$. To argue by contradiction, suppose that this is the case. Then from $\gamma_h^-(\bar{x}_h) = \gamma_h(\bar{x}_h)$ it follows that $\gamma_h^-(\bar{x}_h) = \gamma_h(\bar{x}_h) = \bar{x}_h - \frac{\bar{x}_h(1-F(\bar{x}_h))}{1-F(\bar{x}_h)+\bar{x}_h f(\bar{x}_h)}$ or, equivalently, $\lambda_h = \frac{(1-F(\bar{x}_h))^2}{(1-F(\bar{x}_h)+\bar{x}_h f(\bar{x}_h))}$. The inequality $\bar{x}_h \geq \bar{x}_i$ together with $\gamma_i^-(\bar{x}_i) = \gamma_h^-(\bar{x}_h)$ then imply that $\lambda_i \geq \frac{(1-F(\bar{x}_i))^2}{(1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i))}$. But the latter contradicts $\gamma_i(\bar{x}_i) > \gamma_i^-(\bar{x}_i)$.

Having established part (1) of the Theorem, let us now focus on part (2). So, suppose that bidder $\hat{i}$ is such that $\bar{x}_{\hat{i}} > \bar{x}_j$ for all $j \neq \hat{i}$. By part (i) of the Theorem, it follows that $\gamma_j(\bar{x}_j) = \gamma_j^-(\bar{x}_j)$ for all $j \neq \hat{i}$.

Also, $\min\{\gamma_{\hat{i}}(\bar{x}_{\hat{i}}), \gamma_{\hat{i}}^-(\bar{x}_{\hat{i}})\} \geq \gamma_j(\bar{x}_j) = \gamma_j^-(\bar{x}_j)$ for any $j \neq \hat{i}$. For, if this inequality does not hold then Lemma 11 implies that $\gamma_j(\bar{x}_j) = \gamma_j^-(\bar{x}_j) = \bar{x}_j - \frac{\bar{x}_j(1-F(\bar{x}_j))}{1-F(\bar{x}_j)+\bar{x}_j f(\bar{x}_j)} > \gamma_{\hat{i}}(\bar{x}_{\hat{i}}) = \gamma_{\hat{i}}^-(\bar{x}_{\hat{i}}) = \bar{x}_{\hat{i}} - \frac{\bar{x}_{\hat{i}}(1-F(\bar{x}_{\hat{i}}))}{1-F(\bar{x}_{\hat{i}})+\bar{x}_{\hat{i}} f(\bar{x}_{\hat{i}})}$. But the last inequality cannot hold because $\bar{x}_{\hat{i}} > \bar{x}_j$.

Further, we must have $\max\{\gamma_{\hat{i}}(\bar{x}_{\hat{i}}), \gamma_{\hat{i}}^-(\bar{x}_{\hat{i}})\} > \gamma_j(\bar{x}_j) = \gamma_j^-(\bar{x}_j)$ for any $j \neq \hat{i}$. Otherwise, $\gamma_{\hat{i}}(\bar{x}_{\hat{i}}) = \gamma_{\hat{i}}^-(\bar{x}_{\hat{i}}) = \gamma_{\hat{i}}(\bar{x}_{\hat{i}}) = \gamma_{\hat{i}}^-(\bar{x}_{\hat{i}}) = \bar{x}_{\hat{i}} - \frac{\bar{x}_{\hat{i}}(1-F(\bar{x}_{\hat{i}}))}{1-F(\bar{x}_{\hat{i}})+\bar{x}_{\hat{i}} f(\bar{x}_{\hat{i}})} = \gamma_j(\bar{x}_j) = \gamma_j^-(\bar{x}_j) = \bar{x}_j - \frac{\bar{x}_j(1-F(\bar{x}_j))}{1-F(\bar{x}_j)+\bar{x}_j f(\bar{x}_j)}$. But the latter cannot hold because $\bar{x}_{\hat{i}} > \bar{x}_j$.

Finally, $\gamma_{\hat{i}}^-(\bar{x}_{\hat{i}}) > \gamma_{\hat{i}}(\bar{x}_{\hat{i}}) = \gamma_j(\bar{x}_j) = \gamma_j^-(\bar{x}_j)$ for some j is ruled out by part (a) of Lemma 11, because $\bar{x}_j < 1$ and so in this case the set $A_i(\gamma_i^-(\bar{x}_{\hat{i}}), \gamma_i(\bar{x}_{\hat{i}}))$ has a positive measure.

Thus, we either have $\gamma_{\hat{i}}(\bar{x}_{\hat{i}}) > \gamma_{\hat{i}}^{-}(\bar{x}_{\hat{i}}) = \gamma_j(\bar{x}_j) = \gamma_j^{-}(\bar{x}_j)$ for some j, $j \neq \hat{i}$, or $\min\{\gamma_{\hat{i}}(\bar{x}_{\hat{i}}), \gamma_{\hat{i}}^{-}(\bar{x}_{\hat{i}})\} > \gamma_j(\bar{x}_j) = \gamma_j^{-}(\bar{x}_j)$ for any $j \neq i$. In either case, by Lemma 4 we have $q_{\hat{i}}(x_{\hat{i}}) = 1$ for all $x_{\hat{i}} > \breve{x}_{\hat{i}}$ where $\breve{x}_{\hat{i}}$ is such that $\breve{x}_{\hat{i}} \leq \bar{x}_{\hat{i}}$ and satisfies $\breve{x}_{\hat{i}} - \frac{1-\lambda_{\hat{i}}-F(\breve{x}_{\hat{i}})}{f(\breve{x}_{\hat{i}})} = \max_{j\neq i} \gamma_j(\bar{x}_j)$.

Lemma 4 establishes that the probabilities of trading and hence the mechanism itself are completely determined by the profile of the virtual values $\gamma_j(.)$ for $j \in \{1, ..., n\}$ which, according to (15) depend only on the profiles of the thresholds $\bar{x}_j$ and Lagrange multipliers λ_j, and by the choice of the tie-breaking rule between the bidders with the same threshold $\bar{x}$ when their valuations exceed that threshold.

Therefore, to complete the proof of the Theorem it is sufficient to show that the profiles $(\{\bar{x}_j, \lambda_j\}_{j\neq\hat{i}}, \bar{x}_{\hat{i}}, \lambda_{\hat{i}})$ and $(\{\bar{x}_j, \lambda_j\}_{j\neq\hat{i}}, \breve{x}_{\hat{i}}, \lambda_{\hat{i}})$, together with an arbitrary tie-breaking rule, induce the same probabilities of trading $(q_1(.), ..., q_n(.))$.

The desired conclusion is immediate if $\bar{x}_{\hat{i}} = \bar{x}_j$ for some $j \neq \hat{i}$ since in this case $\bar{x}_{\hat{i}} = \breve{x}_{\hat{i}}$.

Now, suppose that $\bar{x}_{\hat{i}} > \max_{j:j\neq\hat{i}} \bar{x}_j$. Then changing bidder $\hat{i}$'s threshold to $\breve{x}_{\hat{i}}$ from $\bar{x}_{\hat{i}}$ does not affect the virtual value $\gamma_j(.)$ of any player $j \neq \hat{i}$ and also does not affect the virtual value $\gamma_{\hat{i}}(.)$ of player $\hat{i}$ of type $x_{\hat{i}} \in [0, \breve{x}_{\hat{i}})$. However, $\gamma_{\hat{i}}(x_{\hat{i}})$, $x_{\hat{i}} \in [\breve{x}_{\hat{i}}, \bar{x}_{\hat{i}})$, changes from $x_{\hat{i}} - \frac{1-\lambda_{\hat{i}}-F(x_{\hat{i}})}{f(x_{\hat{i}})}$ to $\breve{x}_{\hat{i}} - \frac{\lambda_{\hat{i}}\breve{x}_{\hat{i}}}{1-F(\breve{x}_{\hat{i}})}$, and $\gamma_{\hat{i}}(x_{\hat{i}})$, $x_{\hat{i}} \in [\bar{x}_{\hat{i}}, 1]$, changes from $\bar{x}_{\hat{i}} - \frac{\lambda_{\hat{i}}\bar{x}_{\hat{i}}}{1-F(\bar{x}_{\hat{i}})}$ to $\breve{x}_{\hat{i}} - \frac{\lambda_{\hat{i}}\breve{x}_{\hat{i}}}{1-F(\breve{x}_{\hat{i}})}$.

With threshold $\bar{x}_{\hat{i}}$, we have $q_{\hat{i}}(x_{\hat{i}}) = 1$ for all $x_{\hat{i}} > \breve{x}_{\hat{i}}$, while the tie-breaking rule between i when $x_i = \bar{x}_{\hat{i}}$ and the other bidders can be arbitrary and we can choose it to be $q_{\hat{i}}(\breve{x}_{\hat{i}}) = 1$.

