Optimal and Efficient Mechanisms with Asymmetrically Budget Constrained Buyers*

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This draft: October 30 2018

Abstract

The paper characterizes both the optimal (revenue-maximizing) and constrained-efficient (surplus maximizing) mechanisms for allocating a good to buyers who face asymmetric budget constraints. Both the optimal and efficient mechanisms belong to one of two classes. When the budget differences between the buyers are small, the mechanism discriminates only between high-valuation types for whom the budget constraint is binding. All low valuations buyers are treated symmetrically despite budget differences. When budget differences are sufficiently large, the mechanism discriminates in favor of buyers with small budgets when the valuations are low, and in favor of buyers with larger budgets when the valuations are high.
JEL codes: D44, D82

*We thank the colleagues at UBC and Higher School of Economics, seminar participants at Paris School of Economics, Bocconi University, European University Institute, University of Toronto, Cergy-Pontoise University and numerous conferences for helpful comments and suggestions. Sergei Severinov thanks Social Sciences and Humanities Research Council of Canada for support via grant # 435-2017-0540. A standard disclaimer applies.

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1 Introduction

This paper deals with mechanism design when buyers are budget constrained. Budget constraints often affect participants in trading mechanisms and institutions. In particular, consumers typically face wealth and liquidity constraints which reduce their ability to pay for the goods, especially for big-ticket items like houses and cars. In the keyword search auctions on the internet search platforms (Google, Microsoft's Bing), the advertisers typically face spending limits set by the senior management. The economists have pointed out that budget constraints are an important practical matter affecting bidding and outcomes in spectrum auctions. See, in particular, Rothkopf (2007) and Bulow, Milgrom and Levin (2009). So it is natural that budget constraints should be taken into account in the analysis and design of trading mechanisms and institutions, and there is now a growing literature exploring the implication of budgets constraints in these contexts. With some notable exceptions, discussed below, this literature focuses on the analysis of specific institutions such as different forms of auctions.

In contrast, this paper deals with the design of an optimal mechanism maximizing the seller's revenue and a constrained efficient mechanism maximizing the social surplus. We consider a setting in which several buyers compete for a single good and the seller acts as a mechanism designer. The buyers have private values and commonly known and unequal budgets.

There are several real-world environments in which the bidders' budgets are typically known by the seller and other bidders. First, in large-scale privatization auctions of state assets in Eastern Europe and other countries, as well as in the auctions of publicly-owned stakes in corporations or tracts of natural resources, the bidders are/were typically large corporations whose financial resources were fairly well-known, or could be estimated fairly precisely from their financial and other reports. Alternatively, the sellers of high-value assets may and often do require the bidders to qualify by disclosing their resources and financial situations. In a somewhat different domain, a number of professional sports leagues in North America such as NHL and NFL have salary caps. So when the teams bid for players, their maximal budgets are the available room under their salary caps that are publicly known.

In this paper we characterize both the optimal i.e., revenue-maximizing, and constrained-efficient i.e., surplus maximizing, mechanisms. The motivation for studying optimal mechanisms is broadly recognized in the literature. Designing the most efficient mechanism is also an important objective.[1] Attaining efficiency is especially problematic and full efficiency

[1] A notable example is the privatization of government-controlled assets. Maskin (1992) points out that

often cannot be attained when the buyers have limited budgets, because in this case their willingness to pay cannot be fully translated into their bids.

The optimal and constrained-efficient mechanisms that we derive share a number of interesting and novel qualitative properties. Since these properties are common to both of these mechanisms, we simplify the presentation by focussing on the optimal mechanism below. The constrained-efficient mechanism is presented separately in section 7.

To attenuate the effect of budget asymmetry we will provide main characterization results for the case of identically distributed valuations. However, we show how our results generalize to the environment with asymmetrically distributed valuations in Appendix B at the end of the paper. An important implication of the budget asymmetry is that the designer has to construct ex-ante asymmetric allocation profiles (probability of trading and transfer function), one for each buyer, and do so in a consistent way. In a symmetric situation when all budgets are equal and the bidders' valuations are drawn from the same distribution, a mechanism designer has to construct only a single allocation profile offered to every buyer. This affords a significant analytical simplification, which is not available here.

We show that qualitatively the optimal mechanism belongs to one of the two classes, depending on the profile of budgets.[2] If the budget differences between the buyers are sufficiently small (in the sense made precise below), the optimal mechanism is a so-called "top-auction." It is characterized by a common threshold value $\bar{x}^t$ at which the budget constraint of each bidder becomes binding. All the buyers with values below $\bar{x}^t$ are treated symmetrically: each of them gets the good when she has the highest value and pays a transfer derived by the standard envelope result.

Any bidder with value exceeding $\bar{x}^t$ pays her budget, and gets the good with a probability that typically jumps at x^t but does not change with the bidder's value on $[\bar{x}^t, 1]$. So, all buyers with values above $\bar{x}^t$ are essentially tied. The tie-breaking rule setting the probabilities, with which different bidders with values exceeding $\bar{x}^t$ get the good, plays an important role in this mechanism. In fact, it is the only instrument used by the seller to discriminate between

in privatization auctions, between the objectives of allocative efficiency and maximizing revenue, "the first ... is generally the more urgent in the countries of Eastern Europe." Efficiency has also been one of the goals in telecommunication spectrum auctions. In the US, the Congress mandated the Federal Communications Commission to allocate spectrum in a way that promotes efficiency. The British government has also set efficiency as a goal. For more on this, see Dasgupta and Maskin (2000) and Binmore and Klemperer (2002).

[2]We will assume that each bidder's budget is sufficiently small so that it becomes binding at higher values. As we will show below, a sufficient condition for this is that each budget is less than the price set by a seller facing a single bidder.

different bidders. Particularly, this probability is higher for a richer bidder to compensate her for the higher payment, equal to her budget, to the seller.

The threshold $\bar{x}^t$ is determined by the sum of individual budgets. In turn, $\bar{x}^t$ determines the reservation value which is lower than in the standard case without budget constraints. This happens because the bidders with values above $\bar{x}^t$ pay their budgets, and the seller cannot extract more surplus from them. Therefore, the tradeoff between higher efficiency and leaving greater surplus to the bidders shifts to higher efficiency at lower values.

When the buyers' budgets are sufficiently different, the "top auction" is infeasible because the seller can no longer achieve necessary differentiation between the buyers by discriminating only "at the top." In particular, it becomes impossible to allocate the good to the buyers with valuations above the (endogenous) threshold $\bar{x}^t$ in such a way that each buyer pays her budget. The optimal mechanism in this case is what we call a "budget-handicap auction" in which the seller uses two kinds of discrimination between the buyers. First, she sets different thresholds for different buyers or groups of buyers. Naturally, richer buyers have higher thresholds. Not all thresholds have to be different: there may be clusters of buyers with the same threshold. But there is more than one threshold across bidders. A richer bidder with a value above her threshold gets the good with a higher probability than a poorer bidder with a value above her respective threshold. This is like in the top auction, except the thresholds are now different.

Importantly, in the budget-handicap auction the seller also discriminates between buyers with low values. In particular, a poorer bidder gets the good with a higher probability than a richer bidder when they both have the same value below the threshold of the poorer bidder. A poorer bidder also faces a lower reservation value than a richer bidder. This handicapping of richer bidders creates more competition for them from poorer ones, and allows the seller to extract the whole budgets from richer bidders with high values. But it also introduces an additional inefficiency into the mechanism.

In a seminal paper on the optimal auction design, Myerson (1981) has considered bidders whose values are drawn from different distributions and established the optimality of handicapping the buyers whose values are more likely to be high and who, therefore, have lower virtual values than the bidders whose values are more likely to be low.[3] In our model, the bidders' asymmetry comes from another source- budget differences. When these differences

[3] More recently Jehiel and Lamy (2015) have considered the optimality of such discrimination when auction entry is costly. They showed that discrimination is suboptimal if costly entry precedes buyers' learning their values. However, "incumbent" bidders who do not face entry costs should be handicapped.

are sufficiently large, an asymmetry of virtual values arises endogenously and leads to handicapping of richer bidders, even when the bidders' values are identically distributed. While such handicapping occurs at all possible values in Myerson (1981), in our setting only rich bidders with low values are handicapped and lose the good to poorer bidders with the same values. However, richer bidders with high values get the good with a greater probability than poorer bidders with such values.

The optimal mechanism is unique, and the necessary and sufficient conditions for it are provided in Theorem 2. On the way towards this result, we show that the profile of the bidders' thresholds determines all elements of the optimal mechanism, except for the tie-breaking allocation rule for types above the threshold when this threshold is the same for several bidders. Interestingly, the optimality conditions for a profile of thresholds are essentially the feasibility conditions ensuring consistency between the allocation probabilities at the thresholds and the binding budget constraints at the thresholds. We provide the intuition for these conditions in the discussion following Theorem 2.

Building on this result, Theorem 3 presents the conditions for the optimality of the "top auction." The "budget-handicap" auction is optimal in the complementary case. The most challenging part in computing the "budget-handicap" auction is determining the "clusters" of bidders with the same threshold. This problem is not analytically difficult as it only involves checking the conditions of Theorem 2 for a given cluster configuration. However, one may have to go through all such configurations to determine the optimal one, which is a combinatorial problem that can be solved computationally. We provide an illustration by computing the optimal mechanism with two and three bidders, the latter - under uniform type distribution. The example with three bidders is particularly telling as it shows that every possible cluster configuration is optimal for a set of budget profiles of a positive measure.

The optimal mechanism can be implemented via an indirect bidding mechanism which combines the features of an all-pay auction and a lottery. Precisely, a bidder is offered a choice between buying a lottery ticket by paying her whole budget, and participating in an all pay-auction. A bidder chooses to buy a lottery ticket if her value is above her respective threshold, and participates in the all-pay auction otherwise. The difference between the top auction and the budget handicap auction is that in the former the all-pay auction is symmetric: a bidder gets the good if her bid is the highest and no one has chosen to buy a lottery ticket. In contrast, in the budget-handicap mechanism the all-pay auction is asymmetric and handicaps richer bidders. So, a bidder gets the good if her bid exceeds the bids of poorer bidders with lower thresholds by a certain margin, and also exceeds the bids, lowered by a certain margin, of richer bidders with higher thresholds.

A natural question is how the variability of budgets affects the seller's profits. We show that the seller weakly prefers less budget variability and, with a fixed aggregate budget, she gets the highest expected profits when each bidder has the same budget (Theorem 5). However, the seller's revenue does not change after a sufficiently small redistribution of the aggregate budget between the bidders. This is a consequence of the fact that the optimal mechanism (top auction) and the threshold in it are robust to such small redistribution.

Technically, our paper contains a number of interesting elements. Among them - the equivalence between the optimality and feasibility conditions for the mechanism. Another interesting aspect is the uncovered strong connection between the threshold values at which budget constraints become binding and the Lagrange multipliers associated with budget constraints. Not only there is a one-to-one relationship between them, as demonstrated by Theorem 1, but also the strong duality property between them ultimately allows us to derive the optimal mechanism.

Finally, although we focus on the case in which all budget constraints are binding, our results also apply when some bidders have high budgets and do not face budget constraints.

In the related literature, the paper closest to ours is Laffont and Robert (1996) who consider a similar environment with commonly known but equal budgets. They derive an optimal mechanism which is a special case of our top auction. Their optimal mechanism is symmetric and does not allow to figure out what the seller should do when the bidders' budgets are different. Yet, it is important and interesting to understand mechanism design in such ex-ante asymmetric environments as ours, since equal budgets are a knife-edge case.

To highlight the effects of budget asymmetry, a surprising result of our analysis is that the bidders' thresholds remain equal when budget differences are sufficiently small, yet there is discrimination between the bidders with high valuations via the tie-breaking rule at the top. So, our "top auction" provides a generalization of the optimal auction of Laffont and Robert (1996) to a setting with small budget asymmetry. However, a qualitatively different mechanism - "budget-handicap auction"- is optimal when budget differences are large.

Maskin (2000) studies constrained-efficient mechanisms for two and three bidders who have equal and publicly known budgets, and whose valuations are drawn from different distributions. He assumes a common valuation threshold at which each bidder's budget constraint becomes binding. Yet, our analysis shows that this is generally not true in either constrained-efficient or optimal mechanisms when the budgets are sufficiently asymmetric.

Malakhov and Vohra (2008) derive optimal dominant strategy mechanism for two buyers with values distributed over a discrete support, one of whom faces no budget constraint and the other has a known fixed budget. Their mechanism is similar to the one that we derive in the extension of our example with two bidders one of whom has a small budget and the budget of the other is larger than the "monopoly" price for a single buyer.

Pai and Vohra (2014) study optimal mechanism design with private budgets and identically distributed valuations. In their work, the budgets and valuations have a finite support, with a continuous distribution considered in an extension. They provide a significant contribution to multidimensional mechanism design showing how one can work directly with reduced form auctions. They show that in the optimal mechanism some buyer types receive separating allocations and some buyer types are pooled, although it is hard to pin down those intervals exactly. An extension of their paper considers bidders with equal and public budgets.

Although out setting with publicly known budgets is different from the one with privately known budgets and values in Pai and Vohra (2014), it is nevertheless interesting to compare the differential treatment of richer and poorer buyers in these two settings, since most other works focus on bidders with equal budgets. Pai and Vohra (2014) establish that "pooling serves to allot the good to disadvantaged buyer types ... even in profiles where there are buyers with higher valuations and budgets present." In contrast, in our setting handicapping of high-budget bidders occurs - when budget differences are large- in the region of separating allocations at low values, while the region of pooling includes high-value bidders, where richer bidders get the good with a higher probability.

In the earlier literature on auctions with budget constraints, Che and Gale (1998) compare the performance of first- and second-price auctions when the buyers have privately known budgets and values. They show that the first-price auction yields higher expected social surplus and expected revenue. Che and Gale (1996) show that the all-pay auction performs better than the first-price auction under common value and private budgets. Che and Gale (2000) explore optimal nonlinear pricing for a buyer with privately known value and budget. Zheng (2001) studies the first-price auction in which budget-constrained buyers can bid above their budgets. In case of a win such buyer can either use costly financing to cover the deficit, or default and lose her budget. Hafalir, Ravi and Sayedi (2012) focus on a Vickrey auction with budget-constrained bidders. In their framework, the bidders have different and essentially known budgets. Although their mechanism is not optimal, it is "close" to a Pareto efficient mechanism.

Borgs et. al (2005) and Dobzinski, Lavi and Nisan (2012) are concerned with dominant strategy mechanisms for allocating multiple goods. Both papers establish impossibility results under private budgets, the latter- for Pareto optimal allocation, the former- for allocation satisfying other properties that might be desirable. Borgs et. al (2005) then provide an auction that asymptotically (as maximal budgets becomes large) attains the same revenue as the posted price auction. Dobzinski, Lavi and Nisan (2012) demonstrate that with public budgets, a Pareto optimal allocation can be attained by using Ausubel's clinching auction. In contrast to Dobzinski, Lavi and Nisan (2012), Baisa (2015) demonstrates that clinching auction is a Pareto efficient mechanism under private budget constraints when the bidders' beliefs satisfy full support assumption.

Importantly, Pareto optimality is inconsistent with the goal of revenue maximization pursued in this paper, and a revenue maximizing seller would not offer a Pareto optimal mechanism. In particular, handicapping a richer bidder, as in the budget-handicap auction, and allocating the good randomly between the bidders with values above the common threshold, as in the top auction, can not occur in a Pareto optimal mechanism.

Che, Gale and Kim (2013a) and (2013b) and Richter (2016) study revenue-maximizing and welfare-maximizing assignment of a divisible good to a continuum of budget-constrained agents. The nature of the problem studied by these authors is very different from that of our problem. In particular, as discussed in Richter (2016), his model can be reinterpreted as a single-agent problem in which budget and supply must be balanced on average, and transfers between types of this single agent are permitted.

The rest of the paper is organized as follows. Section 2 develops the model. Section 3 presents a two-bidder example. Section 4 presents the main steps in the analysis. Section 5 provides the characterization of the optimal mechanism and its qualitative properties. Section 6 contains additional examples, including the case of three bidders. Section 7 deals with constrained-efficient mechanism and highlights the differences between the latter and the optimal mechanism. Section 8 concludes. The proofs are relegated to Appendix A. Appendix B deals with the case of asymmetrically distributed valuations.

2 Model and Preliminaries

A seller with one unit of the good faces n bidders. Bidder $i \in \{1, ..., n\}$ has privately known value x_i for the good drawn from a common knowledge distribution $F(.)$, which possesses a continuous positive density function $f(.)$ s.t. $f(x) > 0$ for all x in the support of $F(.)$.[4] Without loss of generality, we assume that the support of $F(.)$ is $[0, 1]$.

Bidder i with valuation x_i gets a payoff equal to $x_i q_i - t_i$ if she gets the good with probability q_i and pays t_i to the seller. This bidder is endowed with budget m_i which t_i can never exceed. The budgets are commonly known and are assumed to be sufficiently small, relative to the range of possible valuations.[5] Furthermore, our results apply also when only some bidders have binding budget constraints.

We will impose a standard assumption on the distribution $F(.)$:

Assumption 1 *Increasing Hazard rate:*

$$\frac{f(x)}{1 - F(x)} \quad \textit{is increasing in} \quad x \quad \textit{for all} \quad x \in [0, 1] \tag{1}$$

In fact, a weaker assumption that $x - \frac{1-F(x)}{f(x)}$ is increasing is sufficient, and we make the increasing hazard rate assumption mainly for the sake of conformity with the literature.[6]

The seller has zero value for the good, and her payoff is the sum of all payments from the buyers, $\sum_{i=1,..,n} t_i$. For most part, we focus on the optimal mechanism designed by the seller maximizing the seller's expected revenue. However, we also consider constrained efficient mechanism maximizing the expected surplus from the mechanism.

By the Revelation principle (Myerson 1979) we can restrict attention to direct truthful mechanisms $(Q_1(.), ..., Q_n(.))$, $(T_1(.), ..., T_n(.))$, where $Q_i(\hat{x}_1, ..., \hat{x}_n)$ is the probability that the bidder i gets the good and $T_i(\hat{x}_1, ..., \hat{x}_n)$ is the transfer that she pays to the seller. when the profile of types $(\hat{x}_n, ..., \hat{x}_n)$ is announced by the buyers.

Further, $q_i(x_i) = \int_{x_{-i} \in [0,1]^{n-1}} Q_i(x_i, x_{-i}) \prod_{j \neq i} dF(x_j)$ and
$t_i(x_i) = \int_{x_{-i} \in [0,1]^{n-1}} T_i(x_i, x_{-i}) \prod_{j \neq i} dF(x_j)$ are the expected probability that bidder i gets

[4]We focus on the symmetric distribution case in order to highlight the consequences of the budget differences between the bidders. However, our analysis can be extended to the case where each bidder's valuation is drawn from a different probability distribution $F_i(.)$.

[5]As we show below in Lemma 8, a sufficient condition for all budget constraints to be binding in the optimal mechanism is $\max_i m_i \leq \arg\max p(1 - F(p))$ i.e., the highest budget is below the price set by a seller facing a single buyer without a budget constraint. With multiple bidders, competition causes a bidder's budget constraint to be binding even if her budget exceeds this level.

[6]Pai and Vohra (2014) suggest that a stronger assumption that $f(x)$ is nonincreasing is necessary in the setting with budget constraints because bidder i's virtual value is $x - \frac{1-F(x)-\lambda_i}{f(x)}$ on the interval of x adjacent to zero, where λ_i is a Lagrange multiplier associated with i's budget constraint. However, as we show below, in the optimal mechanism $\lambda_i \leq 1 - F(x)$ on the appropriate interval of x. Therefore, the monotonicity of $x - \frac{1-F(x)}{f(x)}$ guarantees the monotonicity of $x - \frac{1-F(x)-\lambda_i}{f(x)}$ and the extra assumption on $f(x)$ is unnecessary.

the good and her expected payment, respectively, when she announces type x_i and all other bidders announce their types truthfully. With a slight abuse of terminology we will also refer to the collection $(q_i(.), t_i(.))$, $i \in \{1, ..., n\}$ as a mechanism.

An optimal mechanism solves the revenue maximization problem of the seller:

$$\max \sum_{i=1,...,n} \int_{(x_1,...,x_n)\in[0,1]^n} T_i(x_1, ..., x_n) \prod_{i=1,...,n} dF(x_i) \tag{2}$$

subject to the following:

(i) *interim incentive constraints:*

$$x_i q_i(x_i) - t_i(x_i) \geq x_i q_i(\hat{x}_i) - t_i(\hat{x}_i), \quad \text{for all } (x_i, \hat{x}_i) \in [0,1]^2 \text{ and all } i \in \{1, ..., n\}. \tag{3}$$

(ii) *individual rationality constraints:*

$$x_i q_i(x_i) - t_i(x_i) \geq 0 \quad \text{for all } i \quad \text{and} \quad x_i \in [0,1]. \tag{4}$$

(iii) *budget constraints:*

$$T_i(x_i, x_{-i}) \leq m_i \quad \text{for all } i, \ x_i \in [0,1], \ x_{-i} \in [0,1]^{n-1}. \tag{5}$$

(iv) *feasibility constraints*:

$$\sum_i Q_i(x_1, ..., x_n) \leq 1 \text{ and } Q_i(x_1, ..., x_n) \geq 0 \text{ for all } (x_1, ..., x_n) \in [0,1]^n. \tag{6}$$

3 Example

To illustrate our results we first present the optimal mechanism for two bidders, 1 and 2, with budgets m_1 and m_2, respectively, satisfying $m_1 \geq m_2$ without loss of generality.

If the budgets are sufficiently small and close to each other then the optimal mechanism is a "top auction"[7] defined by four parameters: reservation value r^t; threshold value $\bar{x}^t$; and expected probabilities of trading "at the top," $q_1(\bar{x}^t)$ and $q_2(\bar{x}^t)$ (see Theorem 3 for details).

In the "top auction," the budget constraint of either bidder is not binding when her value is below $\bar{x}^t$. Despite budget asymmetry, for bidder i with value in $[r^t, \bar{x}^t)$ the top auction

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[7] By Lemma 8, a sufficient condition for both bidders' budgets to bind in the optimal mechanism is $m_1 \leq \arg\max_p p(1 - F(p))$ i.e., bidder 1's budget is smaller than the seller's optimal price when she faces only this bidder. However, a weaker condition $m_1 < 1 - \int_{r':r'=\frac{1-F(r')}{f(r')}}^{1} F^{n-1}(x)dx$ is necessary and sufficient for both budget constraints to be binding when the budgets are close to each other and so the optimal mechanism is a top auction.

looks exactly like a standard symmetric auction: she gets the good when her competitor has a lower value, although the reservation value r^t is lower than without budget constraints.

At $\bar{x}^t$ both budget constraints become binding: bidder i with value in $[\bar{x}^t, 1]$ pays her whole budget and gets the good with probability $q_i(\bar{x}^t)$. Naturally, a richer bidder has a higher probability of trading at the top, $q_1(\bar{x}^t) > q_2(\bar{x}^t)$. In fact, both $q_1(x)$ and $q_2(x)$ jump at $x = \bar{x}^t$, except in the borderline parameter case in which only $q_1(x)$ jumps (to 1). So the top auction discriminates only between buyers with high values.

Since the net payoff of a bidder with value $x_i \in [\bar{x}^t, 1]$ is equal to $q_i(\bar{x}^t)x_i - m_i = q_i(\bar{x}^t)(x_i - \bar{x}^t) + \int_{r^t}^{\bar{x}^t} F(x)dx$, a high-value bidder with a higher budget gets a higher payoff than the bidder with the same value but a lower budget.

The threshold value $\bar{x}^t$ is found from the aggregate budget constraint (see Theorem 3):

$$m_1 + m_2 = \bar{x}^t(1 + F(\bar{x}^t)) - 2\int_{r^t}^{\bar{x}_t} F(x)dx$$

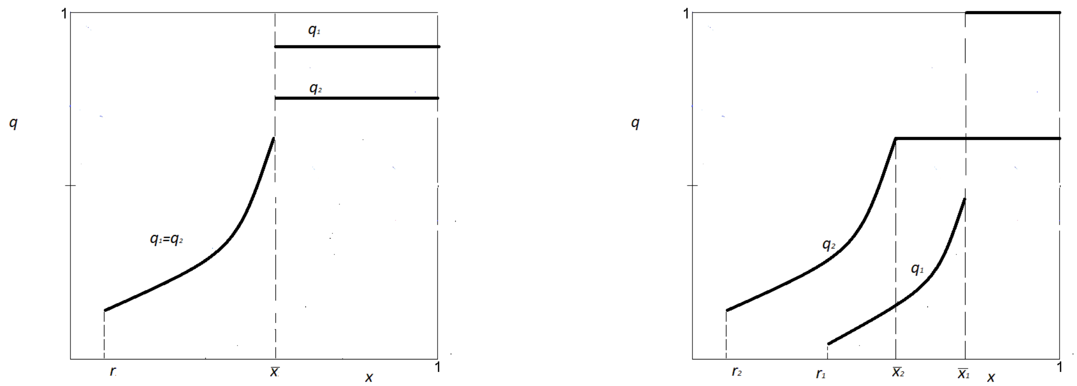
where $(1 + F(\bar{x}^t))$ is the maximal feasible value of $q_1(\bar{x}^t) + q_2(\bar{x}^t)$, and r^t is the level at which a bidder's virtual in this mechanism is zero. Precisely, $r^t = \frac{1-F(r^t) - \frac{(1-F(\bar{x}^t))^2}{1-F(\bar{x}^t)+\bar{x}^t f(\bar{x}^t)}}{f(r^t)}$.