Thus, all that we need to show is that with threshold $\breve{x}_{\hat{i}}$, the virtual value of any type $x_{\hat{i}}$ s.t. $x_i \geq \breve{x}_i$ is strictly greater than $\max_{j\neq\hat{i}} \gamma_j(\bar{x}_j)$ i.e., $\breve{x}_{\hat{i}} - \frac{\lambda_{\hat{i}}\breve{x}_{\hat{i}}}{1-F(\breve{x}_{\hat{i}})} > \max_{j\neq i} \gamma_j(\bar{x}_j)$.

The argument depends on the value of $\lambda_{\hat{i}}$. If $\lambda_{\hat{i}} \geq \frac{(1-F(\bar{x}_{\hat{i}}))^2}{1-F(\bar{x}_{\hat{i}})+\bar{x}_{\hat{i}}f(\bar{x}_{\hat{i}})}$, then $\frac{\partial\left(x_{\hat{i}} - \frac{\lambda_{\hat{i}}x_{\hat{i}}}{1-F(x_{\hat{i}})}\right)}{\partial x_{\hat{i}}} = 1 - \frac{\lambda_{\hat{i}}}{\left(1-F(x_{\hat{i}})\right)^2}\left(1 - F\left(x_{\hat{i}}\right) + x_{\hat{i}} f\left(x_{\hat{i}}\right)\right) \leq 1 - \frac{(1-F(\bar{x}_{\hat{i}}))^2\left(1-F\left(x_{\hat{i}}\right)+x_{\hat{i}}f\left(x_{\hat{i}}\right)\right)}{\left(1-F\left(x_{\hat{i}}\right)\right)^2\left(1-F(\bar{x}_{\hat{i}})+\bar{x}_{\hat{i}}f(\bar{x}_{\hat{i}})\right)} < 0$, where the last inequality relies on the increasing hazard rate assumption. Therefore, $\breve{x}_{\hat{i}} - \frac{\lambda_{\hat{i}}\breve{x}_{\hat{i}}}{1-F(\breve{x}_{\hat{i}})} > \bar{x}_{\hat{i}} - \frac{\lambda_{\hat{i}}\bar{x}_{\hat{i}}}{1-F(\bar{x}_{\hat{i}})} > \max_{j\neq\hat{i}} \gamma_j(\bar{x}_j)$, as required.

Finally, if $\lambda_{\hat{i}} < \frac{(1-F(\bar{x}_{\hat{i}}))^2}{1-F(\bar{x}_{\hat{i}})+\bar{x}_{\hat{i}}f(\bar{x}_{\hat{i}})}$, then $\breve{x}_{\hat{i}} < \bar{x}_{\hat{i}}$ and the increasing hazard rate assumption imply that $\lambda_{\hat{i}} < \frac{(1-F(\breve{x}_{\hat{i}}))^2}{1-F(\breve{x}_{\hat{i}})+\breve{x}_{\hat{i}}f(\breve{x}_{\hat{i}})}$. The last inequality is equivalent to $\frac{1-\lambda-F(\breve{x}_{\hat{i}})}{f(\breve{x}_{\hat{i}})} > \frac{\lambda\breve{x}_{\hat{i}}}{1-F(\breve{x}_{\hat{i}})}$, which again implies the desired result, as we have: $\breve{x}_{\hat{i}} - \frac{\lambda_{\hat{i}}\breve{x}_{\hat{i}}}{1-F(\breve{x}_{\hat{i}})} > \breve{x}_{\hat{i}} - \frac{1-\lambda-F(\breve{x}_{\hat{i}})}{f(\breve{x}_{\hat{i}})} = \max_{j\neq i} \gamma_j(\bar{x}_j)$.

Finally, note that the solution with the threshold value $\breve{x}_{\hat{i}}$ for player $\hat{i}$ is the appropriate one because it satisfies $\breve{x}_{\hat{i}} = \inf\{x_{\hat{i}} | t_{\hat{i}}(x_{\hat{i}}) = m_{\hat{i}}\}$. *Q.E.D.*

Proof of Lemma 5:

Lemma 4 implies that, for fixed $x_{-i} \in [0,1]^{n-1}$, $Q_i(x_i, x_{-i})$ is increasing in x_i if $\gamma_i(x_i)$ is strictly increasing in x_i. Since $q_i(x_i) = \int_{x_{-i}\in[0,1]^{n-1}} Q_i(x_i, x_{-i}) \prod_{j\neq i} dF(x_j)$, it follows that $q_i(x_i)$ is also increasing in x_i if $\gamma_i(x_i)$ is strictly increasing in x_i. Thus, to complete the proof we need to show that $\gamma_i(.)$ is strictly increasing on $[0, \bar{x}_i]$.

By Theorem 1, λ_i satisfies $0 < \lambda_i < 1 - F(\bar{x}_i)$ for all i. Since $\gamma_i(x_i) = x_i - \frac{1-\lambda-F(x_i)}{f(x_i)}$ for $x_i \in [0, \bar{x}_i)$, it is immediate that $\gamma_i'(x_i) > 0$ if $f'(x_i) \geq 0$. If $f'(x_i) < 0$, then $\gamma_i'(x_i) > \frac{d\left(x_i - \frac{1-F(x_i)}{f(x_i)}\right)}{dx_i} > 0$. The last inequality holds by the increasing hazard rate property. So, $\gamma_i(x_i)$ is strictly increasing in x_i for $x_i \in [0, \bar{x}_i)$ and all $i \in \{1, ..., n\}$. Also by Theorem 1, $\gamma_i(\bar{x}_i) > \gamma_i(x_i)$ for all $x_i \in [0, \bar{x}_i)$. *Q.E.D.*

Proof of Lemma 6: To prove the Lemma we argue by contradiction, so suppose that $\bar{x}_j > \bar{x}_i$. Recall that the binding budget constraints of types $\bar{x}_i$ and $\bar{x}_j$ imply that $m_i = \bar{x}_i q_i(\bar{x}_i) - \int_0^{\bar{x}_i} q_i(s)ds$ and $m_j = \bar{x}_j q_j(\bar{x}_j) - \int_0^{\bar{x}_j} q_j(s)ds$. Using the above equations we have:

$$m_j = \bar{x}_j q_j(\bar{x}_j) - \int_0^{\bar{x}_j} q_j(s)ds = \bar{x}_i q_j(\bar{x}_j) + \int_{\bar{x}_i}^{\bar{x}_j} (q_j(\bar{x}_j) - q_j(s))ds - \int_0^{\bar{x}_i} q_j(s)ds \geq \bar{x}_i q_j(\bar{x}_j) - \int_0^{\bar{x}_i} q_j(s)ds \tag{48}$$

Note that the inequality in (48) follows from $\int_{\bar{x}_i}^{\bar{x}_j} (q_j(\bar{x}_j) - q_j(s))ds \geq 0$, and the latter holds because $q_j(s)$ is nondecreasing in s. So we will have established a contradiction to $m_i > m_j$ if we can show both of the following: (a) $q_j(\bar{x}_j) \geq q_i(\bar{x}_i)$; (b) $q_j(s) \leq q_i(s)$ for all $s \in [0, \bar{x}_i]$. Now, (a) holds because, as established in Theorem 1, $\bar{x}_j > \bar{x}_i$ implies that $\gamma_j(\bar{x}_j) > \gamma_i(\bar{x}_i)$. In turn, the latter implies that $q_j(\bar{x}_j) > q_i(\bar{x}_i)$ by Lemma 4.

Finally, to establish that $q_j(s) \leq q_i(s)$ for all $s \in [0, \bar{x}_i]$, note that by Theorem 1, $\bar{x}_j > \bar{x}_i$ implies that $\lambda_j < \lambda_i$. Therefore, $\gamma_j(x) < \gamma_i(x)$, and hence by Lemma 4 $q_j(x) \leq q_i(x)$ for all $x \in [0, \bar{x}_i]$. *Q.E.D.*

Proof of Lemma 7: The inequalities in (18) hold since $q_i(x_i) = \int_{x_{-i}\in[0,1]^{n-1}} Q_i(x_i, x_{-i}) \prod_{j\neq i} dF(x_j)$ and $Q_i(x_i, x_{-i})$ satisfies the properties in Lemma 4.

If $\bar{x}_i \neq \bar{x}_j$ for all $j \neq i$, then the set $\{x_{-i} \in [0,1]^{n-1} : \gamma_i(\bar{x}_i) = \max\{0, \max_{j\neq i} \gamma_j(x_j)\}\}$ has measure zero. Therefore, the left- and right-hand side of (18) are equal to each other.

If $x_i \in [0, \bar{x}_i)$, then the probability of type x_i is zero and so the tie-breaking rule between i and any other bidder j can be chosen arbitrarily. In particualr, by choosing not to award the good to bidder i whenever $\gamma_i(x_i) = \gamma_j(\bar{x}_j)$, we ensure that the first inequality in (18) holds as equality. *Q.E.D.*

Proof of Lemma 8: By Lemma 7, $r_i = \inf\{x_i \in [0,1] | \gamma_i(x_i) > 0\}$. Hence r_i is implicitly defined by the equation $\gamma_i(r_i) = r_i - \frac{1-F(r_i)-\lambda_i}{f(r_i)} = 0$. Substituting the values of λ_i derived in Theorem 1 yields the expression for r_i in equation (19) for $i \in \{2, ..., n\}$ and $i = 1$ when $\bar{x}_1 = \bar{x}_2$ and equation (20) for $i = 1$ when $\bar{x}_1 > \bar{x}_2$.

Using the increasing hazard rate assumption it is easy to check that r_i is well-defined and lies in $(0, \bar{x}_i)$ for all i. Using this assumption and the fact that $\frac{(1-F(\bar{x}_i))^2}{1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i)}$ is decreasing in $\bar{x}_i$ we can also check that r_i defined in (19) is decreasing in $\bar{x}_i$. *Q.E.D.*

Proof of Lemma 9: It is well-known (see e.g. Proposition 1.3.7, page 76, Chapter 1 in Bertsekas (2001)) that the strong duality property holds and (x^*, λ^*) is the solution to both the primal problem, $\max_x \min_\lambda \mathcal{L}(\bar{x}, \lambda)$, and the dual problem, $\min_\lambda \max_x \mathcal{L}(\bar{x}, \lambda)$, if and only if (x^*, λ^*) is a saddle point of the Lagrangian (16) i.e.,

$$\mathcal{L}(x, \lambda^*) \leq \mathcal{L}(x^*, \lambda^*) \leq \mathcal{L}(x^*, \lambda) \tag{49}$$

To establish the existence of a saddle point, we will make use of the Lagrange dual function $g(\lambda) \equiv \max_{\bar{x} \in [0,1]^n} \mathcal{L}(\lambda, \bar{x})$. Note that $g(\lambda) = \mathcal{L}(\lambda, \bar{x}(\lambda))$, where $\bar{x}(\lambda)$ is the solution to the problem $\max_{\bar{x} \in [0,1]^n} \mathcal{L}(\lambda, \bar{x})$, which is characterized in Theorem 1. In particular, $\bar{x}(\lambda)$ is the inverse of the function $\lambda(\bar{x})$ given in the statement of Theorem 1.

By Danskin's Theorem (Bertsekas (2001), Ch. 1, p. 131), the Lagrange dual function $g(\lambda)$ is convex and hence has a unique minimizer which we denote by λ^*. Define $x^* = \bar{x}(\lambda^*)$. Let us show that the saddle-point property (49) holds for the pair (x^*, λ^*).

Since $x^* = \bar{x}(\lambda^*)$, $\mathcal{L}(x, \lambda^*) \leq \mathcal{L}(x^*, \lambda^*)$, holds for all $x \in [0,1]^n$ by Theorem 1.

To show that $\mathcal{L}(\bar{x}(\lambda^*), \lambda^*) \leq \mathcal{L}(\bar{x}(\lambda^*), \lambda)$ we start by arguing that $\mathcal{L}(\bar{x}, \lambda)$ is convex in λ for fixed $\bar{x}$. To see this, recall that as defined in (15), the virtual value function $\gamma_i(x_i)$ is linear in λ_i. Since $\max\{0, \max_i\{\gamma_i(x_i)\}\}$ is convex in $(\gamma_1(x_1), ..., \gamma_n(x_n))$, it follows that $\max\{0, \max_i\{\gamma_i(x_i)\}\}$ is also convex in $(\lambda_1, ..., \lambda_n)$. The integration operator over x preserves convexity of the integrand $\max\{0, \max_i\{\gamma_i(x_i)\}\} \prod_i f(x_i)$ in the parameters $(\lambda_1, ..., \lambda_n)$. Therefore, the Lagrangian $\mathcal{L}(\overline{x}, \lambda)$ is convex in $(\lambda_1, ..., \lambda_n)$.

The convexity of $\mathcal{L}(\bar{x}(\lambda^*), \lambda)$ in λ implies that it has a unique minimum which can be found as a unique solution to the first-order conditions $\frac{\partial \mathcal{L}(\bar{x}(\lambda^*), \lambda+\epsilon h)}{\partial h}_{\epsilon=0} \geq 0$ for all $h \in \mathbf{R}^n$. Again by Danskin's Theorem (Bertsekas (2001), Ch. 1, p. 131), $\frac{\partial \mathcal{L}(\bar{x}(\lambda), \lambda+\epsilon h)}{\partial h}_{\epsilon=0} = \frac{\partial g(\lambda+\epsilon h)}{\partial h}_{\epsilon=0}$ for all λ, h. Since by definition $\lambda^* = \arg\min_\lambda g(\lambda)$, we have $\frac{\partial g(\lambda^*+\epsilon h)}{\partial h}_{\epsilon=0} \geq 0$. So, $\lambda^* = \arg\min_\lambda \mathcal{L}(\bar{x}(\lambda^*), \lambda)$, and hence the second inequality in (49) holds for (x^*, λ^*). This completes the proof that (x^*, λ^*) is a saddle point. *Q.E.D.*

Proof of Theorem 2: By Lemma 9 (strong duality), the solution to our problem can be obtained by minimizing the dual Lagrange function $g(\lambda) \equiv \mathcal{L}(\lambda, \bar{x}^*(\lambda))$ with respect to λ. By Danskin's Theorem (Bertsekas (2001), Ch. 1, p.131), $g(\lambda)$ is convex in λ. So it has a unique minimum attained at λ satisfying the first-order conditions $g'(\lambda; h) \equiv \frac{\partial \mathcal{L}(\lambda+\epsilon h, \bar{x}(\lambda))}{\partial \epsilon}_{\epsilon=0} \geq 0$ for any vector $h \in \mathbf{R}^n$. In the rest of the proof we will focus on these first-order conditions.

To begin with, consider i such that $\bar{x}_i \neq \bar{x}_j$ for any $j \neq i$. In this case, the only variation h in the vector λ that we need to consider to characterize the optimal λ_i involves a change in λ_i only. So we have the following first-order condition:

$$\frac{\partial g(\lambda)}{\partial \lambda_i} = m_i - \bar{x}_i \int_{x_{-i} \in [0,1]^{n-1}: \gamma_i(\bar{x}_i) > \max_{j \neq i} \gamma_j(x_j)} \prod_{j \neq i} dF(x_j)$$
$$+ \int_0^{\bar{x}_i} \int_{x_{-i} \in [0,1]^{n-1}: \gamma_i(s) > \max\{0, \max_{j \neq i} \gamma_j(x_j)\}} \prod_{j \neq i} dF(x_j) ds = m_i - \bar{x}_i q_i(\bar{x}_i) + \int_0^{\bar{x}_i} q_i(s) ds \qquad (50)$$

The second equality in (50) holds by Lemma 7. Thus, we obtain condition (22).

Next suppose that there is a "cluster" $\{k_1, ..., k_l\} \subset \{1, ..., n\}$, with $l \in \{2, ..., n\}$, such that $\bar{x}_{k_1} = ... = \bar{x}_{k_l} = \bar{x}^c \neq \bar{x}_j$ for any $j \notin \{k_1, ..., k_l\}$. Since the threshold $\bar{x}^c$, the corresponding λ^c and the set of bidders in the cluster are all chosen optimally, no variation from it should decrease the value of $g(\lambda)$. Formally, we have to consider all variations of the vector λ, $\epsilon \times \mathbb{I}_J$, where $J \in \{k_1, ..., k_l\}$ is the set of bidders in the "cluster" and $\mathbb{I}_J$ is an n-vector with entries corresponding to bidders in J equal to 1 and other entries equal to zero. Then the following first-order conditions must hold for any J: $\frac{\partial g(\lambda+\epsilon \mathbb{I}_J)}{\partial \epsilon}_{\epsilon=0+} \geq 0$ and $\frac{\partial g(\lambda+\epsilon \mathbb{I}_J)}{\partial \epsilon}_{\epsilon=0-} \leq 0$.

Although there are $2^l - 1$ subsets $J \in \{k_1, ..., k_l\}$, it will be sufficient to consider only $2l$ of them, as will be shown below.