The top auction is not always feasible. It has to satisfy $q_1(\bar{x}^t) \leq 1$ and $q_2(\bar{x}^t) \geq F(\bar{x}^t)$. These conditions together with binding budget constraints at $\bar{x}^t$ i.e., $m_i = q_i(\bar{x}^t)\bar{x}^t - \int_{r^t}^{\bar{x}^t} F(x)dx$, imply that $m_1 - m_2 \leq \bar{x}^t(1 - F(\bar{x}^t))$. If this inequality holds, the top auction is the optimal mechanism. If it fails, the top auction is infeasible. Instead, the threshold $\bar{x}_1$ at which the budget constraint of the richer bidder 1 becomes binding has to be greater than the corresponding threshold $\bar{x}_2$ of the poorer bidder 2.

This has a number of consequences for the mechanism. First, bidder 1's probability of trading jumps to 1 at $\bar{x}_1$, while bidder 2's probability of trading continuously reaches its maximal value $F(\bar{x}_1)$ at $\bar{x}_2$. Significantly, the bidders no longer face a symmetric auction at lower values. Instead, richer bidder 1 is handicapped. She faces a higher reservation value i.e., $r_1 > r_2$. Also, bidder 1 with value $x \in [r_1, \bar{x}_1)$ gets the good with a lower probability than bidder 2 with the same value. Because of this, we refer to this mechanism as a "budget-handicap" auction (Theorem 4). The handicapping of bidder 1 generates more competition from bidder 2 allowing the seller to extract the whole budget from high-value bidder 1.

For some values of the budgets neither budget constraint or only bidder 2's budget constraint is binding. The condition for the former is simple: the poorer bidder 2 with valuation 1 must not be budget constrained in a standard symmetric auction i.e., $m_2 \geq 1 - \int_{r^s}^1 F(x)dx$ with r^s satisfying $r^s - \frac{1-F(r^s)}{f(r^s)} = 0$. Only bidder 2's budget constraint is binding if m_2 fails this condition, while m_1 is above the "monopoly" price $p^m = \arg\max_p p(1 - F(p))$.

Figure 1: Expected Probabilities of Trading with Two Bidders



(a) Top Auction

(b) Budget-Handicap Auction

In this case, the optimal mechanism is like the "budget-handicap" auction, except that bidder 1 with value above her threshold $\bar{x}_1$ gets the good with probability 1 but pays less than m_1.

The implementation of the "top-auction" and "budget-handicap auction" via an indirect mechanism combining an all-pay auction with a lottery is discussed in Section 5 for n bidders. With two bidders, this implementation involves an all-pay auction for the poorer bidder 2, while richer bidder 1 is offered a choice between the all-pay-auction and a "buy-it-now" option: she can get the good for sure by paying his budget.

The expected probabilities of trading in the "top auction" and "budget-handicap auction" are depicted in Figure 1. Figure 2 summarizes how the nature of the optimal mechanism depends on the budgets m_1 and m_2 when the bidders' types are distributed uniformly.

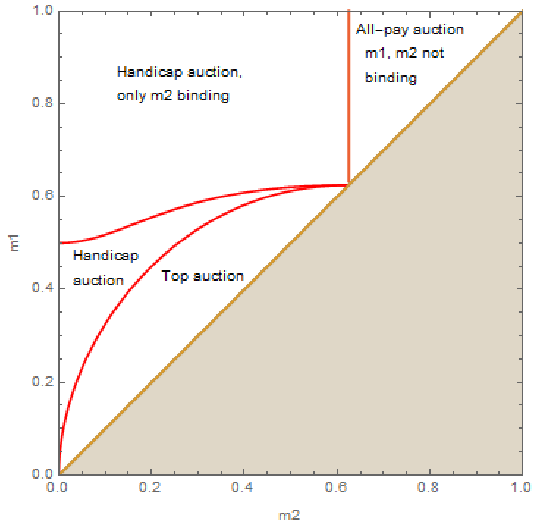
4 Analysis

Our first result establishes the existence and uniqueness of the optimal mechanism.

Lemma 1 *There exists an (almost everywhere) unique optimal mechanism* $(Q_1(.), ..., Q_n(.), T_1(.), ..., T_n(.))$ *solving the problem (2) subject to (3)-(6).*

Proof of Lemma 1: The objective of the maximization problem (2) is a continuous linear functional in the Hilbert space $L^2([0,1]^n)$. Its admissible set specified by constraints (3)-(6) is convex. Therefore, by Theorem 2.6.1 in Balakrishnan (1993) the solution to our problem exists. The uniqueness almost everywhere follows by standard arguments, in particular, the linearity of the objective and the convexity of the constraints. *Q.E.D.*

Figure 2: The Optimal Mechanism and Bidders' Budgets.



Next, let $U_i(x_i) \equiv q_i(x_i)x_i - t_i(x_i)$ be the net expected payoff of buyer i of type x_i in the optimal mechanism. The following result is standard and is left without proof:

Lemma 2 *A mechanism $(Q_1(.), ..., Q_n(.), T_1(.), ..., T_n(.))$ is incentive compatible and individually rational if any only if the expected trading probability $q_i(x_i)$ is nondecreasing in x_i for all i and $x_i \in [0, 1]$, and:*

$$U_i(x_i) = \int_0^{x_i} q_i(s)ds + c_i \ \ \textit{for some} \ \ c_i \in \mathbb{R}_+ \tag{7}$$

The individual rationality requires the constant c_i to be nonnegative. The optimality then implies that $c_i = 0$. Given this, we drop c_i altogether from the analysis.

Combining $U_i(x_i) = x_i q_i(x_i) - t_i(x_i)$ with (7) yields the following expression:

$$t_i(x_i) = x_i q_i(x_i) - \int_0^{x_i} q_i(s)ds \tag{8}$$

Consider now the budget constraints. First, we can replace the ex-post budget constraint in (5) with the interim one, $t_i(x_i) \leq m_i$ for all i and x_i. Indeed, the interim budget constraints obviously hold when (5) holds. In the opposite directions, if $t_i(x_i) \leq m_i$ for all i and x_i, then (5) holds if we set $T_i(x_i, x_{-i}) = t_i(x_i)$ for all i, x_i and x_{-i}. Doing so does not affect the seller's objective, the incentive or individual rationality constraints, since these depend only on the expected transfers $t_i(.)$, but it can potentially relax the budget constraint in some states of the world since the maximal payment by bidder i weakly decreases.

Next, suppose that $(q_i(.), t_i(.))$ are the expected probability of trading and transfer of bidder i in some individually rational incentive compatible mechanism satisfying budget constraints. Then define threshold $\bar{x}_i$ as follows:

$$\bar{x}_i = \inf\{x_i \in [0,1] | t_i(x_i) = t_i(1)\} \quad (9)$$

Lemma 3 *Suppose that $(q_i(.), t_i(.))$ is an incentive compatible individually rational mechanism. If $\bar{x}_i < 1$, then $t_i(x_i)$ and $q_i(x_i)$ are constant on the interval $[\bar{x}_i, 1]$.*

Proof of Lemma 3: Since $q_i(x_i)$ is increasing in x_i by Lemma 2, the expected transfer $t_i(x_i)$ must also be increasing in x_i, for otherwise the mechanism cannot be incentive compatible. Therefore, if $\bar{x}_i < 1$ then, for all $x_i \in (\bar{x}_i, 1]$, $t_i(x_i) = t_i(1)$ and hence $q_i(x_i) = \bar{q}_i$ for some $\bar{q}_i$. Since the allocation $(\bar{q}_i, t_i(1))$ satisfies incentive and individual rationality constraints in (3) and (4) of any type $x_i > \bar{x}_i$ it is also incentive compatible and individually rational for type $\bar{x}_i$. So, without loss of generality, we can take $t_i(\bar{x}_i) = t_i(1)$ and $q_i(\bar{x}_i) = \bar{q}_i$. *Q.E.D.*

The threshold values $\bar{x}_i$, $i \in \{1, ..., n\}$, play an important role as the key choice variables which ultimately determine the whole mechanism. Lemma 3 and equation (8) imply that budget constraints $t_i(x_i) \leq m_i$ can be replaced with the following one:

$$m_i \geq \bar{x}_i q_i(\bar{x}_i) - \int_0^{\bar{x}_i} q_i(s) ds \quad (10)$$

So, the budget constraint of bidder i is binding when $\bar{x}_i < 1$ and (10) holds as an equality.

Next, replacing the transfers in the objective (2) with the right-hand side of (8), using $q_i(x) = q_i(\bar{x}_i)$ for all $x_i \in [\bar{x}_i, 1]$, and then integrating by parts yields the modified objective:

$$\sum_{i=1}^{n} \int_0^1 t_i(x_i) dF(x_i) = \sum_{i=1}^{n} \int_0^1 \left(q_i(x_i) x_i - \int_0^{x_i} q_i(x) dx \right) dF(x_i)$$
$$= \sum_{i=1}^{n} \int_0^{\bar{x}_i} q_i(x_i) \left(x_i - \frac{1 - F(x_i)}{f(x_i)} \right) dF(x_i) + \sum_{i=1}^{n} \int_{\bar{x}_i}^1 q_i(\bar{x}_i) \bar{x}_i dF(x_i) \quad (11)$$

By Lemma 2, in order to ensure the incentive compatibility we need to impose on this objective the condition that $q_i(x_i)$ is increasing for all i. Following standard approach, we will solve a relaxed program omitting this condition and then show that the solution is such that $q_i(.)$ is increasing, strictly at $\bar{x}_i$ from the left. The latter guarantees that (9) holds i.e., $\bar{x}_i$ is the lowest type of i who makes the largest transfer. We will take care of the feasibility constraints (6) by imposing them directly on the probabilities of trading.

Since we have imposed the condition that $q_i(\bar{x}_i) = q_i(x_i)$ for all $x_i \in (\bar{x}_i, 1]$ on the objective (11) explicitly, to ensure that the budget constraint is satisfied for all types in $[\bar{x}_i, 1]$ it is enough to impose (10) on it. Doing so yields the following relaxed program Lagrangian:.

$$\mathcal{L}(Q,\bar{x},\lambda)=\sum_{i=1}^{n}\int_{0}^{\bar{x}_i}q_i(x_i)\left(x_i-\frac{1-F(x_i)}{f(x_i)}\right)dF(x_i)+\int_{\bar{x}_i}^{1}q_i(\bar{x}_i)\bar{x}_i dF(x_i)-\lambda_i\left(q_i(\overline{x}_i)\overline{x}_i-\int_{0}^{\overline{x}_i}q_i(x)dx-m_i\right)$$

$$=\sum_{i=1}^{n}\left(\int_{0}^{\bar{x}_i}q_i(x_i)\left(x_i-\frac{1-\lambda_i-F(x_i)}{f(x_i)}\right)dF(x_i)+\int_{\bar{x}_i}^{1}q_i(\bar{x}_i)\left(\bar{x}_i-\frac{\lambda_i\overline{x}_i}{1-F(\bar{x}_i)}\right)dF(x_i)+\lambda_i m_i\right) \tag{12}$$

where λ_i is a Lagrange multiplier associated with bidder i's budget constraint (10).

Next, using $q_i(x_i)=\int_{x_{-i}\in[0,1]^{n-1}}Q_i(x_i,x_{-i})\prod_{j\neq i}dF(x_j)$ and changing the order of summation and integration in (12) we can rewrite it as follows:

$$\mathcal{L}(Q,\bar{x},\lambda)=\int_{(x_1,\ldots,x_n)\in[0,1]^n}\sum_{i=1}^{n}Q_i(x_1,\ldots,x_n)\gamma_i(x_i)\prod_{i=1,\ldots,n}dF(x_i)+\sum_{i=1}^{n}\lambda_i m_i. \tag{13}$$

where $\gamma_i\left(x_i\right)$ is defined as follows for $i\in\{1,\ldots,n\}$:

$$\gamma_i\left(x_i\right)=\begin{cases} x_i-\frac{1-\lambda_i-F(x_i)}{f(x_i)}, & \text{if} \quad x_i<\overline{x}_i,\\ \overline{x}_i-\frac{\lambda_i\overline{x}_i}{1-F(\overline{x}_i)}, & \text{if} \quad x_i\geq\overline{x}_i.\end{cases} \tag{14}$$

As seen from (13), $\gamma_i(.)$ plays the role of the virtual value of bidder i. Recall that without budget constraints, i's virtual value is $x_i-\frac{1-F(x_i)}{f(x_i)}$. So budget constraints cause the virtual value of type $x_i\in[0,\overline{x}_i)$ to increase by an amount proportional to the value of the Lagrange multiplier. Intuitively, this happens because when $\lambda_i>0$, then all types above $\bar{x}_i$ pay their whole budget. So the seller cannot extract more surplus from these types, and increasing the probability with which they get the good depresses the seller's profits by less than without budget constraints. On the other hand, since all types in $[\overline{x}_i,1]$ get the same allocation, every type in this endogenous "atom" has the same virtual value, $\overline{x}_i-\frac{\lambda_i\overline{x}_i}{1-F(\overline{x}_i)}$.

Note that if bidder i's budget constraint is not binding, then $\lambda_i=0$, and so according to (14) we recover the standard formula for the virtual value for all x_i s.t. $x_i<\bar{x}_i$. It is still possible to have $\bar{x}_i<1$ in this case. In particular, this occurs when bidder i's budget is sufficiently higher than the budget of any other bidder. We demonstrate that this may, indeed, occur in our example with two bidders and uniform type distribution. Thus our analysis also applies when only some bidders have binding budget constraints.

Inspection of (13) yields the following Lemma:

Lemma 4 *Any solution to the relaxed program is such that for bidder i and $(x_i,x_{-i})\in[0,1]^n$:*

1. $Q_i(x_i,x_{-i})=1$ *if* $\gamma_i(x_i)>\max\{0,\max_{j\neq i}\gamma_j(x_j)\}$;

2. $Q_i(x_i, x_{-i}) \in [0,1]$ *if* $\gamma_i(x_i) = \max\{0, \max_{j\neq i}\gamma_j(x_j)\}$;

3. $Q_i(x_i, x_{-i}) = 0$ *if* $\gamma_i(x_i) < \max\{0, \max_{j\neq i}\gamma_j(x_j)\}$.

4. $\sum_{i=1}^{n} Q_i(x_1, ..., x_n) = 1$ *if* $\min_i \gamma_i(x_i) > 0$.

According to this Lemma, the profile of virtual values $(\gamma_1(x_1), ..., \gamma_n(x_n))$ determines the trading probabilities $(Q_i(x), ..., Q_n(x))$ uniquely except in the case of ties, when two or more bidders have the highest virtual value. The ties that have zero probability can be ignored, as the designer can resolve them arbitrarily without affecting her expected profits. In particular, this applies to ties that involve a bidder-type $x_i \in [0, \bar{x}_i)$. However, all bidder types in $[\bar{x}_i, 1]$ have the same virtual value $\gamma_i(\bar{x}_i)$ and essentially constitute an atom of probability $1-F(\bar{x}_i)$. If $\gamma_i(\bar{x}_i) = \gamma_j(\bar{x}_j)$ for some $j \neq i$, then every bidder type in $[\bar{x}_i, 1]$ is tied with every bidder type in $[\bar{x}_j, 1]$. This tie has a positive probability. As we show below, there may, in fact, exist clusters of bidders with the same threshold $\bar{x} < 1$ and the same virtual value function $\gamma(x)$, $x \in [0, \bar{x}]$, even if they have unequal budgets. However, the tie-breaking rule between the bidders with valuations above the threshold $\bar{x}$ in a cluster will be uniquely defined by their binding budget constraints (10) at $\bar{x}$.

Significantly, Lemma 4 implies that it is optimal to set $\sum_{i=1}^{n} Q_i(x_1, ..., x_n)\gamma_i(x_i) = \max\{0, \max_i \gamma_i(x_i)\}$ for all $x = (x_1, ..., x_n) \in [0,1]^n$. Therefore, we can replace Lagrangian (13) with the following one that depends only on $(\bar{x}_1, ..., \bar{x}_n)$ and $(\lambda_1, ..., \lambda_n)$:

$$\mathcal{L}(\bar{x}, \lambda) = \max_{Q:\, 0\leq Q_i(x)\leq 1,\, \sum_i Q_i(x)\leq 1} \mathcal{L}(Q, \bar{x}, \lambda) = \int_{x\in[0,1]^n} \max\{0, \max_{i=1,\dots,n} \gamma_i(x_i)\} \prod_i dF(x_i) + \sum_{i=1}^{n} \lambda_i m_i. \tag{15}$$

Thus, solving the relaxed program boils down to finding the profile $(\bar{x}_1, ..., \bar{x}_n, \lambda_1, ..., \lambda_n)$ maximizing (15) from which we then recover the probabilities of trading $Q_i(.)$ using Lemma 4. The following Theorem provides an important step towards solving this problem. To display it, define

$$\gamma_i^-(\bar{x}_i) \equiv \lim_{x_i \uparrow \bar{x}_i} \gamma_i(x_i) = \bar{x}_i - \frac{1 - \lambda_i - F(\bar{x}_i)}{f(\bar{x}_i)}. \tag{16}$$

Then we have:

Theorem 1 *In any solution to the relaxed program, the profile* $(\bar{x}_1, ..., \bar{x}_n, \lambda_1, ..., \lambda_n)$ *is such that:*

1. *For all i s.t. $\bar{x}_i \leq \bar{x}_j$ for some $j \neq i$, $\gamma_i(x_i)$ is continuous at $x_i = \bar{x}_i$ or, equivalently,*

$$\lambda_i = \frac{\left(1 - F\left(\overline{x}_i\right)\right)^2}{\left(1 - F\left(\overline{x}_i\right) + \overline{x}_i f\left(\overline{x}_i\right)\right)}, \tag{17}$$

So, $\gamma_i(x_i) = x_i - \frac{1 - \frac{(1-F(\overline{x}_i))^2}{(1-F(\overline{x}_i)+\overline{x}_i f(\overline{x}_i))} - F(x_i)}{f(x_i)}$ *for* $x_i \in [0, \bar{x}_i]$.

2. *For bidder h_1 such that $\max_{j \neq i} \bar{x}_j < \bar{x}_{h_1} < 1$, we have: $\gamma_{h_1}(\bar{x}_{h_1}) > \gamma^-_{h_1}(\bar{x}_{h_1}) = \max_{j \neq h_1} \gamma_j(\bar{x}_j)$ or, equivalently, $\lambda_{h_1} < \frac{\left(1-F\left(\overline{x}_{h_1}\right)\right)^2}{\left(1-F\left(\overline{x}_{h_1}\right)+\overline{x}_{h_1} f\left(\overline{x}_{h_1}\right)\right)}$ and*

$$\overline{x}_{h_1} - \frac{1 - F(\overline{x}_{h_1}) - \lambda_{h_1}}{f(\overline{x}_{h_1})} = \max_{j \neq h_1} \frac{\overline{x}_j^2 f(\overline{x}_j)}{1 - F(\overline{x}_j) + \overline{x}_j f(\overline{x}_j)}. \tag{18}$$

Although the proof of Theorem 1 is fairly intricate, it relies on an intuitive observation: if some bidder i's virtual value is not continuous at her threshold $\bar{x}_i$, then the seller can attain a higher payoff by modifying $\bar{x}_i$ slightly.

Theorem 1 is consistent both with binding and non-binding budget constraints of any player i. Particularly, if the budget constraint of bidder i is non-binding, then $\lambda_i = 0$. In this case, we either have $\bar{x}_i = 1$ or, if the only bidder whose budget constraint is non-binding is h_1, then $\bar{x}_{h_1} - \frac{1-F(\bar{x}_{h_1})}{f(\bar{x}_{h_1})} = \max_{j \neq h_1} \frac{\bar{x}_j^2 f(\bar{x}_j)}{1-F(\bar{x}_j)+\bar{x}_j f(\bar{x}_j)}$, and bidder h_1 of type above $\bar{x}_{h_1}$ gets the good with probability 1 and pays a transfer which is below her budget. Also, from (17) and (18) it follows that $\lambda_i > 0$ and so the budget constraint of bidder i is binding if $\bar{x}_i < 1$ and there exists j s.t. $\bar{x}_i \leq \bar{x}_j$. So there can be at most one bidder with threshold $\bar{x}$ strictly below 1 whose budget constraint is not binding.

Theorem 1 has two important implications. First, given a profile $\bar{x} = (\bar{x}_1, ..., \bar{x}_n)$ equations (17) and (18) obviously define a profile $\lambda(\bar{x})$ uniquely. Importantly, as stated in the following Corollary, the converse is also true i.e., given a profile $\lambda = (\lambda_1, ..., \lambda_n)$ there is a unique profile $\bar{x}(\lambda)$ defined by equations (17) and (18). This result is significant as it allows us to reduce the number of choice variables from $2n$ to n.

Corollary 1 *Equations (17) and (18) in Theorem 1 define a bijection between the set of thresholds $(\bar{x}_1, ..., \bar{x}_n)$ and the set of Lagrange multipliers $(\lambda_1, ..., \lambda_n)$.*

Next, one can observe from equations (17) and (18) that $\lambda_i \leq 1 - F(\bar{x}_i)$ for all i. This observation is behind the other important implication of Theorem 1 stated below.

Lemma 5 *Any solution to the relaxed problem is such that $\gamma_i(x_i)$ is strictly increasing and $q_i(x_i)$ is increasing on $[0, \bar{x}_i]$ for all i. Therefore, this solution also solves the full unrelaxed program.*

We can now establish the following intuitive relationship between budgets and thresholds:

Lemma 6 *If $m_i > m_j$ for some $i, j \in \{1, ..., n\}$, then in an optimal mechanism $\bar{x}_i \geq \bar{x}_j$.*

An immediate implication of this Lemma is that bidder h_1, who has the highest threshold and lowest λ, is in fact the highest-budget bidder 1.

Combining Lemmas 4 and 5 we can now provide explicit expressions for the expected trading probabilities $q_i(x_i)$.

Lemma 7 *In an optimal mechanism, the expected probability of trading $q_i(x_i)$ satisfies:*

$$\int_{x_{-i}\in[0,1]^{n-1}:\gamma_i(x_i)>\max\{0,\max_{j\neq i}\gamma_j(x_j)\}} \prod_{j\neq i} dF(x_j) \leq q_i(x_i) \leq \int_{x_{-i}:\gamma_i(x_i)\geq\max\{0,\max_{j\neq i}\gamma_j(x_j)\}} \prod_{j\neq i} dF(x_j) \tag{19}$$

The inequalities in (19) hold as equalities for almost all $x_i \in [0, \bar{x}_i)$ and for $x_i = \bar{x}_i$ if $\bar{x}_i \neq \bar{x}_j$ for all j, $j \neq i$. So the profile $(\bar{x}_1, ..., \bar{x}_n, \lambda_1, ..., \lambda_n)$ uniquely defines the function $q_i(.)$ a.e. on $[0, \bar{x}_i]$ and uniquely defines the function $\int_0^{x_i} q_i(s)ds$ everywhere on $[0, \bar{x}_i]$.

Finally, using Lemmas 6 and 7 and Theorem 1 we can formally establish that a sufficient condition for all budget constraints to be binding is that the highest budget m_1 among the bidders is below the price that the seller posts when she faces a single bidder.

Lemma 8 *Suppose that $m_1 \leq \arg\max_p p(1 - F(p))$. Then the budget constraints of all bidders are binding in the optimal mechanism i.e., (10) holds as equality for all $i \in \{1, ..., n\}$.*

To complete the derivation of the optimal mechanism we will make use of the Lagrangian duality theory (see e.g. Boyd and Vandenberghe (2009) and Bertsekas (2001)). First, let $g(\lambda)$ be Lagrange dual function to (15) i.e.:

$$g(\lambda) \equiv \mathcal{L}(\bar{x}(\lambda), \lambda) = \max_{\bar{x}} \mathcal{L}(\bar{x}, \lambda) = max_{\bar{x}} \int_{x\in[0,1]^n} \max\{0, \max_{i=1,...,n} \gamma_i(x_i)\} \prod_i dF(x_i) + \sum_{i=1}^{n} \lambda_i m_i. \tag{20}$$

Note that $\bar{x}(\lambda)$ in the definition of $g(.)$ is the optimal threshold profile under a given profile of multipliers λ and is implicitly defined by the equations in Theorem 1.

Since $\mathcal{L}(\lambda, \bar{x})$ is linear in λ, by Danskin's Theorem (Bertsekas (2001), Ch. 1) the function $g(\lambda) \equiv \mathcal{L}(\lambda, \bar{x}(\lambda)) = \max_{\bar{x}} \mathcal{L}(\lambda, \bar{x})$ is convex and therefore has a unique minimum. Importantly, the next Lemma establishes the strong duality property for our problem implying that its solution can be obtained by minimizing $g(\lambda)$.

Lemma 9 *The problem of maximizing (15)* [8] *has strong duality property i.e.,*

$$\max_{\bar{x}} \min_{\lambda} \mathcal{L}(\bar{x}, \lambda) = \min_{\lambda} \max_{\bar{x}} \mathcal{L}(\bar{x}, \lambda)$$

We prove this Lemma by directly establishing that $\mathcal{L}(\bar{x}, \lambda)$ possesses saddle-point property, which implies strong duality.[9]

5 Main Results

Using Lemma 9 we can now derive the solution to our problem by minimizing the Lagrange dual function $g(\lambda)$. The result is provided in the next Theorem. To state it, let us introduce the following notation. For any set $J \subseteq \{1, ..., n\}$ s.t. $i \notin J$, let $Prob.[\gamma_i(x_i) > \max_{j \in J} \gamma_j] = \prod_{j \in J} \int_{x_j \in [0,1]:\, \gamma_i(x_i) > \gamma_j(x_j)} dF(x_j)$. Also note that by Lemma 7 the value of $\int_0^{\bar{x}_i} q_i(x_i) dx_i$ for any i is uniquely defined by the vector $(\bar{x}, \lambda)$. To simplify the exposition we will focus on the case where all bidders' budget constraints are binding (by Lemma 8 a sufficient condition for this is that $m_1 \leq \arg\max p(1 - F(p))$ which, under monotone hazard rate, is equivalent to $m_1 - \frac{1-F(m_1)}{f(m_1)} \leq 0$).

Now we can state the following Theorem:

Theorem 2 *The optimal profile of threshold values $(\bar{x}_1, ..., \bar{x}_n)$ is unique and is characterized by the following necessary and sufficient conditions:*

For i such that $\bar{x}_i \neq \bar{x}_j$, $j \neq i$, budget constraint must hold i.e.:

$$m_i = \bar{x}_i q_i(\bar{x}_i) - \int_0^{\bar{x}_i} q_i(s) ds \tag{21}$$

For bidders $k_1, ..., k_l$ that form a cluster $C(\bar{x}^c) \equiv \{i | \bar{x}_i = \bar{x}^c\}$ with threshold $\bar{x}^c$ i.e., $\bar{x}_{k_1} =$

[8] It is well know that the primal problem of maximizing (15), $\max_{\bar{x}} \mathcal{L}(\bar{x}, \lambda)$, is equivalent to the following one: $\max_{\bar{x}} \min_{\lambda} \mathcal{L}(\bar{x}, \lambda)$.

[9] As stated in Boyd and Vandenberghe (2009) p.239., "if x and λ are primal and dual optimal points for a problem in which strong duality obtains, they form a saddle-point for the Lagrangian. The converse is also true: If (x, λ) is a saddle-point of the Lagrangian, then x is primal optimal, λ is dual optimal, and the optimal duality gap is zero." Notably, the primal problem is not required to be a convex problem for this result to hold (Boyd and Vandenberghe (2009), p 215). So, in particular, the constraint set in our problem does not have to be convex.

$... = \bar{x}_{k_l} = \bar{x}^c \neq \bar{x}_j$ *for any* $j \notin \{k_1, ..., k_l\}$*, the following two conditions must hold:*[10,11]

$$(i) \qquad \sum_{h \in \{1,...,l\}} m_{k_h} = \bar{x}^c \frac{1 - F(\bar{x}^c)^l}{1 - F(\bar{x}^c)} Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j \notin \{k_1,...,k_l\}} \gamma_j] - l \int_0^{\bar{x}^c} q_{k_1}(s) ds \tag{22}$$

$$(ii) \quad \text{for all } r \in \{2, ..., l-1\}, \quad \frac{m_{k_1} + ... + m_{k_r}}{r} - \frac{m_{k_{r+1}} + ... + m_{k_l}}{l - r} \leq$$

$$\bar{x}^c \left(\frac{1 - F(\bar{x}^c)^r}{r(1 - F(\bar{x}^c))} - F(\bar{x}^c)^r \frac{1 - F(\bar{x}^c)^{l-r}}{(l-r)(1 - F(\bar{x}^c))} \right) Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j \notin \{k_1,...,k_l\}} \gamma_j] \tag{23}$$

The proof of Theorem 2 shows that conditions (21)-(23) are equivalent to the first-order conditions for minimizing the Lagrange dual function $g(\lambda) \equiv \mathcal{L}(\lambda, \bar{x}(\lambda))$, (48) and (49) in the proof. Since $g(\lambda)$ is convex, its minimum is unique and (21)-(23) are necessary and sufficient conditions for it. Therefore, the threshold profile $\bar{x}$ solving the system (21)-(23) is unique and constitutes the solution to our mechanism design problem.

In particular, the threshold profile uniquely determines $q_i(x_i)$ for all i and almost all $x_i \in [0, \bar{x}_i)$ and $q_i(\bar{x}_i)$ for all i s.t. $\bar{x}_i \neq \bar{x}_j$ for all $j \neq i$ according to Lemma 7. If i belongs to some cluster of bidders with a common threshold $\bar{x}^c$ then, given $\int_0^{\bar{x}_i} q_i(x_i) dx_i$, the value $q_i(\bar{x}^c)$ is uniquely defined via the budget constraint of player i. As we explain below, the first-order conditions (22)-(23) guarantee the feasibility of this choice of $q_i(\bar{x}_i^c)$.

Let us now provide more detailed intuition behind Theorem 2. To begin, condition (21) says that in the optimal mechanism the only necessary and sufficient condition for bidder i who does not belong to any cluster (i.e. $\bar{x}_i \neq \bar{x}_j$ for all $i \neq j$) is that her budget constraint is binding at her threshold valuation $\bar{x}_i$.

Condition (22) is the aggregate budget constraint for the bidders in a cluster with threshold $\bar{x}^c$. The probability that one of them gets the good, $\sum_{r=1,...,l} q_{k_r}(\bar{x}^c)$, is equal to $\frac{1-F(\bar{x}^c)^l}{1-F(\bar{x}^c)} Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j \notin \{k_1,...,k_l\}} \gamma_j]$. This can be shown by summing individual budget constraints (10) of the bidders in the cluster and comparing the result to (22).

Condition (23) is the feasibility condition for the existence of a cluster with threshold $\bar{x}^c$. Although it may appear non-transparent, it has a clear and intuitive economic interpretation. Its left-hand side is the difference between the average budget of the richest r bidders and the average budget of the poorest $l - r$ bidders in the cluster. For the cluster to exist, this

[10]Note that without loss of generality we may assume here that indexes $k_1, ..., k_l$ are ordered according to the budgets i.e., $k_1 < k_2 ... < k_{l-1} < k_l$ and $m_{k_1} \geq m_{k_2} ... \geq m_{k_{l-1}} \geq m_{k_l}$. This is so because when (23) holds for this ordering, it also holds for any alternative ordering.

[11]Note that by Lemma 4 $q_{k_1}(x) = ... q_{k_l}(x)$ for all x, since bidders $\{k_1, ..., k_l\}$ have the same threshold $\bar{x}^c$ and therefore, by Theorem 1, have the same λ and hence the same "virtual values" $\gamma(.)$.

difference cannot be too large, for otherwise it would be impossible to satisfy the necessary condition (10) that budget constraints of all bidders in the cluster are binding when they have threshold values $\bar{x}^c$. Precisely, this difference cannot exceed the largest possible difference between the average transfers paid by the bidders in these two groups with valuations above the threshold $\bar{x}^c$. The latter difference is equal to the maximal difference between the average expected gross surpluses of buyers with valuations $\bar{x}^c$ in these two groups (because the net surplus of each bidder with value $\bar{x}^c$ is the same and equal to $\int_{r^c}^{\bar{x}^c} q_{k_1}(s)ds$), which is the right-hand side of (23). Indeed, the maximal average gross surplus of the richest r bidders with valuation $\bar{x}^c$ is equal to $\bar{x}^c$ times the maximal average probability of trading in this group. The latter is a product of the probability $Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j\notin\{k_1,...,k_l\}} \gamma_j]$ that no bidder outside the cluster has a virtual value exceeding the virtual value of a cluster member of type $\bar{x}^c$, $\gamma_{k_1}(\bar{x}^c)$, and the average probability that at least one among these r bidders has value of at least $\bar{x}^c$, $\frac{1-F(\bar{x}^c)^r}{r(1-F(\bar{x}^c))}$. Similarly, the minimal average gross surplus of the poorest $l-r$ bidders with valuation $\bar{x}^c$ is equal to $\bar{x}^c$ times the minimal average probability of trading for that group. The latter probability is a product of $Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j\notin\{k_1,...,k_l\}} \gamma_j]$ and the average probability that at least one among $l-r$ bidders has value of at least $\bar{x}^c$ while the values of the other r bidders in the cluster are below $\bar{x}^c$, $F(\bar{x}^c)^r \frac{1-F(\bar{x}^c)^{l-r}}{(l-r)(1-F(\bar{x}^c))}$.

To summarize, when conditions (22) and (23) hold, then the vector of trading probabilities "at the top" $(q_{k_1}(\bar{x}^c), ..., q_{k_l}(\bar{x}^c))$ of the bidders in the cluster $C(\bar{x}^c)$ defined by the budget constraint $m_{k_j} = \bar{x}^c q_{k_j}(\bar{x}^c) - \int_0^{\bar{x}^c} q_{k_j}(s)ds$ is feasible.

In fact, to highlight that conditions (22) and (23) are feasibility constraints for our mechanism and to connect them to more familiar notions of feasibility of expected ("reduced form") trading probability functions in an auction, in particular, the ones in Border (1991) and (2007), note the following. If we multiply both sides of inequality (23) by $(l-r)$, add them to (22) and then simplify using the budget constraint $m_{k_j} = \bar{x}^c q_{k_j}(\bar{x}^c) - \int_0^{\bar{x}^c} q_{k_j}(s)ds$ then we obtain:

$$\sum_{j=1,...,h} q_{k_j}(\bar{x}^c) \leq \frac{1-F(\bar{x}^c)^h}{1-F(\bar{x}^c)} Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{i\notin\{k_1,...,k_l\}} \gamma_i] \quad \text{for all } h \in \{1,...,l\} \tag{24}$$

On the other hand, subtracting (23) multiplied by r from (22) and using the same budget constraint yields:

$$\sum_{j=h,...,l} q_{k_j}(\bar{x}^c) \geq F(\bar{x}^c)^{h-1} \frac{1-F(\bar{x}^c)^{l-h+1}}{1-F(\bar{x}^c)} Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j\notin\{k_1,...,k_l\}} \gamma_j] \quad \text{for all } h \in \{1,...,l\} \tag{25}$$

Condition (24) says that the probability of assigning the good to any subset of bidders from the cluster $C(\bar{x}^c)$ with values above $\bar{x}^c$ cannot exceed the probability that a bidder from this subset has value in $[\bar{x}^c, 1]$ and bidders outside the cluster have lower virtual values than $\gamma_{k_1}(\bar{x}^c) = ... = \gamma_{k_l}(\bar{x}^c)$. Clearly, this feasibility condition - which is similar to the condition in Theorem 3 of Border (2007)- must hold for every subset of size $h \in \{1, ..., l\}$ of bidders in a cluster, but it is sufficient to check it for the subset including h richest bidders $k_1, ..., k_h$ since they have higher trading probabilities at the threshold i.e., $q_{k_1}(\bar{x}^c) \geq ... \geq q_{k_l}(\bar{x}^c)$.

Similarly, (25) provides a lower bound on the probability of assigning the good to any subset of bidders in a cluster. Specifically, fixing an arbitrary subset of bidders of size $l-h+1$ in the cluster $C(\bar{x}^c)$, the good should be assigned to a bidder from this subset when at least some bidder in it has value in $[\bar{x}^c, 1]$, the rest of the bidders in the cluster have values below $\bar{x}^c$, and the bidders outside the cluster have lower virtual values than $\gamma_{k_j}(\bar{x}^c)$. It is sufficient that this condition hold for the subsets of every size composed of the bidders with the lowest budgets because they have lower probabilities of trading $q_{k_j}(\bar{x}^c)$ "at the top."

5.1 Top and Budget-Handicap Auctions

In this section, we use the results established above to characterize qualitative properties of the optimal mechanism and show how these properties depend on the profile of budgets.

Qualitatively, we will distinguish between two kinds of optimal mechanisms. A mechanism of the first kind is called a "top auction." In a top auction all thresholds are equal i.e., $\bar{x}_1 = ... = \bar{x}_n = \bar{x}^t$, and all bidders with valuations below $\bar{x}^t$ are treated symmetrically: any bidder with valuation in $[r^t, \bar{x}^t)$ pays the same transfer and gets the good if she has the highest valuation, where the "reservation value" r^t is given in Definition 1. Bidders with valuations below r^t are excluded. But, because the bidders have unequal budgets, the seller discriminates between them "at the top:" a richer high valuation bidder gets the good with a higher probability and pays a higher transfer than a poorer high valuation bidder.

The mechanism of the second kind is called "budget-handicap auction." In a "budget-handicap auction" the mechanism designer sets different thresholds for different bidders or groups of bidders. There may exist clusters of bidders with the same threshold, but not all bidders belong to the same cluster. In this mechanism, there are two types of price discrimination. First, a richer bidder with a value above her threshold has a higher probability of trading than a poorer bidder with a value above her respective threshold. This type of price discrimination applies to any two bidders with different budgets, irrespectively of whether they belong to the same cluster or different clusters.

The second type of price discrimination works in the opposite direction. A poorer bidder with a lower value (below her threshold) has a higher probability of trading and pays a higher transfer than a richer bidder with the same value. So richer bidders are handicapped, and poorer bidders are given an advantage at lower valuations via a lower reserve price and a higher probability of trading. This motivates the use of the term "budget-handicap."

Which mechanism is offered by the designer - a top auction or a budget-handicap auction- ultimately depends on the budget profile. The designer offers a top auction whenever it is feasible, namely, when the budget differences across buyers are not too large. However, when these differences are large, price-discrimination only at the "top" is no longer feasible, as all budget constraints cannot be made binding at the same threshold. So, different thresholds have to be set across bidders, and the seller has to handicap richer bidders at lower valuations.

We will start from the "top auction" which is defined as follows:

Definition 1 *A "top auction" for n bidders with budgets $m_1, ...m_n$ is a mechanism characterized by a common threshold $\bar{x}^t = \bar{x}_1 = ...\bar{x}_n$ uniquely solving*

$$\sum_{i=1,\dots,n} m_i = \bar{x}^t \frac{1-F(\bar{x}^t)^n}{1-F(\bar{x}^t)} - n\int_{r_t}^{\bar{x}^t} F(s)^{n-1} ds, \tag{26}$$

with common reservation value r_t defined by $r_t = \frac{1-F(r_t)-\frac{(1-F(\bar{x}^t))^2}{1-F(\bar{x}^t)+\bar{x}^t f(\bar{x}^t)}}{f(r_t)}$, and expected trading probabilities $q_i(x_i) = F(x_i)^{n-1}$ for all $x_i \in [r, \bar{x}^t)$ and $q_i(\bar{x}^t)$ satisfying:

$$m_i = \bar{x}^t q_i(\bar{x}^t) - \int_{r_t}^{\bar{x}^t} F(s)^{n-1} ds \tag{27}$$

Observe that equation (26) defining the common threshold $\bar{x}^t$ is the equivalent of (22) for the case of the top auction where all bidders belong to a single cluster. The solution to (26) is unique because its right-hand side is increasing in x^t,[12] is equal to zero when $x^t = 0$, and exceeds $\sum_i m_i$ when $x^t = 1$, since by assumption $m_1 \leq 1 - \int_{r:r-\frac{1-F(r)}{f(r)}=0}^{1} F^{n-1}(x)dx$.

Note that (26) and (27) together imply that

$$\sum_{i=1,\dots,n} q_i(\bar{x}^t) = \frac{1-F(\bar{x}^t)^n}{1-F(\bar{x}^t)}.$$

So, with probability 1 the good is given to a bidder with value above the threshold if there is at least one such bidder.

The following Theorem, which is a direct consequence of Theorem 2, shows that the "top auction" is optimal whenever it is feasible i.e., whenever the feasibility condition (23) in Theorem 2 adapted to the top auction is satisfied.

[12] Indeed, the derivative of the right-hand side w.r.t $\bar{x}^t$ is equal to $\frac{1-F(\bar{x}^t)^n}{1-F(\bar{x}^t)} + \frac{\bar{x}^t f(\bar{x}^t)}{(1-F(\bar{x}^t))^2}(1+(n-1)F(\bar{x}^t)^n - nF(\bar{x}^t)^{n-1}) - nF(\bar{x}^t)^{n-1} + nF(r(\bar{x}^t))^{n-1}\frac{dr(\bar{x}^t)}{d\bar{x}^t}$, which is positive, in particular, because $\frac{dr(\bar{x}^t)}{d\bar{x}^t} > 0$.

Theorem 3 *The unique optimal mechanism is a "top auction" with a common threshold* $\bar{x}^t$ *solving (26) if and only if for every* $k = 1, 2, ..., n-1$ *we have:*

$$\frac{m_1 + ... + m_k}{k} - \frac{m_{k+1} + ... + m_n}{n-k} \leq \bar{x}^t \left(\frac{1 - F(\bar{x}^t)^k}{k(1 - F(\bar{x}^t))} - F(\bar{x}^t)^k \frac{1 - F(\bar{x}^t)^{n-k}}{(n-k)(1 - F(\bar{x}^t))} \right) \tag{28}$$

As the discussion following Theorem 2 points out, condition (28) (equivalently, condition (23) in Theorem 2), says that the difference between the average budget of the richest k bidders and the average budget of the poorest $n-k$ bidders does not exceed the maximal difference between the average expected surpluses of these two groups. This allows to allocate the good "at the top" in such a way that all budget constraints hold at the threshold $\bar{x}^t$.

The top auction allocates the good efficiently when all buyers' valuations lie in $[r_t, \bar{x}^t]$. The reservation value r_t is below the reservation value in the optimal auction without budget constraints r, which satisfies $r - \frac{1-F(r)}{f(r)} = 0$. This is so because with budget constraints the tradeoff between higher efficiency at lower values and leaving greater surplus to the bidders with higher values shifts towards the former, since the bidders with values above their thresholds pay the whole budgets, so no more surplus can be extracted from them. However there is an additional inefficiency compared to the standard optimal auction: when several bidders have valuations above $\bar{x}^t$ the good is allocated randomly among them, with probabilities increasing in their budgets. So a bidder with a lower value in $[\bar{x}^t, 1]$ may end up getting the good even if another bidder has a higher value.

The following Corollary of Theorem 3 shows that the seller's expected revenue in the top auction depends only on the aggregate budget $\sum_i m_i$, and not on the distribution of the budgets across the bidders:

Corollary 2 *Suppose that the top auction is the optimal mechanism under budget profiles* $(m_1, ..., m_n)$ *and* $(m'_1, ..., m'_n)$ *such that* $\sum_i m_i = \sum_i m'_i$*. Then the optimal threshold* $\bar{x}^t$ *and the expected seller's revenue is the same in both cases.*

As an application of this Corollary, suppose first that all bidders have the same budgets. So the seller offers a top auction which in this case coincides with the optimal mechanism of Laffont and Robert (1996). Then the nature of the optimal mechanism and its profitability for the seller do not change after a sufficiently small budget redistribution among the bidders that does not violate (28).

Top auction can be implemented via an indirect mechanism which combines an all-pay auction with a lottery. Specifically, each bidder is offered a choice between the former and the latter. If bidder i chooses the lottery, she pays m_i for a "lottery ticket" which wins her the good with probability $q_i(\bar{x}^t)$. If i chooses the all-pay auction she submits a bid b_i and gets the good if she is the highest bidder, her bid is above the "reserve price," and no bidder has chosen to take part in the lottery. The reserve price in the all-pay auction is equal to $t_i(r^t) = r^t F^{n-1}(r^t)$. This mechanism implements the same allocation as the top auction with optimal threshold $\bar{x}^t$. Indeed, it is easy to see that the optimal strategy of bidder i is to buy the lottery ticket if $x_i \in [\bar{x}^t, 1]$; to bid $b_i(x_i) = x_i F^{n-1}(x_i) - \int_{r^t}^{x_i} F^{n-1}(s)ds$ if $x_i \in [r^t, \bar{x}^t]$; and not to participate if $x_i < r^t$. Note that this mechanism is envy-free since any two bidders i and j get equal payoffs if $x_i = x_j \in [0, \bar{x}^t]$, while richer bidder i gets a higher payoff than poorer bidder j when $x_i = x_j > \bar{x}^t$, but j cannot afford i's lottery ticket which costs m_i.

Now suppose that the feasibility condition (28) fails. In this case, the seller has to use additional tools to discriminate between the bidders and, in particular, set different thresholds for them. Naturally, lower-budget bidders have lower thresholds (Lemma 6), although there may still exist some clusters of bidders with the same threshold. Richer bidders with valuations above their higher thresholds get the good with a higher probability and pay more than poorer bidders with valuations above their lower thresholds.

Importantly, there is another type of "price discrimination" in this second kind of mechanism, which we call "budget handicap auction:" a poorer bidder with a low value has a higher probability of trading and pays a higher transfer than a richer bidder with the same value. This handicapping of higher-budget bidders creates a stronger competition for them from lower-budget bidders, and extracts higher payments from the former when they have high values. It also unavoidably increases inefficiency. Formally, we have:

Theorem 4 *Suppose that (28) fails for some k. Then the optimal auction is a "budget handicap auction" which is uniquely defined by a vector of thresholds $(\bar{x}_1, ..., \bar{x}_n)$ s.t. $\bar{x}_i \geq \bar{x}_{i+1}$ for all $i \in \{1, .., n-1\}$, with strict inequality for at least some i.*

In budget handicap auction, if $\bar{x}_i > \bar{x}_j$ then $r_i > r_j$ and $q_i(x) < q_j(x)$ for all $x \in [r_j, \bar{x}_j]$.

If bidders $k_1, ...k_l$, $k_1 \leq k_2... \leq k_l$ form a cluster with threshold $\bar{x}^c$, then these bidders have the closest budgets i.e., there exists $j \in \{0, n-l\}$ such that $k_h = j + h$ for all $h \in \{1, ..., l\}$.

By Theorem 2, the vector of thresholds $(\bar{x}_1, ..., \bar{x}_n)$ is uniquely defined by conditions (21)-(23). The most challenging part in applying this result and computing the optimal "budget handicap" auction is determining which groups of bidders form clusters with common thresholds. Theorem 4 simplifies this task by showing that any cluster contains only "adjacent" bidders with the smallest budget differences. So the number of possible cluster configurations is 2^{n-1}, and potentially one may have to go over all of them to compute the solution. Our results provide a tractable method to check whether a particular cluster configuration is optimal. For example, the optimal mechanism is a budget-handicap auction without any clusters if the following system of n equations has a solution $(\bar{x}_1, ..., \bar{x}_n)$ satisfying $\bar{x}_i > \bar{x}_{i+1}$ for all $i \in \{1, ..., n-1\}$:

$$m_1 = \bar{x}_1 - \int_0^{\bar{x}_1} \int_{x_{-1}:\gamma_1(x_1) > \max\{0, \max_{j\neq 1} \gamma_j(x_j)\}} \prod_{j\neq 1} dF(x_j) dx_1$$
$$m_i = \bar{x}_i \int_{x_{-i}:\gamma_i(\bar{x}_i) > \max\{0, \max_{j\neq i} \gamma_j(x_j)\}} \prod_{j\neq i} dF(x_j) - \int_0^{\bar{x}_i} \int_{x_{-i}:\gamma_i(x_i) > \max\{0, \max_{j\neq i} \gamma_j(x_j)\}} \prod_{j\neq i} dF(x_j) dx_i \tag{29}$$

Similarly, we can write down necessary and sufficient conditions for the optimality of any other cluster configuration. In the next section we will consider an example with three bidders and exhibit conditions for optimality of various cluster configurations in that case.

As in the case of the top auction, the cluster configuration in the budget-handicap auction and the seller's revenue remain robust to certain sufficiently small redistributions of the budgets, as the following Corollary demonstrates.

Corollary 3 *Suppose that under budget profile $(m_1, ..., m_n)$ the optimal mechanism is a budget-handicap auction with thresholds $(\bar{x}_1, ..., \bar{x}_n)$. Consider a budget profile $(m'_1, ..., m'_n)$ such that $|m_i - m'_i|$ is sufficiently small for all i and the aggregate budget of any cluster of bidders with a common threshold under $(m_1, ..., m_n)$ is the same under both budget profiles.*[13]

Then the optimal mechanism under $(m'_1, ..., m'_n)$ is a budget-handicap auction with the same threshold profile and the same expected seller's revenue as under profile $(m_1, ..., m_n)$.