So, let $J = \{k'_1, ..., k'_r\} \subset \{k_1, ..., k_l\}$. Then we have: $\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_J)}{\partial\epsilon}_{|\epsilon=0+} =$

$$\sum_{h=1,...,r} m_{k'_h} + \int_{x:\, \max_{h\in\{1,...,r\}}\gamma_{k'_h}(x_{k'_h})>\max\{0,\max_{j\notin\{k'_1,...,k'_r\}}\gamma_j(x_j)\}} \frac{\partial \max_{h=1,...,r}\gamma_{k'_h}(x_{k'_h})}{\partial\lambda_{k'_h}}_{|\lambda_{k'_h}=\lambda^c} \prod_i dF(x_i)$$

$$= \sum_{h=1,...,r}\left(m_{k'_h} + \int_0^{\bar{x}^c}\int_{x_{-k'_h}:\gamma_{k'_h}(s)>\max\{0,\max_{j\neq k'_h}\gamma_j(x_j)\}} \frac{\partial\gamma_{k'_h}(s)}{\partial\lambda_{k'_h}}_{|\lambda_{k'_h}=\lambda^c}\prod_{j\neq k'_h} dF(x_j)dF(s)\right)$$

$$+ F(\bar{x}^c)^{l-r}\frac{1-F(\bar{x}^c)^r}{1-F(\bar{x}^c)}\int_{\bar{x}^c}^1\int_{x_{-k_1...-k_l}\in[0,1]^{n-l}:\gamma_{k'_1}(\bar{x}^c)>\max_{j\notin\{k_1,...,k_l\}}\gamma_j(x_j))} \frac{\partial\gamma_{k'_1}(s)}{\partial\lambda_{k'_1}}_{|\lambda_{k'_1}=\lambda^c}\prod_{j\notin\{x_{k_1},...,x_{k_l}\}} dF(x_j)dF(s) =$$

$$= \sum_{h=1,...,r} m_{k'_h} + \sum_{h=1,...,r}\int_0^{\bar{x}^c}\int_{x_{-k'_h}\in[0,1]^{n-1}:\gamma_{k'_h}(s)>\max\{0,\max_{j\neq k'_h}\gamma_j(x_j)\}}\prod_{j\neq k'_h} dF(x_j)ds$$

$$-\bar{x}^c F(\bar{x}^c)^{l-r}\frac{1-F(\bar{x}^c)^r}{1-F(\bar{x}^c)}\int_{x_{-k_1...-k_l}\in[0,1]^{n-l}:\gamma_{k'_1}(\bar{x}^c)>\max_{j\notin\{k_1,...,k_l\}}\gamma_j(x_j)}\prod_{j\notin\{x_{k_1},...,x_{k_l}\}} dF(x_j)$$

$$= \sum_{h=1,...,r} m_{k'_h} + r\int_0^{\bar{x}^c} q_{k'_1}(s)ds - \bar{x}^c F(\bar{x}^c)^{l-r}\frac{1-F(\bar{x}^c)^r}{1-F(\bar{x}^c)} Prob[\gamma_{k'_1}(\bar{x}^c) > \max_{j\notin\{k_1,...,k_l\}}\gamma_j(x_j)] \quad (51)$$

The first equality in (51) holds by definition. The second equality uses the properties of the max operator. In particular, the factor $1 - F(\bar{x}^c)^r$ in the last term after the second equality reflects conditioning on the event that at least one of the bidders in $J = \{k'_1, ..., k'_r\}$ has value above $\bar{x}^c$, and the factor $F(\bar{x}^c)^{l-r}$ reflects conditioning on the event that the bidders in $C(\bar{x}^c) \setminus J$ have values below $\bar{x}^c$. We use $\gamma_{k'_1}(s)$ as the integrand in this term, because $\gamma_{k'_1}(s) = \gamma_{k'_h}(s)$ for all $h \in \{1, ..., r\}$.

To obtain the third equality we use the definition (15) and, in particular, $\frac{\partial\gamma_{k'_1}(s)}{\partial\lambda_{k'_1}} = \frac{1}{f(\bar{x}^c)}$ if $s < \bar{x}^c$ and $\frac{\partial\gamma_{k'_1}(s)}{\partial\lambda_{k'_1}} = -\frac{\bar{x}^c}{1-F(\bar{x}^c)}$ if $s > \bar{x}^c$. The final equality uses Lemma 7.

Similarly, we have: $\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_J)}{\partial\epsilon}_{|\epsilon=0-} =$

$$\sum_{h=1,...,r} m_{k'_h} + \int_{x:\ \max_{h\in\{1,...,r\}}\gamma_{k'_h}(x_{k'_h})\geq\max\{0,\max_{j\notin\{k'_1,...,k'_r\}}\gamma_j(x_j)\}} \frac{\partial\max_{h=1,...,r}\gamma_{k'_h}(x_{k'_h})}{\partial\lambda_{k'_h}}_{|\lambda_{k'_h}=\lambda^c} \prod_i dF(x_i)$$

$$= \sum_{h=1,...,r}\left(m_{k'_h} + \int_0^{\bar{x}^c}\int_{x_{-k'_h}:\gamma_{k'_h}(s)\geq\max\{0,\max_{j\neq k'_h}\gamma_j(x_j)\}} \frac{\partial\gamma_{k'_h}(s)}{\partial\lambda_{k'_h}}_{|\lambda_{k'_h}=\lambda^c}\prod_{j\neq k'_h} dF(x_j)dF(s)\right)$$

$$+\frac{1-F(\bar{x}^c)^r}{1-F(\bar{x}^c)}\int_{\bar{x}^c}^1\int_{x_{-k_1...-k_l}\in[0,1]^{n-l}:\gamma_{k'_1}(\bar{x}^c)\geq\max_{j\notin\{k_1,...,k_l\}}\gamma_j(x_j))} \frac{\partial\gamma_{k'_1}(s)}{\partial\lambda_{k'_1}}_{|\lambda_{k'_1}=\lambda^c}\prod_{j\notin\{x_{k_1},...,x_{k_l}\}} dF(x_j)dF(s) =$$

$$= \sum_{h=1,...,r} m_{k'_h} + \sum_{h=1,...,r}\int_0^{\bar{x}^c}\int_{x_{-k'_h}\in[0,1]^{n-1}:\gamma_{k'_h}(s)\geq\max\{0,\max_{j\neq k'_h}\gamma_j(x_j)\}}\prod_{j\neq k'_h} dF(x_j)ds$$

$$-\bar{x}^c\frac{1-F(\bar{x}^c)^r}{1-F(\bar{x}^c)}\int_{x_{-k_1...-k_l}\in[0,1]^{n-l}:\gamma_{k'_1}(\bar{x}^c)\geq\max_{j\notin\{k_1,...,k_l\}}\gamma_j(x_j)}\prod_{j\notin\{x_{k_1},...,x_{k_l}\}} dF(x_j)$$

$$= \sum_{h=1,...,r} m_{k'_h} + r\int_0^{\bar{x}^c} q_{k'_1}(s)ds - \bar{x}^c\frac{1-F(\bar{x}^c)^r}{1-F(\bar{x}^c)}Prob[\gamma_{k'_1}(\bar{x}^c) > \max_{j\notin\{k_1,...,k_l\}}\gamma_j(x_j)] \tag{52}$$

The only difference between $\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_J)}{\partial\epsilon}_{|\epsilon=0-}$ in (52) and $\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_J)}{\partial\epsilon}_{|\epsilon=0+}$ in (51) is that the factor $F(\bar{x}^c)^{l-r}$ in the very last term of $\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_J)}{\partial\epsilon}_{|\epsilon=0+}$ does not appear in the corresponding term of $\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_J)}{\partial\epsilon}_{|\epsilon=0-}$ in (52). This is due to the fact that a negative variation ($\epsilon<0$) in λ^c does increase the value of $\gamma_{k'_h}(x)$ for $x\geq\bar{x}^c$, and so $\max_{h\in\{1,...,l\}}\gamma_{k_h}(x)$ changes irrespective of whether the maximal value among the $l-r$ bidders in the cluster $C(\bar{x}^c)$ who are not in set J is above or below $\bar{x}^c$. On the other hand, a positive variation ($\epsilon>0$) in λ^c does decrease the value of $\gamma_{k'_h}(x)$ for $x\geq\bar{x}^c$, and so the $\max_{h\in\{1,...,l\}}\gamma_{k_h}(x)$ changes only if the maximal value among the other $l-r$ bidders in the cluster is below $\bar{x}^c$. The latter occurs with probability $F(\bar{x}^c)^{l-r}$, the factor in the very last term of $\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_J)}{\partial\epsilon}_{|\epsilon=0+}$.

Note that by Lemma 7 the second term in the equality before last in both (51) and (52) is equal to $r\int_0^{x^c} q_{k'_h}(s)$ for any $h\in\{1,...,r\}$. This is so, in particular, because for almost all $s\in[0,\bar{x}^c)$, the set of $x_{-k'_h}$ such that $\gamma_{k'_h}(s)=\max_{j\neq k'_h}\gamma_j(x_j)$ has measure zero. Therefore, for all $h\in\{1,...,r\}$ we have: $q_{k'_h}(s) = q_{k'_1}(s) = \int_{x_{-k'_h}\in[0,1]^{n-1}:\gamma_{k'_h}(s)\geq\max\{0,\max_{j\neq k'_h}\gamma_j(x_j)\}} dF(x_{-k'_h}) = \int_{x_{-k'_h}\in[0,1]^{n-1}:\gamma_{k'_h}(s)>\max\{0,\max_{j\neq k'-h}\gamma_j(x_j)\}} dF(x_{-k'_h})$. Likewise, because $\bar{x}^c\neq\bar{x}_j$ for any $j\notin\{k_1,...,k_l\}$, we have $Prob[\gamma_{k'_1}(\bar{x}^c)>\max_{j\notin\{k_1,...,k_l\}}\gamma_j(x_j)] = Prob[\gamma_{k'_1}(\bar{x}^c)\geq\max_{j\notin\{k_1,...,k_l\}}\gamma_j(x_j)]$.

Note that the first-order conditions $\frac{\partial g(\lambda+\epsilon\mathbb{I}_J)}{\partial\epsilon}_{\epsilon=0+}\geq 0$ and $\frac{\partial g(\lambda+\epsilon\mathbb{I}_J)}{\partial\epsilon}_{\epsilon=0-}\leq 0$ have to hold for any subset J of the cluster $C(\bar{x}^c)=\{k_1,...,k_l\}$.

In particular, for $J=\{k_1,...,k_l\}$, (51) and (52) yield equation (23) since we have:

$$\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_{\{k_1,...,k_l\}})}{\partial\epsilon}_{|\epsilon=0+}=\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_{\{k_1,...,k_l\}})}{\partial\epsilon}_{|\epsilon=0-}=$$

$$\sum_{h=1,...,l} m_{k_h}+l\int_0^{\bar{x}^c} q_{k_1}(s)ds-\bar{x}^c\frac{1-F(\bar{x}^c)^l}{1-F(\bar{x}^c)}Prob[\gamma_{k_1}(\bar{x}^c)>\max_{j\notin\{k_1,...,k_l\}}\gamma_j(x_j)]=0$$

To obtain (24), note that by (51) $\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_{\{k_{r+1},...,k_l\}})}{\partial\epsilon}_{|\epsilon=0+}$ can be rewritten as:

$$\frac{m_{k_{r+1}}+...+m_{k_l}}{l-r}-\bar{x}^c\frac{F(\bar{x}^c)^r}{l-r}\frac{1-F(\bar{x}^c)^{l-r}}{1-F(\bar{x}^c)}Prob[\gamma_{k_1}(\bar{x}^c)>\max_{j\notin\{k_1,...,k_l\}}\gamma_j(x_j)]\geq-\int_0^{\bar{x}^c} q_{k_1}(s)ds,$$

while by (52) $\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_{\{k_1,...,k_r\}})}{\partial\epsilon}_{|\epsilon=0-}\leq 0$ can be rewritten as follows:

$$\frac{m_{k_1}+...+m_{k_r}}{r}-\bar{x}^c\frac{1}{r}\frac{1-F(\bar{x}^c)^r}{1-F(\bar{x}^c)}Prob[\gamma_{k_1}(\bar{x}^c)>\max_{j\notin\{k_1,...,k_l\}}\gamma_j(x_j)]\leq-\int_0^{\bar{x}^c} q_{k_1}(s)ds.$$

Combining the last two inequalities yields (24) for any $r\in\{2,...,l-1\}$.

To complete the proof and confirm the uniqueness of the solution, let us show that (23) and (24) together imply (51) and (52). Let us fix the size $\#J=r$ of J. By inspection of (51), if it holds for $J=\{k_{l-r+1},...,k_l\}$ including r lowest-budget bidders from $C(\bar{x}^c)$, then (51) also holds for any other J of size r. Likewise, by inspection of (52), if it holds for $J=\{k_1,...,k_r\}$ including r highest-budget bidders from $C(\bar{x}^c)$, then this condition also holds for any other J of size r.

Therefore, it is sufficient to show that (51) holds for the subsets $J=\{k_{l-r+1},...,k_l\}$ of $C(\bar{x}^c)$, $r\in\{2,...,l-1\}$, consisting of the lowest-budget bidders. Similarly, it is sufficient to show that (52) holds for the subsets $J=\{k_1,...,k_{l-r}\}$ consisting of $l-r$ highest budget bidders in the cluster $k_1,...,k_l$, $r\in\{2,...,l-1\}$.

To show that (51) holds for $J=\{k_{l-r+1},...,k_l\}$, combine (23) and (24) to obtain:

$$\frac{\left(m_{k_{l-r+1}}+...+m_{k_l}\right)l}{l-r}\geq-\frac{\bar{x}^c}{1-F(\bar{x}^c)}\left(1-F(\bar{x}^c)^r-rF(\bar{x}^c)^r\frac{1-F(\bar{x}^c)^{l-r}}{l-r}\right)Prob.[\gamma_{k_1}(\bar{x}^c)>\max_{j\notin\{k_1,...,k_l\}}\gamma_j]$$
$$+\sum_{j=1,...,l} m_{k_j}=-\frac{\bar{x}^c}{1-F(\bar{x}^c)}\left(1-F(\bar{x}^c)^r-rF(\bar{x}^c)^r\frac{1-F(\bar{x}^c)^{l-r}}{l-r}\right)Prob.[\gamma_{k_1}(\bar{x}^c)>\max_{j\notin\{k_1,...,k_l\}}\gamma_j]$$
$$+\bar{x}^c\frac{1-F(\bar{x}^c)^l}{1-F(\bar{x}^c)}Prob.[\gamma_{k_1}(\bar{x}^c)>\max_{j\notin\{k_1,...,k_l\}}\gamma_j]-l\int_0^{\bar{x}^c} q_{k_1}(s)ds=$$
$$=\frac{l}{l-r}\bar{x}^cF(\bar{x}^c)^r\frac{1-F(\bar{x}^c)^{l-r}}{1-F(\bar{x}^c)}Prob.[\gamma_{k_1}(\bar{x}^c)>\max_{j\notin\{k_1,...,k_l\}}\gamma_j]-l\int_0^{\bar{x}^c} q_{k_1}(s)ds \qquad (53)$$

The inequality in (53) holds by (24), the first equality holds by (23), the second equality holds by rearrangement. So, (51) holds for $J = \{k_{l-r+1}, ..., k_l\}$.

Now take $J = \{k_1, ..., k_r\}$. Then combining (23) and (24) yields:

$$\frac{(m_{k_1} + ... + m_{k_r})\, l}{r} \leq \bar{x}^c \left((l-r) \frac{1 - F(\bar{x}^c)^r}{(1 - F(\bar{x}^c))\, r} - \frac{F(\bar{x}^c)^r \left(1 - F(\bar{x}^c)^{l-r}\right)}{(1 - F(\bar{x}^c))} \right) Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j \notin \{k_1,...,k_l\}} \gamma_j]$$
$$+ \sum_{j=1,...,l} m_{k_j} = \frac{\bar{x}^c}{1 - F(\bar{x}^c)} \left((l-r) \frac{1 - F(\bar{x}^c)^r}{r} - F(\bar{x}^c)^r \left(1 - F(\bar{x}^c)^{l-r}\right) \right) Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j \notin \{k_1,...,k_l\}} \gamma_j]$$
$$+ \bar{x}^c \frac{1 - F(\bar{x}^c)^l}{1 - F(\bar{x}^c)} Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j \notin \{k_1,...,k_l\}} \gamma_j] - l \int_0^{\bar{x}^c} q_{k_1}(s) ds =$$
$$= \frac{l}{r} \bar{x}^c \frac{1 - F(\bar{x}^c)^r}{1 - F(\bar{x}^c)} Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j \notin \{k_1,...,k_l\}} \gamma_j] - l \int_0^{\bar{x}^c} q_{k_1}(s) ds \tag{54}$$

The inequality in (54) holds by (24). The first equality holds by (23). The second equality holds by rearrangement. So we obtain that (52) holds. *Q.E.D.*

Proof of Theorem 3: **"Only if" Part (Necessity):** Suppose that a top auction with threshold $\bar{x}_1 = ...\bar{x}_n = \bar{x}^t$ and reservation value r_t is an optimal mechanism. We need to show that (30) holds.

By Definition 1, in the top auction $q_i(s) = 0$ for $x_i < r_t$, $q_i(x_i) = F^{n-1}(x_i)$ for $x_i \in [r_t, \bar{x}^t)$, and (27), (28) and (29) hold. Substituting these into (24), which must hold by by Theorem 2, we obtain that (30) holds, as required.

"If" Part (Sufficiency): Suppose that condition (30) holds for all $k \in \{1, ..., n-1\}$ and $\bar{x}^t$ defined by (27). By inspection (27) is equivalent to (23) and (30) is equivalent to (24) in Theorem 2 when $\bar{x}^c = \bar{x}^t$ and the number of bidders in the cluster l is equal to n. Since conditions (23) and (24) are necessary and sufficient for the optimality of a mechanism, we conclude that the top auction is optimal. *Q.E.D.*

Proof of Corollary 2: Since the top auction is the optimal mechanism under both profiles $(m_1, ..., m_n)$ and $(m'_1, ..., m'_n)$ and $\sum_i m_i = \sum_i m'_i = M$, according to (27), the optimal threshold $\bar{x}^t$ is the same in both cases. Hence, by Theorem 1, the Lagrange multipliers of all bidders and in both of these two cases are equal to: $\bar{\lambda}^t = \frac{\left(1 - F\left(\bar{x}^t\right)\right)^2}{(1 - F(\bar{x}^t) + \bar{x}^t f(\bar{x}^t))}$. Therefore, the value of the Lagrangian (16), which gives the seller's expected profits in the mechanism, is the same under these two budget profiles. *Q.E.D.*

Proof of Theorem 4: Theorem 1 shows that the optimal mechanism is uniquely defined by the vector of thresholds $(\bar{x}_1, ..., \bar{x}_n)$. By Theorem 3, the failure of (30) implies that we cannot have $\bar{x}_1 = ... = \bar{x}_n$ in the optimal mechanism. By Lemma 6, $\bar{x}_1 \geq ... \geq \bar{x}_n$. So $\bar{x}_i > \bar{x}_{i+1}$ for some i.