The implementation of a budget-handicap auction via an indirect bidding mechanism is similar to that for the top-auction. As in the latter, bidder i is offered a choice between participating in an all-pay auction (with a handicap) and buying a lottery ticket that costs m_i and wins the good with probability $q_i(\bar{x}_i)$. Bidder i chooses the lottery if her valuation exceeds $\bar{x}_i$ and submits a bid in the auction otherwise. But, unlike in the top auction, the all-pay auction is not symmetric since richer bidders now have to be handicapped. Specifically, bidder i with threshold $\bar{x}_i$ participating in this auction gets the good when her bid: (i) exceeds the bid of any richer bidder j with a higher threshold $\bar{x}_j$ lowered by a certain margin; (ii)

[13]This condition means, in particular, that the budget of a bidder whose threshold is different from the threshold of any other bidder under $(m_1, ..., m_n)$ must remain unchanged.

exceeds the bid of a poorer bidder h with a lower threshold $\bar{x}_h$ by a certain margin. These margins depend both on the bidders' thresholds and type distribution.[14]

In the remainder of this section, we will focus on the properties of the seller's expected payoff function. Recall that by Lemma 9 (strong duality), the seller's expected profit in the optimal mechanism is given by the minimum of the dual Lagrange function $g(\lambda)$ in (20) which can be written as a function of the vector of budgets $(m_1, ..., m_n)$ in the following way:

$$\pi(m_1, ..., m_n) = \min_{\lambda} \left\{ \int_{x \in [0,1]^n} \max\{0, \max_{i=1,...,n} \gamma_i(x_i, \lambda)\} dF(x) + \sum_{i=1}^{n} \lambda_i m_i \right\}. \tag{30}$$

Since $\pi(m_1, ..., m_n)$ is a pointwise minimum in λ of a function affine in $(m_1, ..m_n)$, it is concave in the vector $(m_1, ..., m_n)$.[15]

Concavity of $\pi(.)$ has the following consequences for the seller's revenue:

Theorem 5 *Suppose that the aggregate budget of all bidders is fixed i.e.* $\sum_i m_i = M$.[16]

Then the seller gets the maximal payoff in the optimal mechanism when all bidders' budgets are equal i.e., $m_i = \frac{M}{n}$ *for all* $i = 1, ..., n$.

Moreover, consider two budget profiles $(m_1, ..., m_n)$ *and* $(m'_1, ..., m'_n)$*, ordered from the highest to the lowest, and suppose that* $\sum_{j=1}^{n} m_j = \sum_{j=1}^{n} m'_j$ *and* $\sum_{j=i}^{n} m_j \leq \sum_{j=i}^{n} m'_j$ *for all* $i \in \{2, ..., n\}$*. Then* $\pi(m_1, ..., m_n) \leq \pi(m'_1, ..., m'_n)$.

Intuitively, the second part of the Theorem says that if budget profile $(m_1, ..., m_n)$ is a mean preserving spread of the profile $(m'_1, ..., m'_n)$, in the sense of Rothschild and Stiglitz (1970), then the seller's expected revenue is greater under the latter than under the former. At the same time, Corollaries and 2 and 3 show that the inequality $\pi(m_1, ..., m_n) \leq \pi(m'_1, ..., m'_n)$ is strict only if the difference between the two budget profiles is sufficiently large that the sets of thresholds under the two budget profiles are different.

[14]The mapping of bids into the allocation of the good is defined via the formula for the transfers $t_i(x_i) = q_i(x_i)x_i - \int_{r_i}^{x_i} q_i(s)ds$. In the optimal budget-handicap mechanism $t_i(x_i)$ is strictly increasing in x_i on $[r_i, \bar{x}_i)$ and therefore b_i uniquely defines x_i on this interval via $b_i = t_i(x_i)$. Thus, when i submits bid $b_i = t_i(x_i)$ and bidder $j \neq i$ submit a bid $b_j = t_j(x_j)$, bidder i gets the good whenever $\gamma_i(x_i) \geq \max_{j \neq i} \gamma_j(x_j)$ which occurs with probability $q_i(x_i)$ from i's point of view.

[15]Note that this is true even if some bidder i's budget constraint is not binding. In this case $\lambda_i = 0$ and $\pi(m_1, ..., m_n)$ does not depend on m_i.

[16]To make this result non-trivial M has to be sufficiently small. In particular, we will assume that $M \leq np^m$ where p^m is a monopoly price i.e., $p^m = \arg\max_p p(1 - F(p))$.

6 Examples with Uniform Type Distribution.

6.1 Two Bidders.

We have described the qualitative properties of the optimal mechanism for two bidders in section 3. Below we compute its exact parameters under uniform type distribution on $[0,1]$.

Starting with the top auction, equation (26) defining common threshold value $\bar{x}^t$ becomes:

$$m_1 + m_2 = \bar{x}^t + \left(\bar{x}^t\right)^2 - \left(\bar{x}^t\right)^3 + \frac{\left(\bar{x}^t\right)^4}{4}$$

Condition (28) for the feasibility of the top auction simplifies to $m_1 - m_2 \leq \bar{x}^t(1-\bar{x}^t)$. If it holds, then by Lemma 7 $q_i(x_i) = 0$ if $x_i < \bar{x}^t - \frac{\left(\bar{x}^t\right)^2}{2}$; $q_i(x_i) = x_i$ if $x_i \in [\bar{x}^t - \frac{\left(\bar{x}^t\right)^2}{2}, \bar{x}^t)$; $q_i(x_i) = \frac{1+\bar{x}^t}{2} + \frac{m_i - m_j}{2}$ if $x_i \geq \bar{x}^t$.

So, $q_1(x)$ and $q_2(x)$ jump upwards at $x = \bar{x}^t$, except in the borderline case $m_1 - m_2 = \bar{x}^t(1-\bar{x}^t)$ where $q_1(x)$ jumps to 1 at $\bar{x}^t$, and $q_2(x)$ is continuous at $\bar{x}^t$, with $q_2(\bar{x}^t) = F(\bar{x}^t) = \bar{x}^t$.

If $m_1 - m_2 > \bar{x}^t(1-\bar{x}^t)$, then the optimal mechanism is a budget-handicap auction with thresholds $\bar{x}_1$ and $\bar{x}_2$ such that $\bar{x}_1 > \bar{x}_2$. Then the reservation value of bidder i (where her virtual value is zero) satisfies $r_i = \bar{x}_i - \frac{\bar{x}_2^2}{2}$, $i \in \{1,2\}$, while by Lemma 7, $q_1(\bar{x}_1) = 1$, $q_2(\bar{x}_2) = F(\bar{x}_1) = \bar{x}_1$, and for $x_i \in [\bar{x}_i - \frac{\bar{x}_2^2}{2}, \bar{x}_i)$:

$$q_i(x_i) = \int_{\gamma_i(x_i) > \gamma_j(s)} ds = \int_{x_i - \bar{x}_i > s - \bar{x}_j} ds = x_i - \bar{x}_i + \bar{x}_j \ \text{ for } \ i,j \in \{1,2\} \ i \neq j.$$

So $q_1(x_1)$ increases continuously on $[\bar{x}_1 - \frac{\bar{x}_2^2}{2}, \bar{x}_1)$ and jumps at $\bar{x}_1$ from $\bar{x}_2$ to 1, while $q_2(x_2)$ increases continuously on $[\bar{x}_2 - \frac{\bar{x}_2^2}{2}, \bar{x}_2]$ to its maximum $\bar{x}_1$. Note that $q_1(x) - q_2(x) = 2(\bar{x}_2 - \bar{x}_1) < 0$ for $x \in [\bar{x}_1 - \frac{\bar{x}_2^2}{2}, \bar{x}_2]$, as buyer 1 is handicapped in the intermediate range of values.

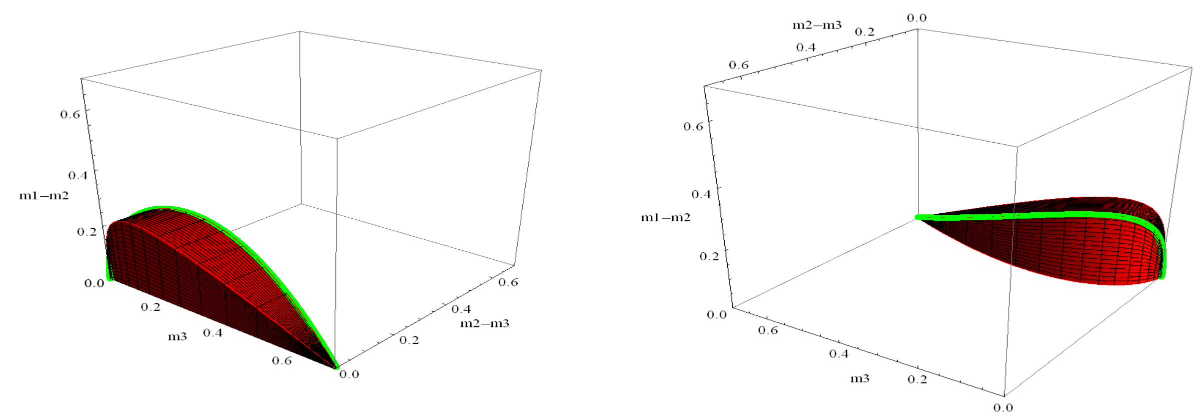
By Theorem 2 the thresholds $\bar{x}_1$ and $\bar{x}_2$ solve the following equations:

$$m_1 = \bar{x}_1 - \int_{r_1}^{\bar{x}_1} \int_{\gamma_1(x_1) > \gamma_2(x_2)} dx_2 dx_1 = \bar{x}_1 - \frac{\bar{x}_2^3}{2} + \frac{\bar{x}_2^4}{8} \tag{31}$$

$$m_2 = \bar{x}_2 \bar{x}_1 - \int_{r_2}^{\bar{x}_2} \int_{\gamma_2(x_2) > \gamma_1(x_1)} dx_1 dx_2 = \bar{x}_1 \bar{x}_2 - \bar{x}_1 \frac{\bar{x}_2^2}{2} + \frac{\bar{x}_2^4}{8} \tag{32}$$

By Theorem 1, $\lambda_2 = (1-\bar{x}_2)^2$ and $\lambda_1 = 1 - 2\bar{x}_1 + \bar{x}_2^2$. Thus, $\lambda_1 > 0$ and hence bidder 1's budget constraint is binding if and only if the solutions to (31) and (32) are such that $\bar{x}_1 < \frac{1+\bar{x}_2^2}{2}$. Substituting $\bar{x}_1 = \frac{1+\bar{x}_2^2}{2}$ into (31) and (32) yields the necessary and sufficient condition for binding budget constraint of bidder 1: consider $\bar{x}_2$ that solves (32) i.e., $m_2 = \frac{\bar{x}_2 + \bar{x}_2^3}{2} - \frac{\bar{x}_2^2}{4} - \frac{\bar{x}_2^4}{8}$. Then $m_1 \leq \frac{1+\bar{x}_2^2}{2} - \frac{\bar{x}_2^3}{2} + \frac{\bar{x}_2^4}{8}$.

Figure 3: Region of Optimality of The Top Auction



(a) View 1

(b) View 2

If the latter inequality fails, then bidder 1's budget constraint is never binding and so $\lambda_1 = 1 - 2\bar{x}_1 + \bar{x}_2^2 = 0$ and $\lambda_2 = (1-\bar{x}_2)^2$. Thus, $\bar{x}_1 = \frac{1+\bar{x}_2^2}{2}$, where $\bar{x}_2$ solves $m_2 = \frac{\bar{x}_2+\bar{x}_2^3}{2} - \frac{\bar{x}_2^2}{4} - \frac{\bar{x}_2^4}{8}$. Substituting these values into the right-hand side of (31) yields $\frac{1+\bar{x}_2^2}{2} - \frac{\bar{x}_2^3}{2} + \frac{\bar{x}_2^4}{8}$ as the transfer paid by bidder 1 of type $x \in \left[\frac{1+\bar{x}_2^2}{2}, 1\right]$, which is less than m_1 in this case.

6.2 Three-Bidder Mechanism Under the Uniform Distribution

The optimal mechanism with three bidders can be of four kinds:

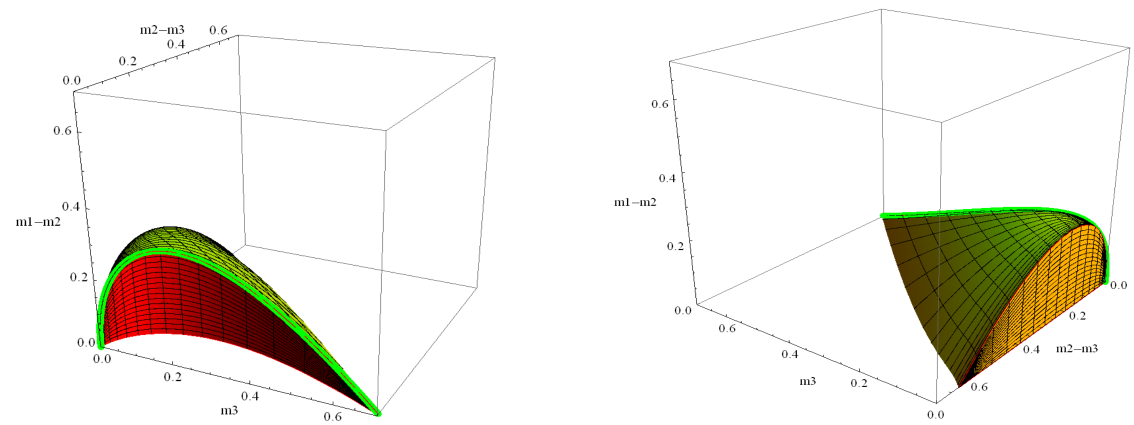
- "top-auction:" $\bar{x}_1 = \bar{x}_2 = \bar{x}_3 = \bar{x}^t$;
- "budget-handicap auctions" with:
  - "top cluster:" $\bar{x}_1 = \bar{x}_2 > \bar{x}_3$.
  - "lower cluster:" $\bar{x}_1 > \bar{x}_2 = \bar{x}_3$.
  - "no clusters:" $\bar{x}_1 > \bar{x}_2 > \bar{x}_3$.

Interestingly, each of these mechanisms is optimal for a set of budgets of a positive measure, as illustrated in Figures 3-6.

To save space, the conditions on the budgets for the optimality of these mechanisms and the details of the derivations are provided in the online Appendix available at *http : //www.severinov.com/bca_uniform*.

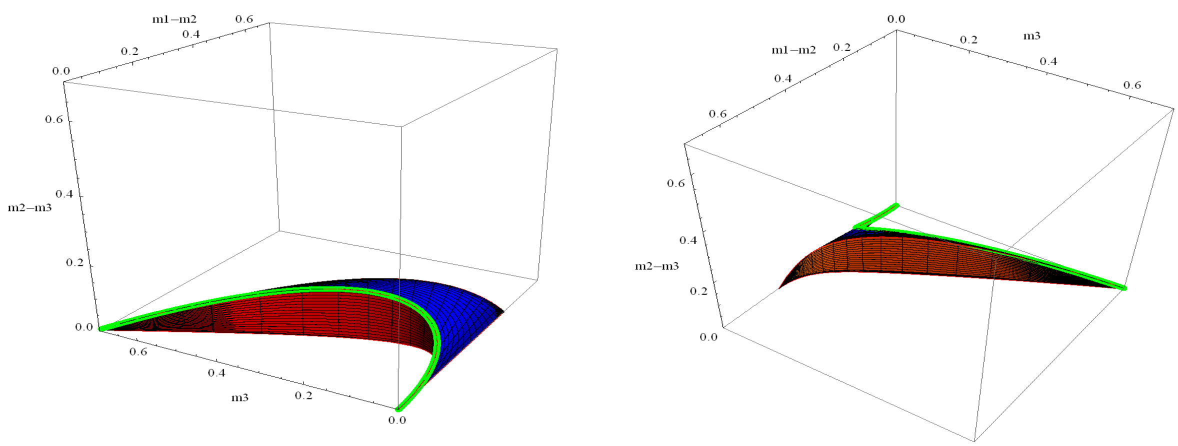
Figure 6 depicts the set of budgets for which the optimal mechanism has no clusters.

Figure 4: Region of Optimality of the Budget Handicap Auction with Top Cluster



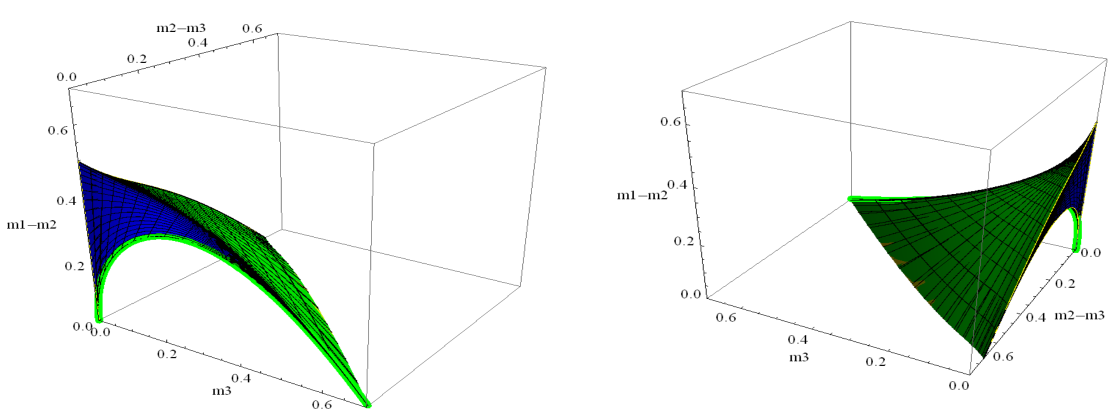
(a) View 1 (b) View 2

Figure 5: Region of Optimality of the Budget Handicap Auction with Lower Cluster



(a) View 1 (b) View 2

Figure 6: Region of Optimality of the Budget Handicap Auction with No Clusters



(a) View 1 (b) View 2

7 Constrained-Efficient Mechanism

In this section, we will characterize the constrained efficient mechanism which maximizes the expected social surplus, rather than the expected seller's revenue. It is well-known that without budget constraints VCG mechanism attains full efficiency under private values. But with budget constraints, VCG mechanism does not work since the bidders' willingness to pay cannot be fully translated into their bids. High-value bidder types can no longer afford to pay the value of the externality that they impose on the others.[17]

The constrained-efficient mechanism in our set-up is qualitatively similar to the optimal one, so we will omit the details and only highlight the main elements and the differences between these two mechanisms. First off, the constrained-efficient mechanism maximizes the expected social surplus $\sum_{i=1}^{n} \int_0^1 q_i(x_i)x_i dF(x_i)$ subject to the same constraints (3)-(6) as the optimal mechanism. Repeating the steps of the analysis of the optimal mechanism, and in particular, imposing the constraint (10) on the objective, and letting x_i^e denote the threshold valuation at which bidder i's budget constraint becomes binding, and λ_i^e denote the corresponding Lagrange multiplier yields the following Lagrangian:

$$\mathcal{L}^e = \sum_{i=1}^{n} \int_0^1 q_i(x_i)x_i dF(x_i) - \lambda_i^e \left(q_i(\overline{x}_i^e)\overline{x}_i^e - \int_0^{\overline{x}_i^e} q_i(x)dx - m_i \right) =$$

$$\sum_{i=1}^{n} \left(\int_0^{\bar{x}_i^e} q_i(x_i) \left(x_i + \frac{\lambda_i^e}{f(x_i)} \right) dF(x_i) + \int_{\bar{x}_i^e}^{1} q_i(\bar{x}_i) \left(E(x_i | x_i \geq \bar{x}_i^e) - \frac{\lambda_i^e \overline{x}_i^e}{1 - F(\bar{x}_i^e)} \right) dF(x_i) + \lambda_i^e m_i \right) \tag{33}$$

[17]One route towards efficiency in our set-up is to have the principal subsidize the bidders. Yet, in important real-world situations such approach is infeasible or politically unacceptable. In the course of large-scale privatization in Eastern Europe, the governments have been concerned with efficiency of asset allocation to spur the economic growth. However, these governments have been experiencing financial problems that made such subsidization practically infeasible. Although the governments in high-income North American and European countries possess sufficient financial resources, subsidizing the bidders in auctions of government assets would arguably be very unpopular and politically unacceptable. In fact, public discontent and political embarrassment have followed the results of second-price auctions where the prices paid by the winner i.e., the second-highest bids, were significantly below the winners' bids, as it was seen as a loss of public revenue. McMillan (1994) provides several examples illustrating this from spectrum auctions in New Zealand and Australia. Not surprisingly, it is hard to find examples of privatization auctions in which public funds are used to subsidize or finance bidders. Another and formal argument against subsidizing or financing bidders is provided by Zheng (2001), albeit in a different framework with common values and first-price auctions. He shows that, when seller provides financing at low interest rates, low budget bidders bid more and then declare bankruptcy with a sufficiently high probability that overall hurts the seller.

This Lagrangian function is the counterpart of (12) for the optimal mechanism. Hence, (33) yields the same modified Lagrangian (13) for the constrained efficiency problem, albeit with the new virtual values $\gamma_i^e(x_i)$:

$$\gamma_i^e(x_i) = \begin{cases} x_i + \frac{\lambda_i^e}{f(x_i)}, & \text{if} \quad x_i < \overline{x}_i^e, \\ E(x_i|x_i \geq \overline{x}_i^e) - \frac{\lambda_i^e \overline{x}_i^e}{1-F(\overline{x}_i^e)}, & \text{if} \quad x_i \geq \overline{x}_i^e. \end{cases} \tag{34}$$

Theorems 1 -4 and Lemma 7 apply to the constrained-efficient mechanism verbatim. Particularly, by Theorem 1 the virtual values $\gamma_i^e(.)$ must be continuous at $\bar{x}_i^e$, and there is a 1-to-1 relationship between the multipliers vector $(\lambda_1^e, ..., \lambda_n^e)$ and the thresholds vector $(\bar{x}_1^e, ..., \bar{x}_n^e)$. Namely, for all $i \in \{2, ..., n\}$ and for $i = 1$ if $\bar{x}_1^e = \bar{x}_2^e$ we have:

$$\lambda_i^e = \frac{(E(x_i|x_i \geq \bar{x}_i^e) - \bar{x}_i^e)(1 - F_i(\bar{x}_i^e))f(\bar{x}_i^e)}{\bar{x}_i^e f(\bar{x}_i^e) + 1 - F_i(\bar{x}_i^e)}.$$

When $\bar{x}_1 > \bar{x}_2$, then

$$\lambda_1^e = f(\bar{x}_1^e)(\gamma_2^e(\bar{x}_2^e) - \bar{x}_1^e).$$

From these expressions the virtual values $\gamma_i^e(x)$ are easily obtained.

By Theorem 2, constrained-efficient mechanism is unique and is characterized by the threshold profile $(\bar{x}_1^e, ..., \bar{x}_n^e)$ satisfying the first-order conditions of this Theorem. Consequently this mechanism, as the optimal one, is either a top auction or a budget-handicap auction, depending on the budget profile. Also, by Theorem 5, under fixed aggregate budget maximal efficiency is attained when all bidders have equal budgets.

However, for each budget profile the parameters of the constrained-efficient mechanism are different from the optimal one, because the "virtual values" γ_i^e differ from the virtual values for the optimal mechanism in (14). Consider first the top auction. The equation determining the common threshold in the constrained-efficient top auction, $\bar{x}^{te}$, is:

$$\sum_{i=1,\ldots,n} m_i = \bar{x}^{te} \frac{1 - F(\bar{x}^{te})^n}{1 - F(\bar{x}^{te})} - n \int_0^{\bar{x}^{te}} F(s)^{n-1} ds \tag{35}$$

Equation (35) implies that x^{te} is increasing in $\sum_{i=1,\ldots,n} m_i$, while its counterpart for the optimal mechanism, equation (26), implies that x^t is increasing in $\sum_{i=1,\ldots,n} m_i$. However, under the same budget profile we have $x^{te} < x^t$ because the reservation value r is positive (zero) in the optimal (constrained-efficient) top auction. Since the same family of inequalities (28) must be satisfied in both optimal and constrained-efficient mechanisms and the right-hand side of (28) is non-monotone in its argument, it follows that under the same budget profile top-auction may be a solution to one of the problems, but not to the other one. In fact, as we illustrate below by an example, the respective sets of budget profiles are non-nested.

Similarly, the constrained-efficient mechanism is a handicap auction when the budgets are small but sufficiently different so that (28) fails, and the corresponding budget set is non-nested with the budget set under which the optimal mechanism is a handicap auction. However, the lowest budget under which all budget constraints are non-binding in the optimal mechanism is higher than in the efficient mechanism. This is so because in the efficient mechanism this budget level is equal to $1 - \int_0^1 F^{n-1}(s)ds$, while in the optimal mechanism it equals $1 - \int_r^1 F^{n-1}(s)ds$ where r is positive and solves $r - \frac{1-F(r)}{f(r)} = 0$.

To conclude, the constrained-efficient mechanism does not attain full efficiency. In the top auction inefficiency arises because all bidders with valuations above the common threshold $\bar{x}^{te}$ are tied. In the budget-handicap auction, a richer bidder is handicapped and loses in competition with poorer bidders if the latter have lower valuations by some margin.