So take bidders i and j such that $\bar{x}_i > \bar{x}_j$ (which implies that $m_i > m_j$). By Theorem 1, $\lambda_i < \lambda_j$, and $\gamma_i(x) < \gamma_j(x)$ for $x \in [0, \bar{x}_j]$. Therefore by Lemma 8 $r_i > r_j$, and by Lemma 7 $q_i(x) < q_j(x)$ for all $x \in [r_j, \bar{x}_j]$.

The last claim of the Theorem follows from Lemma 6. *Q.E.D.*

Proof of Corollary 3: By assumption, if bidder i does not belong to a cluster under $(m_1, ..., m_n)$, i.e. $\bar{x}_i \neq \bar{x}_j$ for all $i \neq j$, then $m_i = m'_i$. Also, if under $(m_1, ..., m_n)$ bidders i, $i+1$,...,$i+l$ constitute a cluster with a common threshold $\bar{x}^c$, then $\sum_{j=i}^{i+l} m_j = \sum_{j=i}^{i+l} m'_j$. So, if we assign the same profile of thresholds $(\bar{x}_1, ..., \bar{x}_n)$ to the bidders with budgets $(m'_1, ..., m'_n)$, then conditions (22) and (23) of Theorem 2 would still hold. To confirm that the threshold profile $(\bar{x}_1, ..., \bar{x}_n)$ remains optimal under $(m'_1, ..., m'_n)$, it remains to verify that inequalities in (24) also still hold. This follows because: (i) (24) holds for $(m_1, ..., m_n)$ and $(\bar{x}_1, ..., \bar{x}_n)$; (ii) the right-hand side of (24) depends only on $(\bar{x}_1, ..., \bar{x}_n)$ and hence remains unchanged; (iii) the left-hand side of the family of inequalities (24) depends only on the budgets; (iv) $|m_i - m'_i|$ is sufficiently small for all i.

Next, Theorem 1 implies that the same Lagrange multiplier is associated with bidder i under both budget profiles $(m_1, ..., m_n)$ and $(m'_1, ..., m'_n)$. Therefore, the value of the Lagrangian (16), which gives the seller's expected profits in the mechanism, is the same under these two budget profiles. *Q.E.D.*

Proof of Lemma 10: Since the bidders' valuations are identically distributed, the seller's revenue function $\pi(m_1, ..., m_n)$ is exchangeable i.e., $\pi(m_1, ..., m_n) = \pi(P(m_1, ..., m_n))$ where $P(m_1, ..., m_n)$ is a permutation of $(m_1, ..., m_n)$. Let PM^m denote the set of permutations of $(m_1, ..., m_n)$. Its cardinality (the total number of permutations) is equal to $n!$.

Fixing a budget profile $(m_1, ..., m_n)$ such that $\sum_i m_i = M$, by concavity of $\pi(.)$ we obtain:

$$\pi\left(\frac{1}{M}, ..., \frac{1}{M}\right) \geq \sum_{P \in PM^m} \frac{\pi(P)}{\#PM^m} = \pi(m_1, ..., m_n)$$

To prove the second statement of the Lemma, let $\sum_{i=1}^n m_i = \sum_{i=1}^n m_i' = M$. We will construct a procedure that consists of a sequence of permutations of the budget profile $(m_1, ..., m_n)$ and linear combinations of the permuted profiles that starts with $(m_1, ..., m_n)$ and ends with $(m_1', ..., m_n')$, so that seller's revenue increases at every step.

First, assume that $\sum_{j=i}^n m_j < \sum_{j=i}^n m_i'$ for all $i \in \{2, ..., m\}$. For otherwise i.e., if there exists $i \in \{2, ..., m\}$ such that $\sum_{j=i}^n m_j = \sum_{j=i}^n m_i'$, then we also have $\sum_{j=1}^{i-1} m_j = \sum_{j=1}^{i-1} m_i'$. Then we will apply our procedure separately to $(m_1, ..., m_{i-1})$ to arrive at a budget profile $(m_1', ..., m_{i-1}', m_i, ..., m_n)$ and show that $\pi(m_1', ..., m_{i-1}', m_i, ..., m_n) \geq \pi(m_1, ..., m_n)$. Further, we will apply our procedure to budget profile $(m_1', ..., m_{i-1}', m_i, ..., m_n)$ to arrive at $(m_1', ..., m_n')$ and show that $\pi(m_1', ..., m_n') \geq \pi(m_1', ..., m_{i-1}', m_i, ..., m_n)$.

In step 1, consider two budget profiles $(m_1, ..., m_n)$ and $(m_n, m_2, ..., m_{n-1}, m_1)$. Let $\lambda_1^1 \in (0, 1]$ be such that $\lambda_1^1 m_1 + (1-\lambda_1^1) m_n = m_1'$. Note that λ_1^1 exists because $m_1 \geq m_1' > \frac{M}{n} > m_n$. Also, let $\lambda_1^2 = \max \lambda \in [0, 1]$ s.t. there exists $\tilde{i} \in \{3, ..., n\}$ satisfying $\sum_{j=\tilde{i}}^{n-1} m_j + \lambda m_n + (1 - \lambda) m_1 = \sum_{j=\tilde{i}}^n m_j'$.

Next, set $\lambda_1 = \max\{\lambda_1^1, \lambda_1^2\} \in (0, 1)$ and consider a budget profile $(m_1^2, ..., m_n^2) = \lambda_1(m_1, ..., m_n) + (1 - \lambda_1)(m_n, m_2, ..., m_{n-1}, m_1)$. Note that by concavity of $\pi(.)$, we have: $\pi(m_1^2, ..., m_n^2) \geq \lambda_1 \pi(m_1, ..., m_n) + (1 - \lambda_1)\pi(m_n, m_2, ..., m_{n-1}, m_1) = \pi(m_1, ..., m_n)$.

If $\lambda_1 = \lambda_1^1 \geq \lambda_1^2$, then $m_1^2 = m_1'$ and $\sum_{j=i}^n m_j^2 \leq \sum_{j=i}^n m_j'$ for all $i \in \{2, ..., n\}$. So, we can now proceed to Step 2 in which we apply the same method to a shorter budget profile $(m_2^2, ..., m_n^2)$ keeping m_1^2 constant in Step 2 and all subsequent steps.

If $\lambda_1 = \lambda_1^2 > \lambda_1^1$, then there exists $\tilde{i} \in \{3, ..., n\}$ such that $\sum_{j=\tilde{i}}^n m_j^2 = \sum_{j=\tilde{i}}^n m_j'$. So, we also have $\sum_{j=1}^{\tilde{i}-1} m_j^2 = \sum_{j=1}^{\tilde{i}-1} m_j'$.

At the same time, for all $i > \tilde{i}$, we have $\sum_{j=i}^n m_j^2 \leq \sum_{j=\tilde{i}}^n m_j'$, and for all $k \in \{2, ..., \tilde{i}-1\}$, $\sum_{j=k}^{\tilde{i}-1} m_j^2 < \sum_{j=k}^{\tilde{i}-1} m_j'$. So, we can now apply the same method as in Step 1 separately, to shorter budget profiles $(m_1^2, ..., m_{\tilde{i}-1}^2)$ leaving the rest of the budget profile constant $(m_{\tilde{i}}^2, ..., m_n^2)$ until we arrive iteratively at the budget profile $(m_1', ..., m_{\tilde{i}-1}', m_{\tilde{i}}^2, ..., m_n^2)$. Then we will apply our procedure separately to the budget profile $(m_{\tilde{i}}^2, ..., m_n^2)$ and iteratively arrive at the budget profile $(m'_1, ..., m'_n)$. Since at each step $\pi(.)$ weakly increases, out proof is complete. *Q.E.D.*

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Online Appendix (Not for Publication)

Computations of the Budget-Handicap Auction with Three Bidders, under Uniform Type Distribution.

8.0.1 Budget-Handicap Auction with Top cluster

Since $\bar{x}_1 = \bar{x}_2$ in the top cluster, we will simplify the notation and let $\bar{x}_1$ denote the threshold of bidders 1 and 2 in the rest of this subsection. So, we have $\bar{x}_1 > \bar{x}_3$, $\gamma_1(x) = \gamma_2(x) = 2x - 2\bar{x}_1 + \bar{x}_1^2$ for $x < \bar{x}_1$, $\gamma_1(\bar{x}_1) = \gamma_2(\bar{x}_1) = \bar{x}_1^2$; $\gamma_3(x) = 2x - 2\bar{x}_3 + \bar{x}_3^2$ for $x < \bar{x}_3$, $\gamma_3(\bar{x}_3) = \bar{x}_3^2$. The bidders' reservation values are given by $r_1 = r_2 = \bar{x}_1 - \frac{\bar{x}_1^2}{2}$, $r_3 = \bar{x}_3 - \frac{\bar{x}_3^2}{2}$.