7.1 Example of Constrained-Efficient Mechanism

In this subsection we compute the constraint-efficient mechanism for two bidders whose types are distributed uniformly on $[0, 1]$ and who have budgets m_1 and m_2, respectively.

First, neither budget constraint is binding and the constrained-efficient mechanism is a standard all-pay auction if $m_2 \geq \frac{1}{2}$, since in this case $m_2 \geq 1 - \int_0^1 sds = \frac{1}{2}$.

Now suppose that $m_2 \leq \frac{1}{2}$. Let us first consider top auction. Using (35) to compute the threshold $\bar{x}^{te}$ and checking the necessary conditions in (28), yields that the constrained-efficient mechanism is a top auction with threshold $\bar{x}^{te} = m_1 + m_2$ if $m_1 \leq \sqrt{2m_2} - m_2$.

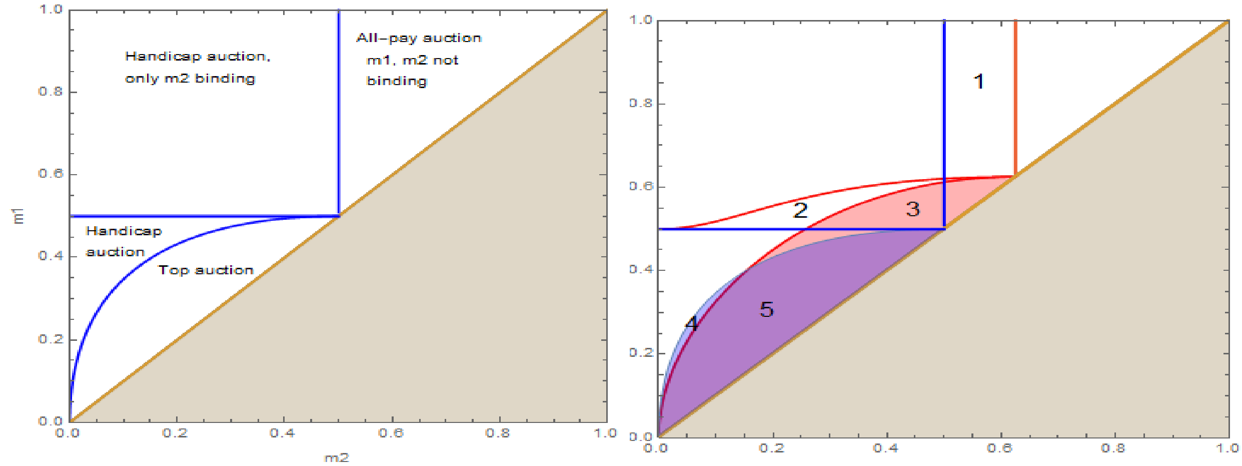
Now suppose that $m_1 > \sqrt{2m_2} - m_2$ and $m_2 \leq \frac{1}{2}$. Then the solution is a budget-handicap auction. First, let us explore the budget-handicap auction with two binding budget constraints. In this case, $\bar{x}_2^e < \bar{x}_1^e < 1$, $\lambda_2^e(x) = \frac{(1-\bar{x}_2^e)^2}{2}$, $\lambda_1^e = -\bar{x}_1^e + \bar{x}_2^e + \frac{(1-\bar{x}_2^e)^2}{2} = -\bar{x}_1^e + \frac{1+(\bar{x}_2^e)^2}{2}$. So $\gamma_2^e(x) = x_2 + \frac{(1-\bar{x}_2^e)^2}{2}$ for $x \leq \bar{x}_2^e$, $\gamma_1^e(x) = x_1 - \bar{x}_1^e + \frac{1+(\bar{x}_2^e)^2}{2}$ for $x < \bar{x}_1^e$, $\gamma_1^e(\bar{x}_1^e) = \frac{1}{2} + \frac{1+\bar{x}_1^e(\bar{x}_1^e-(\bar{x}_2^e)^2)}{2(1-\bar{x}_1^e)} > \frac{1+(\bar{x}_2^e)^2}{2}$. Using Lemma 7 we can now compute the thresholds $\bar{x}_1^e$ and $\bar{x}_2^e$ in the budget-handicap auction:

$$m_1 = \bar{x}_1^e - \int_{x_1 \in [0, \bar{x}_1^e)} q_1(x_1)dx_1 = \bar{x}_1^e - \int_{x_1 \in [0, \bar{x}_1^e)} \int_{\gamma_1^e(x_1) > \gamma_2^e(x_2)} dx_2 dx_1 = \bar{x}_1^e - \frac{(\bar{x}_2^e)^2}{2} \tag{36}$$

$$m_2 = \bar{x}_2^e \bar{x}_1^e - \int_{x_2 \in [0, \bar{x}_2^e)} q_2(x_2)dx_2 = \bar{x}_2^e \bar{x}_1^e - \int_{x_2 \in [0, \bar{x}_2^e)} \int_{\gamma_2^e(x_2) > \gamma_1^e(x_1)} dx_1 dx_2 = \frac{(\bar{x}_2^e)^2}{2} \tag{37}$$

Solving (36) and (37) yield thresholds $\bar{x}_1^e = m_1 + m_2$ and $\bar{x}_2^e = \sqrt{2m_2}$ in the budget-handicap auction when both budget constraints are binding, which is true when $\lambda_1^e = -\bar{x}_1^e + \frac{1+(\bar{x}_2^e)^2}{2} > 0$. This inequality is equivalent to $m_1 < \frac{1}{2}$ given that $\bar{x}_1^e = m_1 + m_2$ and $\bar{x}_2^e = \sqrt{2m_2}$.

Figure 7: Constrained-Efficient and Optimal Mechanisms



(a) Constrained-Efficient Mechanism.

(b) Constrained-Efficient vs Optimal

Finally, if $m_1 \geq \frac{1}{2} \geq m_2$, then the budget constraint of bidder 1 is no longer binding so (36) does not hold and $\lambda_1 = -\bar{x}_1^e + \frac{1+(\bar{x}_2^e)^2}{2} = 0$. Using the latter equality and (37) we obtain $\bar{x}_1^e = \frac{1}{2} + m_2$, $\bar{x}_2^e = \sqrt{2m_2}$, and the maximal transfer paid by bidder 1 with valuation in $\left[\frac{1}{2} + m_2, 1\right]$ is equal to $\bar{x}_1^e - \frac{(\bar{x}_2^e)^2}{2} = \frac{1}{2}$.

To summarize, the constrained-efficient mechanism in this example is:

(i) A standard symmetric all-pay auction with zero reservation value for each bidder and non-binding budget constraints if $m_2 \geq \frac{1}{2}$.

(ii) Top auction with zero reserve and threshold $x^{te} = m_1 + m_2$ if $m_1 \leq \sqrt{2m_2} - m_2$ and $m_2 \leq \frac{1}{2}$.

(iii) Budget-handicap auction with both budget constraint binding and thresholds $\bar{x}_1^e = m_1 + m_2$ and $\bar{x}_2^e = \sqrt{2m_2}$ if $\sqrt{2m_2} - m_2 < m_1 < \frac{1}{2}$.

(iv) Budget-handicap auction in which only the budget constraint of bidder 2 is binding, with thresholds $\bar{x}_1^e = \frac{1}{2} + m_2$ and $\bar{x}_2^e = \sqrt{2m_2}$, if $m_1 \geq \frac{1}{2} \geq m_2$.

Figure 7a depicts how constrained-efficient mechanism depends on the budgets. Figure 7b highlights budget regions in which the constrained-efficient (listed first) and optimal mechanisms (listed second) are different. Specifically, these differences are as follows:

Area 1: All-pay auction vs. budget- handicap auction with only m_2 binding;

Area 2: Budget-handicap Auction with m_2 binding only vs. budget-handicap auction with both budget constraints binding;

Area 3: All-pay or handicap auction vs. Top auction;

Area 4: Top auction vs Budget-handicap auction;

Area 5: top auction is both constrained-efficient and optimal mechanism.

8 Conclusions

In this paper, we have characterized the optimal and the constrained-efficient mechanisms when the bidders have commonly known and unequal budgets. We have demonstrated that the designer should use either a "top auction" mechanism of a budget-handicap auction. The former is used when budget differences are small. In this mechanism, the designer discriminates between the bidders only when they have high valuations exceeding an endogenous common threshold at which all budget constraints become binding. Above this threshold richer bidders are awarded the good with a higher probability than poorer bidders.

When budget differences are sufficiently large, the designer has to use a "budget-handicap" auction in which the valuation thresholds at which budget constraints become binding differ across bidders. In addition to discriminating between bidders who have valuations, this mechanism also discriminates between bidders with low valuations favoring low-budget bidders. This feature of the budget-handicap auction provides justification for favoring smaller or minority-owned businesses in public procurement and allocation mechanism, such as spectrum auctions where bidders often do have known and asymmetric budget constraints.

Our mechanisms have features of an all-pay auction, since a bidder always pays her bid. It would be interesting to consider a modification of our set-up and consider mechanisms in which a bidder pays only when (s)he gets the good. We leave this issue for future research.

One other interesting qualitative property of the optimal and constrained-efficient mechanisms emerges from our analysis of the two bidder case. There, we show that when one bidder has a significantly larger budget than the other, the mechanism has "buy-it-now" features. Generalizing this result to a more general set-up with many bidders is another extension which we leave for future research.

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9 Appendix A: Proofs

Proof of Theorem 1: The proof of the Theorem relies on the following Lemma:

Lemma 10 *The following cannot hold in the optimal mechanism for any i s.t. $\bar{x}_i < 1$:*

(a) $\gamma_i^-(\bar{x}_i) > \gamma_i(\bar{x}_i)$, and the set $A_i(\lambda, \bar{\mathbf{x}})= \{x_{-i} \in [0,1]^{n-1} : \gamma_i(\bar{x}_i) \leq \max\{0, \max_{j\neq i} \gamma_j(x_j)\} < \gamma_i^-(\bar{x}_i)\}$ has a positive measure, where $(\lambda, \bar{\mathbf{x}})= (\lambda_1, ..., \lambda_n, \bar{x}_1, ..., \bar{x}_n)$.

(b) $\gamma_i(\bar{x}_i) > \gamma_i^-(\bar{x}_i)$, and the set $B_i(\lambda, \bar{\mathbf{x}})= \{x_{-i} \in [0,1]^{n-1} : \gamma_i^-(\bar{x}_i) < \max\{0, \max_{j\neq i} \gamma_j(x_j)\} \leq \gamma_i(\bar{x}_i)\}$ has a positive measure.

Proof of Lemma 10:

To prove the Lemma, we will use the first-order conditions of objective in (15) with respect to $\bar{x}_i$. Although it may not possess a derivative with respect to $\bar{x}_i$ because of max operator in its second term, it does possess left- and right- derivatives which we denote by $\frac{\partial_-\mathcal{L}}{\partial\bar{x}_i}$ and $\frac{\partial_+\mathcal{L}}{\partial\bar{x}_i}$, respectively. Then since $\bar{x}_i < 1$, the following conditions must hold: $\frac{\partial_+\mathcal{L}}{\partial\bar{x}_i} \geq 0$ and $\frac{\partial_-\mathcal{L}}{\partial\bar{x}_i} \leq 0$. Note that $\bar{x}_i = 0$ is never optimal because in this case $\gamma_i(x_i) = 0$ for all $x_i \in [0,1]$. Differentiating (15) yields:

$$\frac{\partial_+\mathcal{L}}{\partial\bar{x}_i} = f(\bar{x}_i) \int_{x_{-i}\in[0,1]^{n-1}} \left(\max\{0, \gamma_i^-(\bar{x}_i), \max_{j\neq i} \gamma_j(x_j)\} - \max\{0, \gamma_i(\bar{x}_i), \max_{j\neq i} \gamma_j(x_j)\} \right) dF(x_{-i})$$
$$+ \int_{x\in[0,1]^n} \frac{\partial_+ \max\{0, \max_{j=1,...,n} \gamma_j(x_j)\}}{\partial\bar{x}_i} dF(x) \tag{38}$$

The first term in (38) comes from possible discontinuity of the integrand of $\mathcal{L}$ in (15) at $x_i = \bar{x}_i$. The second term comes from differentiating the integrand of $\mathcal{L}$.

First, let us prove part (a) of the Lemma by contradiction. So suppose that $\gamma_i^-(\bar{x}_i) > \gamma_i(\bar{x}_i)$, $\bar{x}_i < 1$, and the set $A_i(\lambda, \bar{\mathbf{x}})$ has a positive measure. The former inequality is equivalent to $\lambda_i > \frac{(1-F(\bar{x}_i))^2}{1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i)}$, and implies that $\gamma_i^-(\bar{x}_i) > \frac{\bar{x}_i^2 f(\bar{x}_i)}{1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i)} > 0$.

Therefore, the first term in (38) can be rewritten as follows:

$$f(\bar{x}_i) \int_{x_{-i}\in[0,1]^{n-1}:\max\{0,\max_{j\neq i}\gamma_j(x_j)\}<\gamma_i(\bar{x}_i)} \gamma_i^-(\bar{x}_i) - \gamma_i(\bar{x}_i) dF(x_{-i}) +$$
$$f(\bar{x}_i) \int_{x_{-i}\in[0,1]^{n-1}:\gamma_i(\bar{x}_i)\leq\max\{0,\max_{j\neq i}\gamma_j(x_j)\}<\gamma_i^-(\bar{x}_i)} \left(\gamma_i^-(\bar{x}_i) - \max\{0, \max_{j\neq i} \gamma_j(x_j)\} \right) dF(x_{-i}) \tag{39}$$

Now, let us consider the second term of (38). From (14) we have:

$$\frac{\partial_+ \gamma_i(x_i)}{\partial \overline{x}_i} = \begin{cases} 0, & \text{if } x_i < \overline{x}_i, \\ 1 - \frac{\lambda_i}{(1-F(\overline{x}_i))^2}\left(1 - F\left(\overline{x}_i\right) + \overline{x}_i f\left(\overline{x}_i\right)\right) = \frac{f(\bar{x}_i)}{1-F(\bar{x}_i)}\left(\gamma_i(\bar{x}_i) - \gamma_i^-(\bar{x}_i)\right), & \text{if } x_i \geq \bar{x}_i, \end{cases} \tag{40}$$

From (40) and $\gamma_i^-(\bar{x}_i) > \gamma_i(\bar{x}_i)$ it follows that $\frac{\partial_+ \gamma_i(x_i)}{\partial \overline{x}_i} < 0$ for $x_i \geq \overline{x}_i$. This and the fact that $\frac{\partial \gamma_j(x_j)}{\partial \bar{x}_i} = 0$ for $j \neq i$ imply that the second term in (38) equals:

$$\int_{x:\, x_i \in [\bar{x}_i, 1] : \max\{0, \max_{j\neq i} \gamma_j(x_j)\} < \gamma_i(\bar{x}_i)} \frac{\partial_+ \gamma_i(x_i)}{\partial \overline{x}_i} dF(x) =$$

$$f(\bar{x}_i) \int_{x_{-i} \in [0,1]^{n-1} : \max\{0, \max_{j\neq i} \gamma_j(x_j)\} < \gamma_i(\bar{x}_i)} \gamma_i(\bar{x}_i) - \gamma_i^-(\bar{x}_i) dF(x_{-i}) \tag{41}$$

Using (39) and (41) in (38) yields:

$$\frac{\partial_+ \mathcal{L}}{\partial \bar{x}_i} = f(\bar{x}_i) \int_{x_{-i} : \gamma_i(\bar{x}_i) \leq \max\{0, \max_{j\neq i} \gamma_j(x_j)\} < \gamma_i^-(\bar{x}_i)} \left(\gamma_i^-(\bar{x}_i) - \max\{0, \max_{j\neq i} \gamma_j(x_j)\} \right) dF(x_{-i}) > 0 \tag{42}$$

where the inequality holds because the set of integration is $A_i(\lambda, \bar{\mathbf{x}})$, which has a positive measure and is compact, and the integrand is positive everywhere on this set. But then $\bar{x}_i$ cannot be an optimal choice.

Next, consider part (b). Again, the proof is by contradiction, so suppose that $\gamma_i(\bar{x}_i) > \gamma_i^-(\bar{x}_i)$, and the set $B_i(\lambda, \bar{\mathbf{x}})$ has a positive measure. The former is equivalent to $\lambda_i < \frac{(1-F(\bar{x}_i))^2}{1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i)}$ and implies that $\gamma_i(\bar{x}_i) > 0$. Then the first term in (38) is equal to:

$$f(\bar{x}_i) \int_{x_{-i} \in [0,1]^{n-1} : \max_{j\neq i} \gamma_j(x_j) \leq \gamma_i(\bar{x}_i)} \left(\max\{0, \gamma_i^-(\bar{x}_i), \max_{j\neq i} \gamma_j(x_j)\} - \gamma_i(\bar{x}_i) \right) dF(x_{-i}) \tag{43}$$

From (40) it follows that $\frac{\partial_+ \gamma_i(x_i)}{\partial \overline{x}_i} > 0$ if $x_i \geq \bar{x}_i$ and $\frac{\partial_+ \gamma_i(\bar{x}_i)}{\partial \bar{x}_i} = 0$ if $x_i < \bar{x}_i$. Since $\frac{\partial \gamma_j(x_j)}{\partial \bar{x}_i} = 0$ for $j \neq i$ and $\gamma_i(\overline{x}_i) > 0$, the second term of (38) in this case equals:

$$f(\bar{x}_i) \int_{x_{-i} \in [0,1]^{n-1} : \max_{j\neq i} \gamma_j(x_j) \leq \gamma_i(\bar{x}_i)} \gamma_i(\bar{x}_i) - \gamma_i^-(\bar{x}_i) dF(x_{-i}) \tag{44}$$

Combining (43) and (44) yields:

$$\frac{\partial_+ \mathcal{L}}{\partial \bar{x}_i} = f(\bar{x}_i) \int_{x_{-i} \in [0,1]^{n-1} : \max_{j\neq i} \gamma_j(x_j) \leq \gamma_i(\bar{x}_i)} \max\{0, \gamma_i^-(\bar{x}_i), \max_{j\neq i} \gamma_j(x_j)\} - \gamma_i^-(\bar{x}_i) dF(x_{-i}) =$$

$$f(\bar{x}_i) \int_{x_{-i} \in [0,1]^{n-1} : \gamma_i^-(\bar{x}_i) < \max\{0, \max_{j\neq i} \gamma_j(x_j)\} \leq \gamma_i(\bar{x}_i)} \max\{0, \max_{j\neq i} \gamma_j(x_j)\} - \gamma_i^-(\bar{x}_i) dF(x_{-i}) > 0. \tag{45}$$

where the equality holds because the integrand under the first integral is nonnegative for all x_{-i} and is positive only if $\gamma_i^-(\bar{x}_i) < \max\{0, \max_{j\neq i}\gamma_j(x_j)\}$, and the set $\{x_{-i} : \max\{0, \max_{j\neq i}\gamma_j(x_j)\} \leq \gamma_i(\bar{x}_i)\}$ is identical to the set $\{x_{-i} : \max_{j\neq i}\gamma_j(x_j) \leq \gamma_i(\bar{x}_i)\}$ because $\gamma_i(\bar{x}_i) > 0$. Finally, note the set of integration under the last integral is $B_i(\lambda, \bar{\mathbf{x}})$ which has a positive measure, is compact and the integrand is positive for any x_{-i} in this set, so the final inequality holds. But then $\bar{x}_i$ cannot be an optimal choice. This completes the proof of the Lemma. *Q.E.D.*

Next, we prove the following *Claim 1: $\lambda_i \leq 1 - F(\bar{x}_i)$ for all $i \in \{1, ..., n\}$.*

Although we need to consider only the case where $\bar{x}_i < 1$ for all $i \in \{1, ..., n\}$, this claim also holds when $\bar{x}_j = 1$ for some j, because in this case $\lambda_j = 0$ and the following proof applies verbatim to establish the result for other i.

As a first step, let $h_1 \in \arg\max_i \gamma_i^-(\bar{x}_i)$ and suppose that $\lambda_{h_1} > 1 - F(\bar{x}_{h_1})$, and so by (14) $\gamma_{h_1}(\bar{x}_{h_1}) < 0 < \gamma_{h_1}^-(\bar{x}_{h_1})$. Let us show that we cannot be at the optimum.

Note that for every i s.t. $\gamma_i(\bar{x}_i) < \gamma_i^-(\bar{x}_i)$, we have $\gamma_{h_1}(\bar{x}_{h_1}) < \max\{0, \gamma_i^-(\bar{x}_i)\} < \gamma_{h_1}^-(\bar{x}_{h_1})$. Also, for every i s.t. $\gamma_i^-(\bar{x}_i) = \gamma_{h_1}^-(\bar{x}_{h_1}) \leq \gamma_i(\bar{x}_i)$, we have $\lambda_i < 1 - F(\bar{x}_i)$, and hence $\gamma_i(x_i)$ is increasing on $[0, \bar{x}_i]$. So, $\gamma_{h_1}(\bar{x}_{h_1}) < \max\{0, \gamma_i(x_i)\} < \gamma_{h_1}^-(\bar{x}_{h_1})$ for all $x_i \in [0, \bar{x}_i)$. Finally, if $\gamma_i^-(\bar{x}_i) < \gamma_{h_1}^-(\bar{x}_{h_1})$, then $\gamma_{h_1}(\bar{x}_{h_1}) < \max\{0, \gamma_i(x_i)\} < \gamma_{h_1}^-(\bar{x}_{h_1})$ for all x_i in some interval $(\bar{x}_i - \delta_i, \bar{x}_i)$ where $\delta_i > 0$. Hence, the set $A_{\bar{i}}(\lambda, \bar{\mathbf{x}})$ has a positive measure, and so by part (a) of Lemma 10 we cannot be at the optimum.

Next, we proceed by induction. Fix $h_k \in \{1, ..., n\}$ and suppose that for every i s.t. $\gamma_{h_k}^-(\bar{x}_{h_k}) < \gamma_i^-(\bar{x}_i)$ we have $\lambda_i \leq 1 - F(\bar{x}_i)$. Let us show that $\lambda_{h_k} \leq 1 - F(\bar{x}_{h_k})$. To prove this, we suppose otherwise i.e., $\lambda_{h_k} > 1 - F(\bar{x}_{h_k})$, and so by definition in (14) $\gamma_{h_k}(\bar{x}_{h_k}) < 0 < \gamma_{h_k}^-(\bar{x}_{h_1})$. Let us show that this cannot be optimal.

By inductive assumption, for every i s.t. $\gamma_{h_k}^-(\bar{x}_{h_k}) < \gamma_i^-(\bar{x}_i)$ we have $\lambda_i \leq 1 - F(\bar{x}_i)$ and hence $\gamma_i(x_i)$ is increasing on $[0, \bar{x}_i]$, with $\gamma_i(0) < 0$. By definition in (14), the same is true for every i such that $\gamma_i^-(\bar{x}_i) \leq \gamma_i(\bar{x}_i)$. So for all such i, $\gamma_{h_k}(\bar{x}_{h_k}) < \max\{0, \gamma_i(x_i)\} < \gamma_{h_k}^-(\bar{x}_{h_k})$ for all x_i in some interval $(\bar{x}_i - \delta_i, \bar{x}_i)$ where $\delta_i > 0$. Finally, if $\gamma_i(\bar{x}_i) < \gamma_i^-(\bar{x}_i)$, then $\lambda_i > 1 - F(\bar{x}_i)$ and so by inductive assumption $\gamma_i^-(\bar{x}_i) \leq \gamma_{h_k}^-(\bar{x}_{h_k})$, and hence $\gamma_{h_k}(\bar{x}_{h_k}) < \max\{0, \gamma_i(\bar{x}_i)\} < \gamma_{h_k}^-(\bar{x}_{h_k})$. Hence, the set $A_{h_k}(\lambda, \bar{\mathbf{x}})$ has a positive measure, and so by part (a) of Lemma 10 we cannot be at the optimum. This completes the proof of Claim 1.

An important consequence of Claim 1 is that $\gamma_i(x_i)$ is increasing on $[0, \bar{x}_i)$ for all i, with $\gamma_i(0) < 0$. Making use of this property, we will establish the next two Claims.

Claim 2: Suppose that $\gamma_i^-(\bar{x}_i) \leq \gamma_j^-(\bar{x}_j)$ for some i and j. Then $\gamma_i^-(\bar{x}_i) \leq \gamma_i(\bar{x}_i)$.

Suppose that $\gamma_i(\bar{x}_i) < \gamma_i^-(\bar{x}_i)$. Let us show that we cannot be at the optimum. Note that $\gamma_i^-(\bar{x}_i) > 0$. By Claim 1, $\gamma_k(x_k)$ is increasing on $[0, \bar{x}_k)$ for all k , with $\gamma_k(0) < 0$. So there exists $\tilde{x}_k \in (0, \bar{x}_k]$ s.t. $\gamma_k(x_k) < \gamma_i^-(\bar{x}_i)$ for all $x_k \in [0, \tilde{x}_k)$. Moreover, there exists $\tilde{x}_j \in (0, \bar{x}_j]$ s.t. $\gamma_j(\tilde{x}_j) = \gamma_i^-(\bar{x}_j)$. Therefore, $\gamma_i(\bar{x}_i) < \max\{0, \gamma_j(x_j), \max_{k \neq i,j} \gamma_k(x_k)\} < \gamma_i^-(\bar{x}_i)$ if $x_k \in [0, \tilde{x}_k)$ for all $k \notin \{i, j\}$, and $x_j \in (\bar{x}_j - \delta_j, \bar{x}_j)$ for some $\delta_j > 0$. So, the set $A_i(\lambda, \bar{\mathbf{x}})$ has a positive measure, and by part (a) of Lemma 10 we cannot be at the optimum.