Then by Lemma 7 for $i \in \{1, 2\}$, $q_i(x) = 0$ for $x < \bar{x}_1 - \frac{\bar{x}_1^2}{2}$, $q_i(x) = x(x - \bar{x}_1 + \frac{\bar{x}_1^2}{2} + \bar{x}_3 - \frac{\bar{x}_3^2}{2})$ for $x \in (\bar{x}_1 - \frac{\bar{x}_1^2}{2}, \bar{x}_1 - \frac{\bar{x}_1^2}{2} + \frac{\bar{x}_3^2}{2}]$, and $q_i(x) = x$ for $x \in (\bar{x}_1 - \frac{\bar{x}_1^2}{2} + \frac{\bar{x}_3^2}{2}, \bar{x}_1)$. The values of $q_1(\bar{x}_1)$ and $q_2(\bar{x}_1)$ are determined by the budget constraints of bidders 1 and 2.

For bidder 3, we have $q_3(x) = 0$ for $x < \bar{x}_3 - \frac{\bar{x}_3^2}{2}$, $q_3(x) = \left(x - \bar{x}_3 + \frac{\bar{x}_3^2}{2} + \bar{x}_1 - \frac{\bar{x}_1^2}{2}\right)^2$ for $x \in (\bar{x}_3 - \frac{\bar{x}_3^2}{2}, \bar{x}_3)$, and $q_3(\bar{x}) = \left(\frac{\bar{x}_3^2}{2} + \bar{x}_1 - \frac{\bar{x}_1^2}{2}\right)^2$.

Note that while $q_3(x)$ is continuous everywhere above r_3, $q_1(x)$ and $q_2(x)$ experience two jumps. First, there is a jump at $\bar{x}_1 - \frac{\bar{x}_1^2}{2} + \frac{\bar{x}_3^2}{2}$, as bidders 1 and 2 with values above this level no longer face the competition from bidder 3 because $\gamma_1(\bar{x}_1 - \frac{\bar{x}_1^2}{2} + \frac{\bar{x}_3^2}{2}) = \gamma_3(\bar{x}_3)$. The second jump happens at the threshold $\bar{x}_1$, since $\lim_{x \to \bar{x}-} q_1(x) + q_2(x) = 2\bar{x} < 1 + \bar{x} = q_1(\bar{x}) + q_2(\bar{x})$.

By Theorem 2 (conditions (22)-(24)), the budget-handicap auction with a top cluster is optimal if the following system of two equations and one inequality has a solution:

$$m_3 = \bar{x}_3 q_3(\bar{x}_3) - \int_{\bar{x}_3 - \frac{\bar{x}_3^2}{2}}^{\bar{x}_3} q_3(x_3) dx_3 \tag{55}$$

$$m_1 + m_2 = (1 + \bar{x}_1) - 2 \int_{\bar{x}_1 - \frac{\bar{x}_1^2}{2}}^{\bar{x}_1} q_1(x_1) dx_1 \tag{56}$$

$$m_1 - m_2 \leq \bar{x}_1(1 - \bar{x}_1).$$

Using the expressions for $q_i(x)$, $i \in \{1,2,3\}$ in (55) and (56) yields:

$$m_3 = \bar{x}_3\left(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_1^2}{2}\right)^2 - \int_{\bar{x}_3 - \frac{\bar{x}_3^2}{2}}^{\bar{x}_3}\left(s - \bar{x}_3 + \bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_1^2}{2}\right)^2 ds = \bar{x}_3\left(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_1^2}{2}\right)^2 - \frac{\left(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_1^2}{2}\right)^3}{3} + \frac{\left(\bar{x}_1 - \frac{\bar{x}_1^2}{2}\right)^3}{3} = -\frac{\bar{x}_3^6}{24} + \frac{\bar{x}_3^5}{4} + \bar{x}_3^3\left(1 - \frac{\bar{x}_3}{4}\right)\left(\bar{x}_1 - \frac{\bar{x}_1^2}{2}\right) + \left(\bar{x}_3 - \frac{\bar{x}_3^2}{2}\right)\left(\bar{x}_1 - \frac{\bar{x}_1^2}{2}\right)^2 \tag{57}$$

$$\begin{aligned} m_1 + m_2 &= \bar{x}_1(1+\bar{x}_1) - 2\int_{\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_1^2}{2}}^{\bar{x}_1} y\,dy - 2\int_{\bar{x}_1 - \frac{\bar{x}_1^2}{2}}^{\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_1^2}{2}} y\left(y - \bar{x}_1 + \bar{x}_3 + \frac{\bar{x}_1^2}{2} - \frac{\bar{x}_3^2}{2}\right))dy \\ &= \bar{x}_1(1+\bar{x}_1) + \frac{\bar{x}_3^4}{4}\left(1 - \bar{x}_3 + \frac{\bar{x}_3^2}{6}\right) - \bar{x}_1^3\left(1 - \frac{\bar{x}_1}{4}\right) + \left(\bar{x}_1 - \frac{\bar{x}_1^2}{2}\right)\bar{x}_3^2\left(1 - \frac{\bar{x}_3}{2}\right)^2 \end{aligned} \tag{58}$$

Equations (57) and (58) implicitly define $\bar{x}_1$ and $\bar{x}_3$. If the solution is such that $m_1 - m_2 \leq \bar{x}_1(1-\bar{x}_1)$, then the optimal mechanism is a handicap auction with a "top cluster." The set of budgets for which this is true is depicted in Figure 4.

8.0.2 Lower cluster

Next, consider the case of the "lower cluster" with $\bar{x}_1 > \bar{x}_2 = \bar{x}_3$. To simplify the presentation, we let $\bar{x}_2$ denote the threshold of bidders 2 and 3 and drop $\bar{x}_3$ from the notation.

Then we have: $\gamma_1(x_1) = 2x - 2\bar{x}_1 + \bar{x}_2^2$ for $x_1 < \bar{x}_1$, $\gamma_1(\bar{x}_1) > \gamma_1^-(\bar{x}_1) = \frac{\bar{x}_2^2}{2}$, $\gamma_2(x) = \gamma_3(x) = 2x - 2\bar{x}_2 + \bar{x}_2^2$ for $x < \bar{x}_2$, $\gamma_2(\bar{x}_2) = \gamma_3(\bar{x}_2) = \bar{x}_2^2$. The reservation values are $r_1 = \bar{x}_1 - \frac{\bar{x}_2^2}{2}$ and $r_2 = r_3 = \bar{x}_2 - \frac{\bar{x}_2^2}{2}$.

The probabilities of trading are given by: $q_1(x_1) = 0$ for $x_1 < \bar{x}_1 - \frac{\bar{x}_2^2}{2}$, $q_1(x_1) = (x_1 - \bar{x}_1 + \bar{x}_2)^2$ for $x_1 \in \left[\bar{x}_1 - \frac{\bar{x}_2^2}{2}, \bar{x}_1\right)$, $q_1(\bar{x}_1) = 1$. For $i \in \{2,3\}$, $q_i(x) = 0$ for $x < \bar{x}_2 - \frac{\bar{x}_2^2}{2}$, and $q_i(x) = x\left(x - \bar{x}_2 + \bar{x}_1\right)$ for $x \in \left[\bar{x}_2 - \frac{\bar{x}_2^2}{2}, \bar{x}_2\right)$. Finally, $q_2(\bar{x}_2)$ and $q_3(\bar{x}_2)$ are determined by the budget constraints of bidders 2 and 3, correspondingly.

By Theorem 2, condition (22) must hold for bidder 1 and conditions (23) and (24) must hold for bidders 2 and 3 i.e.:

$$m_1 = \bar{x}_1 - \int_{\bar{x}_1 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_1} (s - \bar{x}_1 + \bar{x}_2)^2\, ds = \bar{x}_1 - \frac{\bar{x}_2^3}{3} + \frac{\left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)^3}{3}$$

$$= \bar{x}_1 - \frac{\bar{x}_2^2}{6}\left(\bar{x}_2^2 + \bar{x}_2\left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right) + \left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)^2\right) = \bar{x}_1 - \frac{\bar{x}_2^4}{2}\left(1 - \frac{\bar{x}_2}{2} + \frac{\bar{x}_2^2}{12}\right) \tag{59}$$

$$m_2 + m_3 = \bar{x}_2\bar{x}_1(1+\bar{x}_2) - 2\int_{\bar{x}_2 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_2} s\,(s - \bar{x}_2 + \bar{x}_1)\, ds = \bar{x}_1\bar{x}_2(1+\bar{x}_2) - \frac{2\bar{x}_2^3}{3} + \frac{2\left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)^3}{3}$$

$$-(\bar{x}_1 - \bar{x}_2)\left(\bar{x}_2^2 - \left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)^2\right) = \bar{x}_1\bar{x}_2(1+\bar{x}_2) + \frac{\bar{x}_2^5}{4}\left(1 - \frac{\bar{x}_2}{3}\right) - \bar{x}_2^3\bar{x}_1\left(1 - \frac{\bar{x}_2}{4}\right) \tag{60}$$

$$m_2 - m_3 \leq \bar{x}_2(1 - \bar{x}_2)\bar{x}_1 \tag{61}$$

Equations (59) and (60) implicitly define $\bar{x}_1$ and $\bar{x}_2$. If the solution satisfies (61), the optimal mechanism is the handicap auction with the lower cluster and thresholds $\bar{x}_1$ and $\bar{x}_2 = \bar{x}_3$. The set of budgets for which this is true is depicted in Figure 5.