Claim 3: Suppose that in an optimal mechanism $h_1 \in \arg\max_{j \in \{1,\ldots,n\}} \gamma_i^-(\bar{x}_j)$ and i, $i \neq h_1$, is such that either (i) $\gamma_i^-(\bar{x}_i) < \gamma_{h_1}^-(\bar{x}_{h_1})$, or (ii) $\gamma_i^-(\bar{x}_i) = \gamma_{h_1}^-(\bar{x}_{h_1})$ and $\gamma_i(\bar{x}_i) \leq \gamma_{h_1}(\bar{x}_{h_1})$. Then $\gamma_i(\bar{x}_i) \leq \gamma_i^-(\bar{x}_i)$.

Let us suppose that $\gamma_i^-(\bar{x}_i) < \gamma_i(\bar{x}_i)$, and so $\gamma_i(\bar{x}_i) > 0$. We will show that this cannot be optimal. As the Claim is stated, we need to consider two cases:

Case (i): $\gamma_i^-(\bar{x}_i) < \gamma_{h_1}^-(\bar{x}_{h_1})$. Since $\lambda_j \leq 1 - F(\bar{x}_j)$ and hence $\gamma_j(0) < 0$ for all j, the set $\{x_j : \gamma_j(x_j) < \gamma_i(\bar{x}_i)\}$ has a positive measure for all $j \notin \{i, h_1\}$, and the set $\{x_{h_1} : \gamma_i^-(\bar{x}_i) < \gamma_{h_1}(x_{h_1}) < \gamma_i(\bar{x}_i)\}$ also has a positive measure. It follows that the set $B_i(\lambda, \bar{\mathbf{x}})$ has a positive measure. Hence, $\gamma_i^-(\bar{x}_i) < \gamma_i(\bar{x}_i)$ is suboptimal by Lemma 10. Note that this argument also implies that $\gamma_i^-(\bar{x}_i) \leq \gamma_{h_1}(\bar{x}_{h_1})$.

Case (ii): $\gamma_i^-(\bar{x}_i) = \gamma_{h_1}^-(\bar{x}_{h_1})$. We will complete this case by showing that we cannot have $\gamma_i^-(\bar{x}_i) = \gamma_{h_1}^-(\bar{x}_{h_1}) < \gamma_i(\bar{x}_i) \leq \gamma_{h_1}(\bar{x}_{h_1})$. For, suppose that this is so. Then $0 < \gamma_i(\bar{x}_i) \leq \gamma_{h_1}(\bar{x}_{h_1})$ and $\bar{x}_i < 1$, $\bar{x}_{h_1} < 1$. Observe that the set $\{x_j : \gamma_j(x_j) < \gamma_i(\bar{x}_i)\}$ includes $[0, \bar{x}_j)$ for all j. Also, we have $\{x_i : \gamma_{h_1}^-(\bar{x}_{h_1}) < \gamma_i(x_i) \leq \gamma_{h_1}(\bar{x}_{h_1})\} = [\bar{x}_i, 1]$, and this set also has a has a positive measure. It follows that the set $B_{h_1}(\lambda, \bar{\mathbf{x}})$ has a positive measure, which cannot hold at the optimum by part (b) of Lemma 10. This completes the proof of Claim 3.

Summarizing, Claims 2 and 3 imply that $\gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i)$ either if $\gamma_i^-(\bar{x}_i) = \gamma_{h_1}^-(\bar{x}_{h_1})$ and $\gamma_i(\bar{x}_i) \leq \gamma_{h_1}(\bar{x}_{h_1})$, or if $\gamma_i^-(\bar{x}_i) < \gamma_{h_1}^-(\bar{x}_{h_1})$. Observe that in the latter case i.e., when $\gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i) < \gamma_{h_1}^-(\bar{x}_{h_1})$, we must also have $\gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i) < \gamma_{h_1}(\bar{x}_{h_1})$, for otherwise we cannot be at the optimum by part (a) of Lemma 10.

Hence, we may conclude that the vector $\{\gamma_i^-(\bar{x}_i), \gamma_i(\bar{x}_i)\}$, $i = 1, .., n$, in the optimal mechanism has the following properties. First, there exists h_1 s.t. $\min\{\gamma_{h_1}^-(\bar{x}_{h_1}), \gamma_{h_1}(\bar{x}_{h_1})\} \geq \max_{i \neq h_1}\{\gamma_i^-(\bar{x}_i), \gamma_i(\bar{x}_i)\}$. For $i \neq h_1$ we have, $\gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i)$. For h_1, we have either $\min\{\gamma_{h_1}^-(\bar{x}_{h_1}), \gamma_{h_1}(\bar{x}_{h_1})\} > \max_{i \neq h_1} \gamma_i^-(\bar{x}_i)$ or $\gamma_{h_1}(\bar{x}_{h_1}) \geq \gamma_{h_1}^-(\bar{x}_{h_1}) = \gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i)$ for some $i \neq h_1$.

By definition (14), the equality $\gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i)$ is equivalent to $\lambda_i = \frac{(1-F(\bar{x}_i))^2}{(1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i))}$. Substituting this into (14) yields $\gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i) = \frac{\bar{x}_i^2 f(\bar{x}_i)}{1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i)}$, which is increasing in $\bar{x}_i$.

Note that if $\bar{x}_{h_1} = \bar{x}_i$ for some i, then it is easy to see that $\gamma_i^-(\bar{x}_i) = \gamma_i(\bar{x}_i) = \gamma_{h_1}(\bar{x}_{h_1}) = \gamma_{h_1}^-(\bar{x}_{h_1})$, and vice versa.

So to complete the proof of part 1 of the Theorem, we establish the following: *Claim 4:*

$\bar{x}_{h_1} > \bar{x}_i$ *for all* $i \neq h_1$ *if and only if for any such* i *either* $\min\{\gamma^-_{h_1}(\bar{x}_{h_1}), \gamma_{h_1}(\bar{x}_{h_1})\} > \gamma^-_i(\bar{x}_i) = \gamma_i(\bar{x}_i)$*, or* $\gamma_{h_1}(\bar{x}_{h_1}) > \gamma^-_{h_1}(\bar{x}_{h_1}) = \gamma^-_i(\bar{x}_i) = \gamma_i(\bar{x}_i)$.

"Only If": Suppose that $\bar{x}_{h_1} > \bar{x}_i$ for all i. Let us show that $\gamma^-_{h_1}(\bar{x}_{h_1}) \geq \gamma^-_i(\bar{x}_i)$ for any i. For suppose otherwise i.e., $\gamma^-_i(\bar{x}_i) > \gamma^-_{h_1}(\bar{x}_{h_1})$. Then by Steps 2 and 3 above, $\gamma_i(\bar{x}_i) > \gamma^-_{h_1}(\bar{x}_{h_1}) = \gamma_{h_1}(\bar{x}_{h_1}) = \frac{\bar{x}^2_{h_1} f(\bar{x}_{h_1})}{1-F(\bar{x}_{h_1})+\bar{x}_{h_1} f(\bar{x}_{h_1})}$. So, we have $\min\{\gamma^-_i(\bar{x}_i), \gamma_i(\bar{x}_i)\} > \gamma^-_{h_1}(\bar{x}_{h_1}) = \gamma_{h_1}(\bar{x}_{h_1})$. However, it is easy to see from (14) that this cannot hold when $\bar{x}_{h_1} > \bar{x}_i$.

Given that $\gamma^-_{h_1}(\bar{x}_{h_1}) \geq \gamma^-_i(\bar{x}_i)$ for all i, we only need to rule out the case $\gamma^-_{h_1}(\bar{x}_{h_1}) = \gamma_{h_1}(\bar{x}_{h_1}) = \gamma^-_i(\bar{x}_i) \leq \gamma_i(\bar{x}_i)$ for some i. Observe that in this case $\gamma^-_{h_1}(\bar{x}_{h_1}) = \gamma_{h_1}(\bar{x}_{h_1}) = \frac{\bar{x}^2_{h_1} f(\bar{x}_{h_1})}{1-F(\bar{x}_{h_1})+\bar{x}_{h_1} f(\bar{x}_{h_1})}$ and $\lambda_i \leq \frac{(1-F(\bar{x}_i))^2}{(1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i))}$. Then is easy to see from (14) that the previous inequality $\gamma^-_{h_1}(\bar{x}_{h_1}) = \gamma_{h_1}(\bar{x}_{h_1}) = \gamma^-_i(\bar{x}_i) \leq \gamma_i(\bar{x}_i)$ cannot hold since $\bar{x}_{h_1} > \bar{x}_i$.

It follows that $\gamma^-_{h_1}(\bar{x}_{h_1}) \geq \gamma^-_i(\bar{x}_i) = \gamma_i(\bar{x}_i)$. So, by Steps 2 and 3, we can only have $\min\{\gamma^-_{h_1}(\bar{x}_{h_1}), \gamma_{h_1}(\bar{x}_{h_1})\} > \gamma^-_i(\bar{x}_i) = \gamma_i(\bar{x}_i)$ or $\gamma_{h_1}(\bar{x}_{h_1}) > \gamma^-_{h_1}(\bar{x}_{h_1}) = \gamma^-_i(\bar{x}_i) = \gamma_i(\bar{x}_i)$, completing the proof of the **Only If** statement.

"If:" First note that $\gamma^-_i(\bar{x}_i) = \gamma_i(\bar{x}_i)$ implies $\gamma^-_i(\bar{x}_i) = \gamma_i(\bar{x}_i) = \frac{\bar{x}^2_i f(\bar{x}_i)}{1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i)}$ and $\lambda_i = \frac{(1-F(\bar{x}_i))^2}{(1-F(\bar{x}_i)+\bar{x}_i f(\bar{x}_i))}$.

So, $\bar{x}_{h_1} > \bar{x}_i$ follows from (14) both if $\min\{\gamma^-_{h_1}(\bar{x}_{h_1}), \gamma_{h_1}(\bar{x}_{h_1})\} > \gamma^-_i(\bar{x}_i) = \gamma_i(\bar{x}_i)$, and if $\gamma_{h_1}(\bar{x}_{h_1}) > \gamma^-_{h_1}(\bar{x}_{h_1}) = \gamma^-_i(\bar{x}_i) = \gamma_i(\bar{x}_i)$. In particular, to see this in the latter case, note that $\gamma^-_h(\bar{x}_h) < \gamma_h(\bar{x}_h)$ implies that $\lambda_h < \frac{(1-F(\bar{x}_h))^2}{(1-F(\bar{x}_h)+\bar{x}_h f(\bar{x}_h))}$ and $\gamma^-_h(\bar{x}_h) < \frac{\bar{x}^2_h f(\bar{x}_h)}{1-F(\bar{x}_h)+\bar{x}_h f(\bar{x}_h)}$. But then $\gamma^-_i(\bar{x}_i) = \gamma^-_h(\bar{x}_h)$ implies that $\bar{x}_h > \bar{x}_i$.

This completes the proof of Claim 4 and part 1 of the Theorem.

Now let us establish part 2 of the Theorem. So, suppose that bidder h_1 is such that $\bar{x}_{h_1} > \bar{x}_i$ for all $i \neq h_1$. As shown in Claim 4 above, there are only two possibilities in this case: for any i either $\min\{\gamma^-_{h_1}(\bar{x}_{h_1}), \gamma_{h_1}(\bar{x}_{h_1})\} > \gamma^-_i(\bar{x}_i) = \gamma_i(\bar{x}_i)$, or $\gamma_{h_1}(\bar{x}_{h_1}) > \gamma^-_{h_1}(\bar{x}_{h_1}) = \gamma^-_i(\bar{x}_i) = \gamma_i(\bar{x}_i)$.

So, to prove this part, it remains to rule out the case $\min\{\gamma_{h_1}(\bar{x}_{h_1}), \gamma^-_{h_1}(\bar{x}_{h_1})\} > \gamma_i(\bar{x}_i) = \gamma^-_i(\bar{x}_i)$ for all $i \neq h_1$. Note, however, that this contradicts the definition of $\bar{x}_i$ in (9). Indeed, let $\breve{x}_{h_1}$ be such that $\breve{x}_{h_1} - \frac{1-\lambda_{h_1}-F(\breve{x}_{h_1})}{f(\breve{x}_{h_1})} = \max_{i \neq h_1} \gamma_i(\bar{x}_i)$. Such $\breve{x}_{h_1}$ exists and satisfies $\breve{x}_{h_1} < \bar{x}_{h_1}$ because $\gamma_{h_1}(0) < 0 < \max_{i \neq h_1} \gamma_i(\bar{x}_i) < \gamma_{h_1}(\breve{x}_{h_1})$. Then for all $x > \breve{x}_{h_1}$, we have $\gamma_{h_1}(x) > \max_{j \neq i} \max_{x_i \in [0,1]} \gamma_i(x_i)$ and so $q_{h_1}(x) = 1$ and so $t_{h_1}(x) = t_{h_1}(1)$, contradicting (9).

Finally, note the value of the Lagrangian (15) does not change when we change the threshold of bidder i from $\bar{x}_i$ to $\breve{x}_{h_1}$, as follows from the following equality:

$$\int_{\breve{x}_{h_1}}^{\bar{x}_{h_1}} x - \frac{1-\lambda_{h_1}-F(x)}{f(x)} dF(x) + \int_{\bar{x}_{h_1}}^{1} \bar{x}_{h_1} - \frac{\lambda_{h_1}\bar{x}_{h_1}}{1-F(\bar{x}_{h_1})} dF(x) = \int_{\breve{x}_{h_1}}^{1} \breve{x}_{h_1} - \frac{\lambda_{h_1}\breve{x}_{h_1}}{1-F(\breve{x}_{h_1})} dF(x)$$

Q.E.D.

Proof of Corollary 1:

Given $\bar{x} = (\bar{x}_1, ..., \bar{x}_n)$, it is obvious that the profile $\lambda(\bar{x})$ defined by (17) and (18) is unique. To establish the opposite i.e., a profile $\lambda = (\lambda_1, ..., \lambda_n)$ uniquely defines a profile $\bar{x}(\lambda)$, observe that by (17) and (18) we have $0 \leq \lambda_i \leq 1 - F(\bar{x}_i)$ for all i, so we can restrict consideration to such profiles. Then (17) and (18) also imply that $\bar{x}_i$ is decreasing in λ_i. Hence if $\lambda_i \leq \lambda_j$ for some j, then $\bar{x}_i$ is a solution to (17) which is well-defined because the right-hand side of (17) is decreasing in $\bar{x}_i$, and is equal to 1 (0) if $\bar{x}_i = 0$ ($\bar{x}_i = 1$).

Finally, if the profile $(\lambda_1, ..., \lambda_n)$ is such that there exists h_1 satisfying $\lambda_{h_1} < \lambda_j$ for all $j \neq h_1$, then $\bar{x}_{h_1}$ is a solution to (18) which is well-defined because: (i) the right-hand side of (18) depends only on $\min_{j \neq h_1} \lambda_j$ and is decreasing in it; (ii) its left-hand side is increasing in λ_{h_1} and in $\bar{x}_{h_1}$ by increasing hazard rate property and because $\lambda_{h_1} \leq 1 - F(\bar{x}_{h_1})$. *Q.E.D.*

Proof of Lemma 5: By Theorem 1, λ_i satisfies $0 < \lambda_i < 1 - F(\bar{x}_i)$ for all i. Since $\gamma_i(x_i) = x_i - \frac{1-\lambda-F(x_i)}{f(x_i)}$ for $x_i \in [0, \bar{x}_i)$, it is immediate that $\gamma_i'(x_i) > 0$ if $f'(x_i) \geq 0$. If $f'(x_i) < 0$, then $\gamma_i'(x_i) > \frac{d\left(x_i - \frac{1-F(x_i)}{f(x_i)}\right)}{dx_i} > 0$. The last inequality holds by the increasing hazard rate property. So, $\gamma_i(x_i)$ is strictly increasing on $[0, \bar{x}_i)$. Equations (17)-(18) in Theorem 1 also imply that $\gamma_i(\bar{x}_i) > \gamma_i(x_i)$ for all $x_i \in [0, \bar{x}_i)$.

But then Lemma 4 implies that $Q_i(x_i, x_{-i})$ and hence $q_i(x_i)$ are both increasing in x_i. So a solution to the relaxed program satisfies the condition of monotonicity of $q_i(.)$ for all i and is therefore also a solution to the full program. *Q.E.D.*

Proof of Lemma 6: To prove the Lemma we argue by contradiction, so suppose that $\bar{x}_j > \bar{x}_i$. Then by part 1 of Theorem 1, $\lambda_i(\bar{x}_i) > 0$. Hence, the budget constraint of bidder i is binding i.e., $m_i = \bar{x}_i q_i(\bar{x}_i) - \int_0^{\bar{x}_i} q_i(s)ds$.

Further, since $\bar{x}_j > \bar{x}_i$, by Theorem 1 we have $\lambda_j < \lambda_i$ and so $\gamma_j(\bar{x}_j) > \gamma_i(\bar{x}_i)$ and $\gamma_j(x) < \gamma_i(x)$ for all $x \in [0, \bar{x}_i]$. So by Lemma 4, $q_j(\bar{x}_j) \geq q_i(\bar{x}_i)$ and $q_j(x) \leq q_i(x)$ for all $x \in [0, \bar{x}_i]$. Therefore, we have:

$$m_j \geq \bar{x}_j q_j(\bar{x}_j) - \int_0^{\bar{x}_j} q_j(s)ds = \bar{x}_i q_j(\bar{x}_j) + \int_{\bar{x}_i}^{\bar{x}_j} (q_j(\bar{x}_j) - q_j(s))ds - \int_0^{\bar{x}_i} q_j(s)ds \geq$$
$$\bar{x}_i q_j(\bar{x}_j) - \int_0^{\bar{x}_i} q_j(s)ds \geq \bar{x}_i q_i(\bar{x}_i) - \int_0^{\bar{x}_i} q_i(s)ds = m_i \tag{46}$$

The second inequality in (46) holds because $q_j(.)$ is nondecreasing, and the third inequality holds because, as established above, $q_j(\bar{x}_j) \geq q_i(\bar{x}_i)$ and $q_j(x) \leq q_i(x)$ for all $x \in [0, \bar{x}_i]$. But

(46) contradicts $m_i > m_j$. Hence, we must have $\bar{x}_i \geq \bar{x}_j$. *Q.E.D.*

Proof of Lemma 7: Since $q_i(x_i) \equiv \int_{x_{-i}\in[0,1]^{n-1}} Q_i(x_i, x_{-i}) \prod_{j\neq i} dF(x_j)$, the inequalities in (19) immediately follow from Lemma 4.

By Lemma 5, $\gamma_i(x_i)$ is strictly increasing on $[0, \bar{x}_i)$ for all i. So the set $Z_i = \{x_i | \gamma_i(x_i) = \gamma_j(\bar{x}_j)$ for some $j \neq i\}$ is at most finite. Therefore, for any fixed $x_i \in [0, \bar{x}_i)$, $x_i \notin Z_i$, the left-hand side of (19) is equal to its right-hand side because for any $j \neq i$ there is at most a single type $x_j \in [0, \bar{x}_j)$ s.t. $\gamma_i(x_i) = \gamma_j(x_j)$ and the probability of this type x_j is zero.

Both the left-hand and the right-hand sides of (19) depend only on x_i and, through $\gamma_i(.)$ defined in (14), upon the profile $(\bar{x}_1, ..., \bar{x}_n, \lambda_1, ..., \lambda_n)$. So, when left-hand side and the right-hand sides of (19) are equal, $q_i(x_i)$ is uniquely defined by this profile.

Now, consider $x_i = \bar{x}_i$. If $\bar{x}_i \neq \bar{x}_j$ for all $j \neq i$, then the set $\{x_{-i} \in [0,1]^{n-1} : \gamma_i(\bar{x}_i) = \max\{0, \max_{j\neq i} \gamma_j(x_j)\}\}$ has measure zero. Therefore, the left- and right-hand sides of (19) are equal to each other.

Finally, the values of the left-hand and the right-hand sides of (19) depend only on x_i and the profile $(\bar{x}_1, ..., \bar{x}_n, \lambda_1, ..., \lambda_n)$. So when the left-hand and the right-hand sides of (19) are equal to each other, $q_i(x_i)$ is determined by x_i and the profile $(\bar{x}_1, ..., \bar{x}_n, \lambda_1, ..., \lambda_n)$. Since this is true a.e. on $[0, \bar{x}_i]$, the last statement of the Lemma follows. *Q.E.D.*

Proof of Lemma 8: First, let us show that the budget constraint of the highest-budget bidder 1 with budget m_1 is binding in an optimal mechanism. Note that by Lemma 6 $\bar{x}_1 \geq \bar{x}_i$ for all $i \neq 1$. The proof is by contradiction, so suppose not i.e., $m_1 > \bar{x}_1 q(\bar{x}_1) - \int_0^{\bar{x}_1} q_1(x)dx$. Then $\lambda_1 = 0$. Therefore, $\gamma_1(x_1) = x_1 - \frac{1-F(x_1)}{f(x_1)}$ for $x_1 < \bar{x}_1$, which implies, in particular, that $q_1(x_1) = 0$ for all $x_1 < p^m$ where $p^m = \arg\max_p p(1 - F(p))$ i.e., $p^m - \frac{1-F(p^m)}{f(x^m)} = 0$.

We will consider two cases: $\bar{x}_1 = 1$ and $\bar{x}_1 < 1$. First, suppose that $\bar{x}_1 = 1$. Then $\gamma_1(\bar{x}_1) = 1$. By Theorem 1, $\gamma_i(x_i) < 1$ for all $x_i < 1$. So, $\gamma_1(\bar{x}_1) > \max_{i\neq 1} \gamma_i(x_i)$ with probability 1, and hence $q_1(1) = 1$ by Lemma 7. So we have:

$$m_1 > \bar{x}_1 q_1(\bar{x}_1) - \int_0^{\bar{x}_1} q_1(x_1)dx_1 = 1 - \int_{p^m}^1 q_1(x_1)dx_1 \geq 1 - \int_{p^m}^1 1dx_1 = p_m,$$

which contradicts the assumption that $m_1 \leq p^m$.

Now, suppose that $\bar{x}_1 < 1$. Since $\lambda_1 = 0$, by Theorem 1 we must have $\gamma_1(\bar{x}_1) > \max_{i\neq 1, x_i\in[0,1]} \gamma_i(x_i)$, so $q_1(\bar{x}_1) = 1$ by Lemma 7. Hence we have:

$$m_1 > \bar{x}_1 q_1(\bar{x}_1) - \int_0^{\bar{x}_1} q_1(x_1)dx_1 = 1 - (1 - \bar{x}_1) - \int_0^{\bar{x}_1} q_1(x_1)dx_1 \geq 1 - \int_{p^m}^1 1dx_1 = p_m.$$

So we again obtain a contradiction to the assumption that $m_1 \leq p^m$.

To complete the proof let us establish that the budget constraint of any bidder i, $i \neq 1$, is also binding. By Lemma 6 $\bar{x}_i \leq \bar{x}_1$, so by Theorem 1 equation (17) holds for i. Hence, either $\lambda_i > 0$, in which case the budget constraint of i must be binding or $\bar{x}_i = 1$.

To show that i's budget constraint is binding when $\bar{x}_1 = 1$, we again argue by contradiction and suppose otherwise. Then $\lambda_i = 0$, and the same argument as for bidder 1 in case $\bar{x}_1 = 1$ can be used to show that $q_i(\bar{x}_i) = 1$ and $q_i(x_i) = 0$ for all $x_i < p^m$. Then we have $m_i > \bar{x}_i q_i(\bar{x}_i) - \int_0^{\bar{x}_i} q_i(x_i)dx_i \geq 1 - \int_{p^m}^1 1dx_1 = p_m$. A contradiction. *Q.E.D.*

Proof of Lemma 9: The strong duality property holds and (x^*, λ^*) is the solution to both the primal problem $\max_x \min_\lambda \mathcal{L}(\bar{x}, \lambda)$ and its dual $\min_\lambda \max_x \mathcal{L}(\bar{x}, \lambda)$ if and only if (x^*, λ^*) is a saddle point of the Lagrangian (15) (see e.g. Proposition 1.3.7, page 76, Chapter 1 in Bertsekas (2001)) i.e., for all $\bar{x}$ and λ we have:

$$\mathcal{L}(\bar{x}, \lambda^*) \leq \mathcal{L}(x^*, \lambda^*) \leq \mathcal{L}(x^*, \lambda) \tag{47}$$

So, to complete the proof of the Lemma we will establish that the saddle point property (47) holds. Consider Lagrange dual function $g(\lambda) \equiv \max_{\bar{x} \in [0,1]^n} \mathcal{L}(\bar{x}, \lambda)$. We have $g(\lambda) = \mathcal{L}(\bar{x}(\lambda), \lambda)$, where $\bar{x}(\lambda)$ is the solution to the problem $\max_{\bar{x} \in [0,1]^n} \mathcal{L}(\bar{x}, \lambda)$ characterized in Theorem 1 and is given by the inverse of the function $\lambda(\bar{x})$ in Theorem 1.

By Danskin's Theorem (Bertsekas (2001), Ch. 1, p. 131), the Lagrange dual function $g(\lambda)$ is convex and hence has a unique minimizer λ^*. Define $x^* = \bar{x}(\lambda^*)$. Let us show that the saddle-point property (47) holds for the pair (x^*, λ^*).

Since $x^* = \bar{x}(\lambda^*)$, $\mathcal{L}(\bar{x}, \lambda^*) \leq \mathcal{L}(x^*, \lambda^*)$, holds for all $\bar{x} \in [0,1]^n$ by Theorem 1.

To show that $\mathcal{L}(x^*, \lambda^*) \leq \mathcal{L}(x^*, \lambda)$ we start by arguing that $\mathcal{L}(\bar{x}, \lambda)$ is convex in λ for fixed $\bar{x}$. First, recall that the virtual value function $\gamma_i(x_i)$ defined in (14) is linear in λ_i. Since $\max\{0, \max_i\{\gamma_i(x_i)\}\}$ is convex in $(\gamma_1(x_1), ..., \gamma_n(x_n))$, it follows that $\max\{0, \max_i\{\gamma_i(x_i)\}\}$ is also convex in $(\lambda_1, ..., \lambda_n)$. The integration operator over x preserves convexity of the integrand $\max\{0, \max_i\{\gamma_i(x_i)\}\} \prod_i f(x_i)$ in the parameters $(\lambda_1, ..., \lambda_n)$, so the Lagrangian $\mathcal{L}(\overline{x}, \lambda)$ is, indeed, convex in $(\lambda_1, ..., \lambda_n)$ for all $\bar{x}$.

The convexity of $\mathcal{L}(\bar{x}(\lambda^*), \lambda)$ in λ implies that it has a unique minimum which can be found as a unique solution to the first-order conditions $\frac{\partial \mathcal{L}(\bar{x}(\lambda^*), \lambda + \epsilon h)}{\partial \epsilon}\Big|_{\epsilon=0} \geq 0$ for all $h \in \mathbf{R}^n$. But by Danskin's Theorem (Bertsekas (2001), Ch. 1, p. 131), $\frac{\partial \mathcal{L}(\bar{x}(\lambda), \lambda + \epsilon h)}{\partial \epsilon}\Big|_{\epsilon=0} = \frac{\partial g(\lambda + \epsilon h)}{\partial \epsilon}\Big|_{\epsilon=0}$ for all λ, h. Since by definition $\lambda^* = \arg\min_\lambda g(\lambda)$, we have $\frac{\partial g(\lambda^* + \epsilon h)}{\partial h}\Big|_{\epsilon=0} \geq 0$ for all h. So, $\frac{\partial \mathcal{L}(\bar{x}(\lambda^*), \lambda^* + \epsilon h)}{\partial \epsilon}\Big|_{\epsilon=0} \geq 0$ for all h i.e., $\lambda^* = \arg\min_\lambda \mathcal{L}(\bar{x}(\lambda^*), \lambda)$, and hence the second inequality in (47) holds. This completes the proof that (x^*, λ^*) is a saddle point. *Q.E.D.*

Proof of Theorem 2: By Lemma 9, the solution to our problem can be obtained by minimizing the dual Lagrange function $g(\lambda) \equiv \mathcal{L}(\lambda, \bar{x}(\lambda))$. This function has a unique minimum attained at λ satisfying the first-order conditions $g'(\lambda; h) \equiv \frac{\partial \mathcal{L}(\lambda+\epsilon h, \bar{x}(\lambda))}{\partial \epsilon}_{\epsilon=0} \geq 0$ for any vector $h \in \mathbf{R}^n$. In the rest of the proof we will focus on these first-order conditions.

To begin with, consider i such that $\lambda_i \neq \lambda_j$, and so by Theorem 2, $\bar{x}_i \neq \bar{x}_j$ for any $j \neq i$. In this case, the only variation h in the vector λ that we need to consider to characterize the optimal λ_i involves a change in λ_i only. So we have the following first-order condition:

$$\frac{\partial g(\lambda)}{\partial \lambda_i} = m_i - \bar{x}_i \int_{x_{-i} \in [0,1]^{n-1}: \gamma_i(\bar{x}_i) > \max_{j \neq i} \gamma_j(x_j)} \prod_{j \neq i} dF(x_j)$$
$$+ \int_0^{\bar{x}_i} \int_{x_{-i} \in [0,1]^{n-1}: \gamma_i(s) > \max\{0, \max_{j \neq i} \gamma_j(x_j)\}} \prod_{j \neq i} dF(x_j) ds = m_i - \bar{x}_i q_i(\bar{x}_i) + \int_0^{\bar{x}_i} q_i(s) ds = 0$$

The second equality in this sequence holds by Lemma 7. Thus, we obtain condition (21).

Next suppose that there is a "cluster" $\{k_1, ..., k_l\}$, $l \in \{2, ..., n\}$, such that $\bar{x}_{k_1} = ... = \bar{x}_{k_l} = \bar{x}^c \neq \bar{x}_j$ for any $j \notin \{k_1, ..., k_l\}$. Since the common multiplier λ^c for all bidders in the cluster and the set of bidders in it must be optimal, no variation h of them should decrease the value of $g(\lambda)$. Thus, we have to consider all variations of the vector λ, $\epsilon \times \mathbb{I}_J$, where $J \subseteq \{k_1, ..., k_l\}$ is a subset of bidders in the "cluster" and $\mathbb{I}_J$ is an n-vector with entries corresponding to bidders in J equal to 1 and other entries equal to zero. The following first-order conditions must hold for any J: $\frac{\partial g(\lambda+\epsilon \mathbb{I}_J)}{\partial \epsilon}_{\epsilon=0+} \geq 0$ and $\frac{\partial g(\lambda+\epsilon \mathbb{I}_J)}{\partial \epsilon}_{\epsilon=0-} \leq 0$. Although there are $2^l - 1$ such nonempty subsets J, it will be sufficient to consider only $2l$ of them, as will be shown below.

So, let $J = \{k'_1, ..., k'_r\} \subseteq \{k_1, ..., k_l\}$. Then we have: $\frac{\partial g(\lambda+\epsilon \times \mathbb{I}_J)}{\partial \epsilon}_{|\epsilon=0+} =$

$$\sum_{h=1,...,r} m_{k'_h} + \int_{x:\, \max_{h \in \{1,...,r\}} \gamma_{k'_h}(x_{k'_h}) > \max\{0, \max_{j \notin \{k'_1,...,k'_r\}} \gamma_j(x_j)\}} \frac{\partial \max_{h=1,...,r} \gamma_{k'_h}(x_{k'_h})}{\partial \lambda_{k'_h}}_{|\lambda_{k'_h} = \lambda^c} \prod_i dF(x_i)$$

$$= \sum_{h=1,...,r} \left(m_{k'_h} + \int_0^{\bar{x}^c} \int_{x_{-k'_h}: \gamma_{k'_h}(s) > \max\{0, \max_{j \neq k'_h} \gamma_j(x_j)\}} \frac{\partial \gamma_{k'_h}(s)}{\partial \lambda_{k'_h}}_{|\lambda_{k'_h} = \lambda^c} \prod_{j \neq k'_h} dF(x_j) dF(s) \right) +$$

$$F(\bar{x}^c)^{l-r} \frac{1 - F(\bar{x}^c)^r}{1 - F(\bar{x}^c)} \int_{\bar{x}^c}^1 \int_{x_{-k_1 \cdots -k_l} \in [0,1]^{n-l}: \gamma_{k'_1}(\bar{x}^c) > \max_{j \notin \{k_1,...,k_l\}} \gamma_j(x_j))} \frac{\partial \gamma_{k'_1}(s)}{\partial \lambda_{k'_1}}_{|\lambda_{k'_1} = \lambda^c} \prod_{j \notin \{x_{k_1},...,x_{k_l}\}} dF(x_j) dF(s)$$

$$= \sum_{h=1,...,r} m_{k'_h} + \sum_{h=1,...,r} \int_0^{\bar{x}^c} \int_{x_{-k'_h} \in [0,1]^{n-1}: \gamma_{k'_h}(s) > \max\{0, \max_{j \neq k'_h} \gamma_j(x_j)\}} \prod_{j \neq k'_h} dF(x_j) ds$$

$$- \bar{x}^c F(\bar{x}^c)^{l-r} \frac{1 - F(\bar{x}^c)^r}{1 - F(\bar{x}^c)} \int_{x_{-k_1 \dots -k_l} \in [0,1]^{n-l}: \gamma_{k'_1}(\bar{x}^c) > \max_{j \notin \{k_1,...,k_l\}} \gamma_j(x_j)} \prod_{j \notin \{x_{k_1},...,x_{k_l}\}} dF(x_j)$$

$$= \sum_{h=1,...,r} m_{k'_h} + r \int_0^{\bar{x}^c} q_{k'_1}(s) ds - \bar{x}^c F(\bar{x}^c)^{l-r} \frac{1 - F(\bar{x}^c)^r}{1 - F(\bar{x}^c)} Prob[\gamma_{k'_1}(\bar{x}^c) > \max_{j \notin \{k_1,...,k_l\}} \gamma_j(x_j)] \quad (48)$$

The first equality in (48) holds by definition. The second equality uses the properties of the max operator. In particular, the factor $1 - F(\bar{x}^c)^r$ in the last term after the second equality reflects conditioning on the event that at least one of the bidders in $J = \{k'_1, ..., k'_r\}$ has value above $\bar{x}^c$, and the factor $F(\bar{x}^c)^{l-r}$ reflects conditioning on the event that the bidders in $C(\bar{x}^c) \setminus J$ have values below $\bar{x}^c$. We use $\frac{\partial \gamma_{k'_1}(s)}{\partial \lambda_{k'_1}}$ as the integrand in this term, because $\gamma_{k'_1}(s) = \gamma_{k'_h}(s)$ for all $h \in \{1, ..., r\}$.

To obtain the third equality we use the definition (14) and, in particular, $\frac{\partial \gamma_{k'_1}(s)}{\partial \lambda_{k'_1}} = \frac{1}{f(\bar{x}^c)}$ if $s < \bar{x}^c$ and $\frac{\partial \gamma_{k'_1}(s)}{\partial \lambda_{k'_1}} = -\frac{\bar{x}^c}{1-F(\bar{x}^c)}$ if $s > \bar{x}^c$. The final equality uses Lemma 7.

Similarly, we have: $\frac{\partial g(\lambda + \epsilon \times \mathbb{I}_J)}{\partial \epsilon}_{|\epsilon = 0-} =$

$$\sum_{h=1,...,r} m_{k'_h} + \int_{x:\, \max_{h \in \{1,...,r\}} \gamma_{k'_h}(x_{k'_h}) \geq \max\{0, \max_{j \notin \{k'_1,...,k'_r\}} \gamma_j(x_j)\}} \frac{\partial \max_{h=1,...,r} \gamma_{k'_h}(x_{k'_h})}{\partial \lambda_{k'_h}}_{|\lambda_{k'_h} = \lambda^c} \prod_i dF(x_i)$$

$$= \sum_{h=1,...,r} \left(m_{k'_h} + \int_0^{\bar{x}^c} \int_{x_{-k'_h}: \gamma_{k'_h}(s) \geq \max\{0, \max_{j \neq k'_h} \gamma_j(x_j)\}} \frac{\partial \gamma_{k'_h}(s)}{\partial \lambda_{k'_h}}_{|\lambda_{k'_h} = \lambda^c} \prod_{j \neq k'_h} dF(x_j) dF(s) \right)$$

$$+ \frac{1 - F(\bar{x}^c)^r}{1 - F(\bar{x}^c)} \int_{\bar{x}^c}^1 \int_{x_{-k_1 \dots -k_l} \in [0,1]^{n-l}: \gamma_{k'_1}(\bar{x}^c) \geq \max_{j \notin \{k_1,...,k_l\}} \gamma_j(x_j))} \frac{\partial \gamma_{k'_1}(s)}{\partial \lambda_{k'_1}}_{|\lambda_{k'_1} = \lambda^c} \prod_{j \notin \{x_{k_1},...,x_{k_l}\}} dF(x_j) dF(s)$$

$$= \sum_{h=1,...,r} m_{k'_h} + \sum_{h=1,...,r} \int_0^{\bar{x}^c} \int_{x_{-k'_h} \in [0,1]^{n-1}: \gamma_{k'_h}(s) \geq \max\{0, \max_{j \neq k'_h} \gamma_j(x_j)\}} \prod_{j \neq k'_h} dF(x_j) ds$$

$$- \bar{x}^c \frac{1 - F(\bar{x}^c)^r}{1 - F(\bar{x}^c)} \int_{x_{-k_1 \dots -k_l} \in [0,1]^{n-l}: \gamma_{k'_1}(\bar{x}^c) \geq \max_{j \notin \{k_1,...,k_l\}} \gamma_j(x_j)} \prod_{j \notin \{x_{k_1},...,x_{k_l}\}} dF(x_j)$$

$$= \sum_{h=1,...,r} m_{k'_h} + r \int_0^{\bar{x}^c} q_{k'_1}(s) ds - \bar{x}^c \frac{1 - F(\bar{x}^c)^r}{1 - F(\bar{x}^c)} Prob[\gamma_{k'_1}(\bar{x}^c) > \max_{j \notin \{k_1,...,k_l\}} \gamma_j(x_j)] \quad (49)$$

Note that the second term in the penultimate expression in both (48) and (49) is equal to $r \int_0^{x^c} q_{k'_h}(s) ds$ for any $h \in \{1, ..., r\}$. This is so because: (i) for almost all $s \in [0, \bar{x}^c)$, the set of $x_{-k'_h}$ such that $\gamma_{k'_h}(s) = \max_{j \neq k'_h} \gamma_j(x_j)$ has measure zero, and so by Lemma 7, $q_{k'_h}(s) = \int_{x_{-k'_h} \in [0,1]^{n-1}: \gamma_{k'_h}(s) \geq \max\{0, \max_{j \neq k'_h} \gamma_j(x_j)\}} dF(x_{-k'_h}) = \int_{x_{-k'_h} \in [0,1]^{n-1}: \gamma_{k'_h}(s) > \max\{0, \max_{j \neq k' - h} \gamma_j(x_j)\}} dF(x_{-k'_h})$ for all $h \in \{1, ..., r\}$ and almost all $s \in [0, \bar{x}^c]$; (ii) $\bar{x}^c \neq \bar{x}_j$ for any $j \notin \{k_1, ..., k_l\}$, and so $Prob[\gamma_{k'_1}(\bar{x}^c) > \max_{j \notin \{k_1,...,k_l\}} \gamma_j(x_j)] = Prob[\gamma_{k'_1}(\bar{x}^c) \geq \max_{j \notin \{k_1,...,k_l\}} \gamma_j(x_j)]$.

The expressions for $\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_J)}{\partial\epsilon}_{|\epsilon=0+}$ in (48) and $\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_J)}{\partial\epsilon}_{|\epsilon=0-}$ in (49) differ by the factor $F(\bar{x}^c)^{l-r}$ in the last term of (48), which does not appear in the corresponding term of (49). This is so because a negative variation ($\epsilon<0$) in λ^c does increase $\gamma_{k'_h}(x)$ for $x\geq\bar{x}^c$, and so $\max_{h\in\{1,...,l\}}\gamma_{k_h}(x)$ changes irrespective of the highest type among the other $l-r$ bidders in the cluster $C(\bar{x}^c)$ who are not in set J. On the other hand, a positive variation ($\epsilon>0$) in λ^c lowers the value of $\gamma_{k'_h}(x)$ for $x\geq\bar{x}^c$, and so the $\max_{h\in\{1,...,l\}}\gamma_{k_h}(x)$ changes only if the maximal value among the other $l-r$ bidders in the cluster is below $\bar{x}^c$. The latter occurs with probability $F(\bar{x}^c)^{l-r}$, the factor in the last term of (48).

When $J=C(\bar{x}^c)$, (48) and (49) yield (22) as in this case we have:

$$\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_{\{k_1,...,k_l\}})}{\partial\epsilon}_{|\epsilon=0+}=\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_{\{k_1,...,k_l\}})}{\partial\epsilon}_{|\epsilon=0-}=$$
$$\sum_{h=1,...,l}m_{k_h}+l\int_0^{\bar{x}^c}q_{k_1}(s)ds-\bar{x}^c\frac{1-F(\bar{x}^c)^l}{1-F(\bar{x}^c)}Prob[\gamma_{k_1}(\bar{x}^c)>\max_{j\notin\{k_1,...,k_l\}}\gamma_j(x_j)]=0$$

To obtain (23), first use (48) to rewrite $\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_{\{k_{r+1},...,k_l\}})}{\partial\epsilon}_{|\epsilon=0+}\geq 0$ as follows:

$$\frac{m_{k_{r+1}}+...+m_{k_l}}{l-r}-\bar{x}^c\frac{F(\bar{x}^c)^r}{l-r}\frac{1-F(\bar{x}^c)^{l-r}}{1-F(\bar{x}^c)}Prob[\gamma_{k_1}(\bar{x}^c)>\max_{j\notin\{k_1,...,k_l\}}\gamma_j(x_j)]\geq-\int_0^{\bar{x}^c}q_{k_1}(s)ds. \tag{50}$$

Then use (49) to rewrite $\frac{\partial g(\lambda+\epsilon\times\mathbb{I}_{\{k_1,...,k_r\}})}{\partial\epsilon}_{|\epsilon=0-}\leq 0$ as follows:

$$\frac{m_{k_1}+...+m_{k_r}}{r}-\bar{x}^c\frac{1}{r}\frac{1-F(\bar{x}^c)^r}{1-F(\bar{x}^c)}Prob[\gamma_{k_1}(\bar{x}^c)>\max_{j\notin\{k_1,...,k_l\}}\gamma_j(x_j)]\leq-\int_0^{\bar{x}^c}q_{k_1}(s)ds. \tag{51}$$

Combining (50) and (51) yields (23) for any $r\in\{2,...,l-1\}$.

To complete the proof and establish the uniqueness of the solution, let us show that (22) and (23) imply (50) and (51). It is sufficient to show that (50) holds for the subsets $J_{l-r}=\{k_{r+1},...,k_l\}$ of $C(\bar{x}^c)$, $r\in\{1,...,l-1\}$ including $l-r$ lowest-budget bidders, for then it also holds for any other subset of $C(\bar{x}^c)$ of size $l-r$. Similarly, it is enough to show that (51) holds for the subsets $\hat{J}_r=\{k_1,...,k_r\}$ including r highest-budget bidders in the cluster.

To show that (50) holds for $J=\{k_{l-r+1},...,k_l\}$, combine (22) and (23) to obtain:

$$\frac{(m_{k_{r+1}}+...+m_{k_l})\,l}{l-r}\geq-\frac{\bar{x}^c}{1-F(\bar{x}^c)}\left(1-F(\bar{x}^c)^r-rF(\bar{x}^c)^r\frac{1-F(\bar{x}^c)^{l-r}}{l-r}\right)Prob.[\gamma_{k_1}(\bar{x}^c)>\max_{j\notin\{k_1,...,k_l\}}\gamma_j]$$
$$+\sum_{j=1,...,l}m_{k_j}=-\frac{\bar{x}^c}{1-F(\bar{x}^c)}\left(1-F(\bar{x}^c)^r-rF(\bar{x}^c)^r\frac{1-F(\bar{x}^c)^{l-r}}{l-r}\right)Prob.[\gamma_{k_1}(\bar{x}^c)>\max_{j\notin\{k_1,...,k_l\}}\gamma_j]$$
$$+\bar{x}^c\frac{1-F(\bar{x}^c)^l}{1-F(\bar{x}^c)}Prob.[\gamma_{k_1}(\bar{x}^c)>\max_{j\notin\{k_1,...,k_l\}}\gamma_j]-l\int_0^{\bar{x}^c}q_{k_1}(s)ds=$$
$$=\frac{l}{l-r}\bar{x}^cF(\bar{x}^c)^r\frac{1-F(\bar{x}^c)^{l-r}}{1-F(\bar{x}^c)}Prob.[\gamma_{k_1}(\bar{x}^c)>\max_{j\notin\{k_1,...,k_l\}}\gamma_j]-l\int_0^{\bar{x}^c}q_{k_1}(s)ds. \tag{52}$$

The inequality in (52) holds by (23), the first equality holds by (22), the second equality holds by rearrangement. So, (50) holds for $J = \{k_{l-r+1}, ..., k_l\}$.

Now take $J = \{k_1, ..., k_r\}$. Then combining (22) and (23) yields:

$$\frac{(m_{k_1} + ... + m_{k_r})\, l}{r} \leq \bar{x}^c \left((l-r) \frac{1 - F(\bar{x}^c)^r}{(1 - F(\bar{x}^c))\, r} - \frac{F(\bar{x}^c)^r \left(1 - F(\bar{x}^c)^{l-r}\right)}{(1 - F(\bar{x}^c))} \right) Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j \notin \{k_1,...,k_l\}} \gamma_j]$$
$$+ \sum_{j=1,...,l} m_{k_j} = \frac{\bar{x}^c}{1 - F(\bar{x}^c)} \left((l-r) \frac{1 - F(\bar{x}^c)^r}{r} - F(\bar{x}^c)^r \left(1 - F(\bar{x}^c)^{l-r}\right) \right) Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j \notin \{k_1,...,k_l\}} \gamma_j]$$
$$+ \bar{x}^c \frac{1 - F(\bar{x}^c)^l}{1 - F(\bar{x}^c)} Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j \notin \{k_1,...,k_l\}} \gamma_j] - l \int_0^{\bar{x}^c} q_{k_1}(s) ds =$$
$$= \frac{l}{r} \bar{x}^c \frac{1 - F(\bar{x}^c)^r}{1 - F(\bar{x}^c)} Prob.[\gamma_{k_1}(\bar{x}^c) > \max_{j \notin \{k_1,...,k_l\}} \gamma_j] - l \int_0^{\bar{x}^c} q_{k_1}(s) ds \tag{53}$$

The inequality in (53) holds by (23). The first equality holds by (22). The second equality holds by rearrangement. So (51) also holds. *Q.E.D.*

Proof of Theorem 3: **"Only if" Part:** Suppose that top auction with threshold $\bar{x}^t$ and reservation value r_t is an optimal mechanism. Then by Theorem 2 condition (23) holds for the cluster including all bidders. By Definition 1, $q_i(s) = 0$ for $x_i < r_t$, $q_i(x_i) = F^{n-1}(x_i)$ for $x_i \in [r_t, \bar{x}^t)$, and (26)-(27) hold. Substituting these into (23) yields (28).

"If" Part: Suppose that (28) holds for all $k \in \{1, ..., n-1\}$ and $\bar{x}^t$ defined by (26). By inspection (26) is equivalent to (22) and (28) is equivalent to (23) when $\bar{x}^c = \bar{x}^t$ and cluster C includes all n bidders. By Theorem 2 the optimal mechanism is unique and (22) and (23) are necessary and sufficient conditions characterizing it, so the top auction is a unique optimal mechanism. *Q.E.D.*

Proof of Corollary 2: Since top auction is the optimal mechanism under both profiles $(m_1, ..., m_n)$ and $(m'_1, ..., m'_n)$ and $\sum_i m_i = \sum_i m'_i = M$, according to (26) the optimal threshold $\bar{x}^t$ is the same in both cases. Hence, by Theorem 1, the Lagrange multiplier of each bidder under both budget profiles is $\bar{\lambda}^t = \frac{\left(1 - F(\bar{x}^t)\right)^2}{(1 - F(\bar{x}^t) + \bar{x}^t f(\bar{x}^t))}$. So, the value of the Lagrangian (15) giving the seller's expected profits is the same under both budget profiles. *Q.E.D.*

Proof of Theorem 4: Theorem 1 shows that the optimal mechanism is uniquely defined by the vector of thresholds $(\bar{x}_1, ..., \bar{x}_n)$. By Theorem 3, the failure of (28) implies that we cannot have $\bar{x}_1 = ... = \bar{x}_n$. By Lemma 6, $\bar{x}_1 \geq ... \geq \bar{x}_n$. So $\bar{x}_i > \bar{x}_{i+1}$ for some i.