8.0.3 No Clusters.

Finally, we consider the case with no clusters i.e., $\bar{x}_1 > \bar{x}_2 > \bar{x}_3$.

In this case, $\gamma_1(x_1) = 2x - 2\bar{x}_1 + \bar{x}_2^2$ for $x_1 < \bar{x}_1$, $\gamma_1(\bar{x}_1) > \gamma_1^-(\bar{x}_1) = \frac{\bar{x}_2^2}{2}$, $\gamma_2(x) = 2x - 2\bar{x}_2 + \bar{x}_2^2$ for $x < \bar{x}_2$, $\gamma_2(\bar{x}_2) = \bar{x}_2^2$, $\gamma_3(x) = 2x - 2\bar{x}_3 + \bar{x}_3^3$ for $x < \bar{x}_3$, $\gamma_3(\bar{x}_3) = \bar{x}_3^2$. The reservation values are $r_1 = \bar{x}_1 - \frac{\bar{x}_2^2}{2}$, $r_2 = \bar{x}_2 - \frac{\bar{x}_2^2}{2}$, and $r_3 = \bar{x}_3 - \frac{\bar{x}_3^2}{2}$.

Therefore, the probabilities of trading of bidder 1 are as follows: $q_1(x) = 0$ for $x < \bar{x}_1 - \frac{\bar{x}_2^2}{2}$, $q_1(x) = (x - \bar{x}_1 + \bar{x}_2)\left(x - \bar{x}_1 + \bar{x}_3 + \frac{\bar{x}_2^2}{2} - \frac{\bar{x}_3^2}{2}\right)$ for $x \in \left[\bar{x}_1 - \frac{\bar{x}_2^2}{2}, \bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right]$, $q_1(x) = x - \bar{x}_1 + \bar{x}_2$ for $x \in \left(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}, \bar{x}_1\right)$, and $q_1(\bar{x}_1) = 1$.

For bidder 2, $q_2(x) = 0$ for $x < \bar{x}_2 - \frac{\bar{x}_2^2}{2}$, $q_2(x) = (x - \bar{x}_2 + \bar{x}_1)\left(x - \bar{x}_2 + \bar{x}_3 + \frac{\bar{x}_2^2}{2} - \frac{\bar{x}_3^2}{2}\right)$ for $x \in \left[\bar{x}_2 - \frac{\bar{x}_2^2}{2}, \bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right]$, $q_2(x) = x - \bar{x}_2 + \bar{x}_1$ for $x \in \left(\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}, \bar{x}_2\right)$, $q_2(\bar{x}_2) = \bar{x}_1$.

Finally, for bidder 3, $q_3(x) = 0$ for $x < \bar{x}_3 - \frac{\bar{x}_3^2}{2}$, $q_3(x) = \left(x - \bar{x}_3 + \bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)\left(x - \bar{x}_3 + \bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)$ for $x \in \left[\bar{x}_3 - \frac{\bar{x}_3^2}{2}, \bar{x}_3\right)$, and $q_3(\bar{x}_3) = (\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2})(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2})$.

By Theorem 2, in the "no cluster" case the necessary and sufficient conditions characterizing the optimal thresholds $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$ are the budget constraints (22) i.e., $m_i =$

$\bar{x}_i q_i(\bar{x}_i) - \int_{r_i}^{\bar{x}_i} q_i(s)ds$ for $i = 1, 2, 3$. If the solution to this system of three equations exists and is such that $1 \geq \bar{x}_1 > \bar{x}_2 > \bar{x}_3 \geq 0$, then we have an optimal mechanism with no clusters.

In the rest of this subsection, we will exhibit the system of three equations $m_i = \bar{x}_i q_i(\bar{x}_i) - \int_{r_i}^{\bar{x}_i} q_i(s)ds$ for $i = 1, 2, 3$ explicitly using the expressions for $q_i(.)$ above and then replace it with a simpler system. First, consider $i = 1$. We have:

$$m_1 = \bar{x}_1 - \int_{\bar{x}_1 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}} (x - \bar{x}_1 + \bar{x}_2)\left(x - \bar{x}_1 + \bar{x}_3 + \frac{\bar{x}_2^2}{2} - \frac{\bar{x}_3^2}{2}\right) dx - \int_{\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}}^{\bar{x}_1} x - \bar{x}_1 + \bar{x}_2 ds =$$
$$\bar{x}_1 - \frac{\left(\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)^3}{3} + \frac{\left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)^3}{3} + \frac{\left(\bar{x}_2 - \bar{x}_3 - \frac{\bar{x}_2^2}{2} + \frac{\bar{x}_3^2}{2}\right)}{2}\left(\left(\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)^2 - \left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)^2\right)$$
$$-\frac{\bar{x}_2^2}{2} + \frac{\left(\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)^2}{2} = \bar{x}_1 + \frac{\bar{x}_3^4}{8}\left(1 - \bar{x}_3 + \frac{\bar{x}_3^2}{6}\right) - \frac{\bar{x}_2^3}{2}\left(1 - \frac{\bar{x}_2}{4}\right) + \left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)\frac{\bar{x}_3^2}{2}\left(1 - \frac{\bar{x}_3}{2}\right)^2 \tag{62}$$

Second, using the expressions for $q_2(.)$ and $q_3(.)$ derived above, we obtain:

$$m_2 = \bar{x}_2\bar{x}_1 - \int_{\bar{x}_2 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}} (x - \bar{x}_2 + \bar{x}_1)\left(x - \bar{x}_2 + \bar{x}_3 + \frac{\bar{x}_2^2}{2} - \frac{\bar{x}_3^2}{2}\right) dx - \int_{\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}}^{\bar{x}_2} x - \bar{x}_2 + \bar{x}_1 ds \tag{63}$$

$$m_3 = \bar{x}_3\left(\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)\left(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right) - \int_{\bar{x}_3 - \frac{\bar{x}_3^2}{2}}^{\bar{x}_3} (x - \bar{x}_3 + \bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2})(x - \bar{x}_3 + \bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2})dx \tag{64}$$

Next, we replace (63) and (64) with the equations for $m_1 - m_2$ and $m_2 - m_3$ as follows. First, subtracting (63) from (62) we obtain:

$$m_1 - m_2 = \bar{x}_1(1 - \bar{x}_2) + \int_{\bar{x}_2 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}} (\bar{x}_1 - \bar{x}_2)(x - \bar{x}_2 + \bar{x}_3 + \frac{\bar{x}_2^2}{2} - \frac{\bar{x}_3^2}{2})dx + \int_{\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}}^{\bar{x}_2} \bar{x}_1 - \bar{x}_2 ds$$
$$= \bar{x}_1(1 - \bar{x}_2) + \frac{\bar{x}_1 - \bar{x}_2}{2}\left(\bar{x}_2^2 - \left(\bar{x}_3 - \frac{\bar{x}_3^2}{2}\right)^2\right). \tag{65}$$

Finally, we perform a change of variable of integration in the second term of (63) to $y =$

$x - \bar{x}_2 + \frac{\bar{x}_2^2}{2} + \bar{x}_3 - \frac{\bar{x}_3^2}{2}$ and subtract (64) from the result to obtain:

$$m_2 - m_3 = \bar{x}_1\bar{x}_2 - \frac{\bar{x}_1^2}{2} + \frac{\left(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)^2}{2} - \bar{x}_3\left(\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)\left(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)$$
$$+ \int_{\bar{x}_3 - \frac{\bar{x}_3^2}{2}}^{\bar{x}_3}\left(x - \bar{x}_3 + \bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)\left(\bar{x}_2 - \frac{\bar{x}_2^2}{2} - \bar{x}_3 + \frac{\bar{x}_3^2}{2}\right)dx =$$
$$\bar{x}_1\bar{x}_2 + (\bar{x}_2\bar{x}_3 - \bar{x}_1(1 - \bar{x}_3))\frac{\bar{x}_2^2 - \bar{x}_3^2}{2} + \left(\frac{1}{2} - \bar{x}_3\right)\left(\frac{\bar{x}_2^2}{2} - \frac{\bar{x}_3^2}{2}\right)^2 + \frac{\bar{x}_3^2}{2}\left(\bar{x}_2 - \frac{\bar{x}_2^2}{2} - \bar{x}_3 + \frac{\bar{x}_3^2}{2}\right)\left(\bar{x}_1 + \frac{\bar{x}_3^2}{4} - \frac{\bar{x}_2^2}{2}\right) \quad (66)$$

To conclude, when the solution to the system (62), (65) and (66) satisfies $\bar{x}_1 > \bar{x}_2 > \bar{x}_3$, this is the optimal mechanism. The set of budgets for this case is depicted in Figure 5.