Now, consider bidders i and j such that $\bar{x}_i > \bar{x}_j$. By Theorem 1, $\lambda_i < \lambda_j$, and $\gamma_i(x) < \gamma_j(x)$ for $x \in [0, \bar{x}_j]$. Therefore the reservation values r_i and r_j where the virtual utilities of bidders i and j, respectively, are equal to zero, satisfy $r_i > r_j$. Also, by Lemma

7 $q_i(x) < q_j(x)$ for $x \in [r_j, \bar{x}_j]$. The last claim of the Theorem follows from Lemma 6. *Q.E.D.*

Proof of Corollary 3: If under $(m_1, ..., m_n)$ bidders i, $i+1$,...,$i+l$ constitute a cluster with a common threshold $\bar{x}^c$, then $\sum_{j=i}^{i+l} m_j = \sum_{j=i}^{i+l} m'_j$. This also applies to trivial clusters consisting of a single bidder. So, if we assign the same profile of thresholds $(\bar{x}_1, ..., \bar{x}_n)$ to the bidders with budgets $(m'_1, ..., m'_n)$, then conditions (21) and (22) of Theorem 2 would still hold. To confirm that the threshold profile $(\bar{x}_1, ..., \bar{x}_n)$ remains optimal under $(m'_1, ..., m'_n)$, it remains to verify that condition (23) still hold. This is so because: (i) (23) holds for $(m_1, ..., m_n)$ and $(\bar{x}_1, ..., \bar{x}_n)$; (ii) the right-hand side of (23) depends only on $(\bar{x}_1, ..., \bar{x}_n)$ and hence remains unchanged; (iii) the left-hand side of (23) depend only on the budgets; (iv) $|m_i - m'_i|$ is sufficiently small for all i.

Next, since the same threshold profile $(\bar{x}_1, ..., \bar{x}_n)$ is optimal under both budget profiles $(m_1, ..., m_n)$ and $(m'_1, ..., m'_n)$, Theorem 1 implies that the profile of Lagrange multipliers $(\lambda_1, ..., \lambda_n)$ is the same also. Therefore, the value of the Lagrangian (15), which gives the seller's expected profits, is the same under these two budget profiles. *Q.E.D.*

Proof of Theorem 5: Since the bidders' valuations are identically distributed, the seller's revenue function $\pi(m_1, ..., m_n)$ is exchangeable i.e., $\pi(m_1, ..., m_n) = \pi(P(m_1, ..., m_n))$ where $P(m_1, ..., m_n)$ is a permutation of $(m_1, ..., m_n)$. Let PM^m denote the set of permutations of $(m_1, ..., m_n)$. Its cardinality (the total number of permutations) is equal to $n!$.

Fixing a budget profile $(m_1, ..., m_n)$ such that $\sum_i m_i = M$, by concavity of $\pi(.)$ we obtain:

$$\pi\left(\frac{1}{M}, ..., \frac{1}{M}\right) \geq \sum_{P \in PM^m} \frac{\pi(P)}{\# PM^m} = \pi(m_1, ..., m_n)$$

To establish the second part of the Theorem, note that under its conditions there exists a doubly stochastic matrix S (i.e., all entries of S are nonnegative and all the entries in each row and column sum up to one) such that $m = Sm'$, where $m = (m_1, ..., m_n)$ and $m' = (m'_1, ..., m'_n)$ (see Theorem 3.5.4. in Marcus and Mink (1992).

By Birkhoff Theorem (see e.g. Marcus and Mink (1992)), any doubly stochastic matrix belongs to a convex hull generated by permutation matrices.[18] So, $S = \sum_{k=1}^{n'} \theta_k P_k$, where $\theta_k \geq 0$ for all $k \in \{1, ..., n'\}$, for some $n' > 1$, $\sum_{k=1}^{n'} \theta_k = 1$, and P_k is a permutation matrix. So, $\pi(m) = \pi(\sum_{k=1}^{n'} \theta_k P_k m') \geq \sum_{k=1}^{n'} \theta_k \pi(P_k m') = \pi(m')$, where the inequality holds by concavity of π and the second equality holds by the exchangeability of $\pi(\cdot)$. *Q.E.D.*

[18]A permutation matrix is an $n \times n$ matrix every row and column of which contains a single 1 with zeroes everywhere else.

10 Appendix B: Asymmetrically Distributed Values

In this section we extend our analysis to the case of asymmetrically distributed valuations and show that, for a set of parameter values, the optimal mechanism is a "generalized top auction." In this mechanism, as in the "top auction," bidders with sufficiently high valuations are tied and the good is allocated randomly between them. However, in the generalized top auction the bidders have different valuation thresholds, not a common one as in the top auction, and bidders with the same valuations below the thresholds face different probabilities of trading due to the distribution asymmetry.

For brevity, we will focus on the case of two bidders. Extending the results to n bidders is straightforward but notationally cumbersome. So, suppose that bidder i's, $i \in \{1,2\}$, valuation is distributed according to probability distribution F_i with increasing hazard rate, and her budget m_i satisfies $m_i - \frac{1-F_i(m_i)}{f_i(m_i)} < 0$. As in the symmetric case, this assumption ensures that budget constraints of all bidders are binding. We do not impose an ordering of m_1 and m_2. However, we make the following ranking assumption.

Assumption 2 *(Monotone Likelihood Ratio) For all $x, x' \in [0,1]$, $x' > x$, $\frac{f_1(x')}{f_1(x)} > \frac{f_2(x')}{f_2(x)}$.*

Note that Assumption 2 implies that $F_1(.)$ first-order stochastically dominates $F_2(.)$.

A careful perusal of the derivation of the Lagrangian (13), and of the proofs of Theorem 1, which establishes a 1-to-1 relationship between the vectors of thresholds $\overline{x}$ and the Lagrange multipliers λ, and Lemmas 4-5 and 7-9 confirms that all these results apply verbatim to the case of the asymmetric distributions. We omit rewriting these results in order to save space. Next, let us introduce the following definition:

Definition 2 *A generalized top auction is a mechanism in which the bidders' thresholds $\overline{x}_1$, $\overline{x}_2$ and expected probabilities of trading $q_1(\overline{x}_1)$ and $q_2(\overline{x}_2)$ satisfy the following conditions:*

$$\frac{\overline{x}_1^2 f_1(\overline{x}_1)}{1-F_1(\overline{x}_1)+\overline{x}_1 f_1(\overline{x}_1)} = \frac{\overline{x}_2^2 f_2(\overline{x}_2)}{1-F_2(\overline{x}_2)+\overline{x}_2 f_2(\overline{x}_2)}, \tag{54}$$

$$\sum_{i=1,2}(1-F_i(\overline{x}_i))q_i(\overline{x}_i) = 1 - F_1(\overline{x}_1)F_2(\overline{x}_2), \tag{55}$$

and in which the probabilities of trading $q_i(x_i)$ for $x_i \in [0, \overline{x}_i)$, $i \in \{1,2\}$ is uniquely defined by 19 in Lemma 7, with $\gamma_i(x_i) = x_i - \frac{1-F_i(x_i)-\frac{(1-F_i(\overline{x}_i))^2}{(1-F_i(\overline{x}_i)+\overline{x}_i f_i(\overline{x}_i))}}{f_i(x_i)}$.

Note that equation (54) says that the buyers' virtual values at the thresholds $\overline{x}_1$ and $\overline{x}_2$, $\gamma_1(\overline{x}_1)$ and $\gamma_2(\overline{x}_2)$, are equal, so it is optimal for the seller to allocate the good randomly across the buyers when $x_1 \geq \overline{x}_1$ and $x_2 \geq \overline{x}_2$. Equation (55) is the feasibility condition on $q_1(\overline{x}_1)$ and $q_2(\overline{x}_2)$ which must be satisfied when the good is allocated to a buyer with a valuation above her threshold iff there is at least on such buyer.

Our proof of the existence and optimality of the generalized top auction will proceed as follows. First, we will substitute $q_i(\overline{x}_i)$ out from (55) by using the budget constraint i.e., $q_i(\overline{x}_i) = \frac{m_i + \int_0^{\overline{x}_i} q_i(x_i)dx_i}{\overline{x}_i}$, yielding a system of two equations, (54) and modified (55), which only depends on $\overline{x}_1$ and $\overline{x}_2$. Then we will establish that this system has a solution. Besides condition (55), feasibility requires that $q_i(\overline{x}_i) \leq 1$ and $q_i(\overline{x}_i) \geq F_j(\overline{x}_j)$ for $i \in \{1, 2\}$. The latter condition is necessary to ensure that $q_i(.)$ is increasing at $\overline{x}_i$ for $i \in \{1, 2\}$. However, when (55) holds, any two of these feasibility conditions imply the other two. We will use this property in the following Theorem to establish the feasibility of the generalized top auction.

Theorem 6 *(i) There exist $\delta_1, \delta_2 \in (0, 1)$ such that the system of equations (54) and (56) below has a solution $(\overline{x}_1, \overline{x}_2) \in (\delta_1, 1 - \delta_1) \times (\delta_2, 1 - \delta_2)$:*

$$\sum_{i=1,2} \frac{1 - F_i(\overline{x}_i)}{\overline{x}_i} \left(m_i + \int_0^{\overline{x}_i} q_i(x)dx \right) = 1 - F_1(\overline{x}_1) F_2(\overline{x}_2). \tag{56}$$

where $q_i(\cdot)$ are uniquely defined in (19) in Lemma 7.

(ii) There exists $\epsilon > 0$ s.t. whenever $|F_2(x) - F_1(x)| < 0$ for $x \in [0, 1]$ and $|m_2 - m_1| \leq \epsilon$, then then the solution $(\overline{x}_1, \overline{x}_2) \in (0, 1)^2$ to (54) and (56) is unique and satisfies the feasibility conditions $F_j(\overline{x}_j) \leq \frac{m_i + \int_0^{\overline{x}_i} q_i(x_i)dx_i}{\overline{x}_i} \leq 1$ for $i, j \in \{1, 2\}$.

The optimal mechanism is a generalized top auction with these thresholds, $(\overline{x}_1, \overline{x}_2)$.

Proof: Note that (56) is obtained by substituting $q_i(\overline{x}_i)$ from (55) using the budget constraints of each bidder, $m_i = \overline{x}_i q_1(\overline{x}_1) - \int_0^{\overline{x}_i} q_i(x_i)dx_i$. Thus, the thresholds $(\overline{x}_1, \overline{x}_2)$ in a generalized top auction must satisfy (54)and (56). Claims 1-3 below establish that a solution to this system of two equations exists.

After establishing this, we will need to verify that our solution $(\overline{x}_1, \overline{x}_2)$ to (54)and (56) is such that $q_1(\overline{x}_1) = \frac{m_1 + \int_0^{\overline{x}_1} q_1(x_1)dx_1}{\overline{x}_1}$ and $q_2(\overline{x}_2) = \frac{m_2 + \int_0^{\overline{x}_2} q_2(x_1)dx_1}{\overline{x}_2}$ are feasible i.e., $F_j(\overline{x}_j) \leq q_i(\overline{x}_i) \leq 1$. Claim 4 establishes that these feasibility conditions hold under the conditions of the Theorem. Finally, Claim 5 completes the proof by establishing the uniqueness of the solution.

Claim 1. *Equation (54) defines a continuous, increasing and one-to-one mapping $\overline{x}_2(\overline{x}_1)$ between $\overline{x}_1 \in [0, 1]$ and $\overline{x}_2 \in [0, 1]$ such that $\overline{x}_2(1) = 1$ and $\overline{x}_2(0) = 0$.*

Proof of Claim 1: Note that $\frac{\overline{x}_i^2 f_i(\overline{x}_i)}{1 - F_i(\overline{x}_i) + \overline{x}_i f_i(\overline{x}_i)}$, $i = 1, 2$. is continuous and increasing in $\overline{x}_i$ on $[0, 1]$ by the increasing hazard rate assumption, and is equal to 0 (1) if $\overline{x}_i = 0$ ($\overline{x}_i = 1$), from which Claim 1 follows immediately.

Claim 2. *The mapping $\overline{x}_2(\overline{x}_1)$ defined by (54) is such that $\overline{x}_2(\overline{x}_1) \leq \overline{x}_1$, and $\frac{d\overline{x}_2}{d\overline{x}_1}_{\overline{x}_1=1} = 1$.*

Proof of Claim 2: Let us rewrite (54) as follows:

$$\frac{1}{\overline{x}_1} + \frac{1 - F_1\left(\overline{x}_1\right)}{\overline{x}_1^2 f_1(\overline{x}_1)} = \frac{1}{\overline{x}_2} + \frac{1 - F_2\left(\overline{x}_2\right)}{\overline{x}_j^2 f_2\left(\overline{x}_2\right)}. \tag{57}$$

By the increasing hazard rate assumption, the left-hand (right-hand) side of (57) is monotonically decreasing in $\overline{x}_1$ ($\overline{x}_2$).

Further, since $\frac{1-F_1(x)}{f_1(x)} \geq \frac{1-F_2(x)}{f_2(x)}$, the left-hand side of (57) is greater than its right-hand-side at any $\overline{x}_1 = \overline{x}_2$. Hence, (57) holds as equality only if $\overline{x}_1 \geq \overline{x}_2$.

Next, differentiating (57) yields:

$$\left(-\frac{2}{\overline{x}_1^2} - \frac{f_1'(\overline{x}_1)(1 - F_1\left(\overline{x}_1\right))}{\overline{x}_1^2 f_1(\overline{x}_1)} - \frac{2(1 - F_1\left(\overline{x}_1\right))}{\overline{x}_1^3 f_1(\overline{x}_1)}\right) = \left(-\frac{2}{\overline{x}_2^2} - \frac{f_2'(\overline{x}_2)(1 - F_2\left(\overline{x}_2\right))}{\overline{x}_2^2 f_1(\overline{x}_2)} - \frac{2(1 - F_2\left(\overline{x}_2\right))}{\overline{x}_2^3 f_2(\overline{x}_2)}\right)\frac{d\overline{x}_2}{d\overline{x}_1}. \tag{58}$$

From (58) and $\overline{x}_2(1) = 1$ it follows that $\frac{d\overline{x}_2}{d\overline{x}_1}_{\overline{x}_1=1} = 1$.

Claim 3. *The system of equations (54), (56) has a solution $(\overline{x}_1, \overline{x}_2)$. Any such solution belongs to $(\delta_1, 1-\delta_1) \times (\delta_2, 1-\delta_2)$ for some $\delta_1, \delta_2 \in (0,1)$.*

Proof of Claim 3: Using the mapping $\overline{x}_2(\overline{x}_1)$ defined by equation (54) and described in Claims 1 and 2, we can rewrite equation (56) as follows: $G_1\left(\overline{x}_1\right) = G_2\left(\overline{x}_1\right)$ where

$$G_1\left(\overline{x}_1\right) = \frac{1 - F_1\left(\overline{x}_1\right)}{\overline{x}_1} m_1 + \frac{1 - F_2\left(\overline{x}_2(\overline{x}_1)\right)}{\overline{x}_2(\overline{x}_1)} m_2 \tag{59}$$

$$G_2\left(\overline{x}_1\right) = 1 - F_1(\overline{x}_1)F_2(\overline{x}_2(\overline{x}_1)) - \frac{1 - F_1\left(\overline{x}_1\right)}{\overline{x}_1}\int_0^{\overline{x}_1} q_1\left(x_1\right)dx_1 - \frac{1 - F_2\left(\overline{x}_2(\overline{x}_1)\right)}{\overline{x}_2(\overline{x}_1)}\int_0^{\overline{x}_2(\overline{x}_1)} q_2\left(x_2\right)dx_2 \tag{60}$$

Differentiating (59) and (60) and using $\overline{x}_2(1) = 1$ and $\left(\frac{d\overline{x}_2}{d\overline{x}_1}\right)_{\overline{x}_1=1} = 1$ yields:

$$G_1'\left(1\right) = -\sum_{i=1,2} m_i f_i(1), \tag{61}$$

$$G_2'\left(1\right) = -\sum_{i=1,2} f_i\left(1\right)\left(1 - \int_0^1 q_i\left(x_i\right)dx_i\right). \tag{62}$$

By Lemma 7, when $\overline{x}_1 = 1$ then $q_i(x_i) = 0$ for $x_i < r_i(1)$ where $r_i(1)$ is defined by $r_i(1) - \frac{1-F_i(r_i(1))}{f_i(r_i(1))} = 0$. So, $1 - \int_0^1 q_i(x_i)dx_i \geq r_i(1)$. But since $m_i - \frac{1-F_i(m_i)}{f_i(m_i)} < 0$ for $i \in \{1,2\}$ by assumption, we have $m_i < r_i(1) \leq 1 - \int_0^1 q_i(x_i)dx_i$. So, from (61) and (62) it follows that $0 > G_1'(1) > G_2'(1)$. But since $G_1\left(1\right) = G_2\left(1\right) = 0$, it follows that there exists $\delta' > 0$ s.t. $G_1\left(x_1\right) < G_2\left(x_1\right)$ for all $x_1 \in [1-\delta', 1]$.

On the other hand, $G_1\left(.\right)$ is monotonically decreasing on $[0,1]$, $\lim_{x_1 \to 0} G_1(x_1) \to \infty$ and $G_2\left(0\right) = 1$. So, there exists $\delta'' > 0$ s.t. $G_1\left(x_1\right) > G_2\left(x_1\right)$ if $x_1 \in [0, \delta'']$. Let $\delta_1 = \min\{\delta', \delta''\}$.

So, since $G_1(.)$ and $G_2(.)$ are continuous, there exists $\overline{x}_1 \in (\delta_1, 1-\delta)$ such that $G_1(\overline{x}_1) = G_2(\overline{x}_1)$. Then $\overline{x}_1$ and $\overline{x}_2 = \overline{x}_2(\overline{x}_1)$ constitute a solution to the system (54), (56). Since the mapping $\overline{x}_2(\overline{x}_1)$ is continuous and satisfies $\overline{x}_2(0) = 0$ and $\overline{x}_2(\overline{x}_1) \leq \overline{x}_1$ by Claim 2, we have $\overline{x}_2 = \overline{x}_2(\overline{x}_1) \in (0,1)$. So, with a slight abuse of notation, from now on let $(\overline{x}_1, \overline{x}_2)$ denote such solution.

Next, we set $q_i(\overline{x}_i) = \frac{m_i + \int_0^{\overline{x}_i} q_i(x_i)dx_i}{\overline{x}_i}$ for $i \in \{1,2\}$. So (56) can be rewritten as follows:

$$q_1(\overline{x}_1)(1 - F_1(\overline{x}_1)) + q_2(\overline{x}_2)(1 - F_2(\overline{x}_2)) = 1 - F_1(\overline{x}_1)F_2(\overline{x}_2)). \tag{63}$$

Claim 4. *A solution $(\overline{x}_1, \overline{x}_2)$ to (54) and (56) satisfies the feasibility conditions $F_j(\overline{x}_j) \leq q_i(\overline{x}_i) \leq 1$ for $i, j \in \{1,2\}$ if and only if the inequalities (64) and (65) hold.*

$$m_1 - m_2 \leq \overline{x}_1 - \overline{x}_2 F_1(\overline{x}_1) - \int_0^{\overline{x}_1} q_1(x_1)\,dx_1 + \int_0^{\overline{x}_2} q_2(x_2)\,dx_2, \tag{64}$$

$$m_1 - m_2 \geq \overline{x}_1 F_2(\overline{x}_2) - \overline{x}_2 - \int_0^{\overline{x}_1} q_1(x_1)\,dx_1 + \int_0^{\overline{x}_2} q_2(x_2)\,dx_2, \tag{65}$$

The **"Only If"** part of the claim is obvious. If the feasibility conditions $F_j(\overline{x}_j) \leq q_i(\overline{x}_i) \leq 1$ for $i, j \in \{1,2\}$ hold, then using these conditions in the budget constraints $m_i = \overline{x}_i q_i(\overline{x}_i) - \int_0^{\overline{x}_i} q_i(x_i)dx_i$ yields (64) and (65).

In the opposite direction, note that from (63) it follows immediately that $F_j(\overline{x}_j) \leq q_i(\overline{x}_i)$ if and only if $q_j(\overline{x}_j) \leq 1$ for $i, j \in \{1,2\}$. So, if $q_1(\overline{x}_1) > 1$ then $q_2(\overline{x}_2) < F_1(\overline{x}_1)$. Using these inequalities in the budget constraints yields that (64) fails. A similar argument shows that if $q_2(\overline{x}_2) > 1$ then by (63) $q_1(\overline{x}_1) < F_2(\overline{x}_2)$, and so (65) fails.

Claim 5. There exists $\eta > 0$ such that (64) and (65) hold if $|F_2(x) - F_1(x)| < \eta$ for all $x \in [0,1]$ and $|m_1 - m_2| < \eta$.

Proof: First, we need to introduce some notation. Let r_i be the unique solution for x_i to $\gamma_i(x_i) = 0$ for $i \in \{1,2\}$. Then for $x_i \in [r_i, \overline{x}_i]$ define $\hat{x}_j(x_i)$ as a solution for x_j to the equation $\gamma_i(x_i) = \gamma_j(x_j)$. That is, $\hat{x}_2(x_1)$ ($\hat{x}_1(x_2)$) is the solution in x_2 (x_1) to the following equation:

$$x_1 - \frac{1 - F_1(x_1) - \frac{(1-F_1(\overline{x}_1))^2}{(1-F_1(\overline{x}_1)+\overline{x}_1 f_1(\overline{x}_1))}}{f_1(x_1)} = x_2 - \frac{1 - F_2(x_2) - \frac{(1-F_2(\overline{x}_2))^2}{(1-F_2(\overline{x}_2)+\overline{x}_2 f_2(\overline{x}_2))}}{f_2(x_2)}. \tag{66}$$

Note that both $\hat{x}_1(.)$ and $\hat{x}_2(.)$ are increasing, continuous, and satisfy $\widehat{x}_i(r_j) = r_i$ and $\widehat{x}_i(\overline{x}_j) = \overline{x}_i$ for $i, j \in \{1,2\}$.

Further, let us show that $\hat{x}_2(x_1) < x_1$ for all $x_1 \in (r_1, \overline{x}_1]$. Since $\gamma_i'(x) > 0$, it is sufficient to establish that $\gamma_2(x) > \gamma_1(x)$ for all $x \in [r_2, \overline{x}_2]$.

First, since $\gamma_i(x)$ is continuous in x for $i \in \{1,2\}$, $\gamma_1(\overline{x}_1) = \gamma_2(\overline{x}_2)$ and, as established above, $\overline{x}_1 > \overline{x}_2$, it follows that there exists $\eta > 0$ s.t. $\gamma_2(x) > \gamma_1(x)$ for all $x \in [\overline{x}_2 - \eta, \overline{x}_2]$. So, if $\gamma_2(x) \leq \gamma_1(x)$ for some $x \in [r_2, \overline{x}_2)$, there exists $\tilde{x} \in [r_2, \overline{x}_2)$ s.t. $\gamma_2(\tilde{x}) = \gamma_1(\tilde{x})$ and $\gamma_2'(\tilde{x}) > \gamma_1'(\tilde{x})$ which, by definition of $\gamma_i(.)$, implies that $\frac{f_2'(\tilde{x})}{f_2(\tilde{x})} \geq \frac{f_1'(\tilde{x})}{f_1(\tilde{x})}$. However, the last inequality contradicts Assumption 2 (MLRP). Hence, we must have $\gamma_2(x) > \gamma_1(x)$ for all $x \in [r_2, \overline{x}_2)$ and therefore $\hat{x}_2(x_1) < x_1$ for all $x_1 \in [r_1, \overline{x}_1]$.

Using this notation, we have $q_i(x_i) = F_j(\hat{x}_j(x_i))$ if $x_i \geq r_i$ and $q_i(x_i) = 0$ otherwise.

Our next step is to prove a lower bound for the right-hand sides of (64) and an upper bound for the right-hand side of (65). For this, we need to bound the expression $\int_0^{\overline{x}_1} q_1(x)dx - \int_0^{\overline{x}_2} q_2(x)dx$. We have: $\int_0^{\overline{x}_2} q_2(x_2)dx_2 =$

$$\int_{r_2}^{\overline{x}_2} F_1(\hat{x}_1(x_2))dx_2 = \int_0^{\overline{x}_1} \overline{x}_2 - \max\{r_2, \hat{x}_2(x_1)\}dF_1(x_1) = \overline{x}_2 F_1(\overline{x}_1) - r_2 F_1(r_1) - \int_{r_1}^{\overline{x}_1} \hat{x}_2(x_1)dF_1(x_1)$$
$$\geq \overline{x}_2 F_1(\overline{x}_1) - r_2 F_1(r_1) - \int_{r_1}^{\overline{x}_1} x_1 dF_1(x_1) = (\overline{x}_2 - \overline{x}_1)F_1(\overline{x}_1) + (r_1 - r_2)F_1(r_1) + \int_{r_1}^{\overline{x}_1} F_1(x_1)dx_1, \tag{67}$$

where the first equality has been established above, the second equality is obtained by changing the order of integration, the inequality holds because $\hat{x}_2(x_1) \leq x_1$, and the last equality is obtained by integrating by parts. Combining (67) with $\int_0^{\overline{x}_1} q_1(x_1) = \int_{r_1}^{\overline{x}_1} F_2(\hat{x}_2(x_1))dx_1 \leq \int_{r_1}^{\overline{x}_1} F_2(x_1)dx_1$ yields the following lower bound for the right-hand side of (64):

$$\overline{x}_1(1 - F_1(\overline{x}_1)) + (r_1 - r_2)F_1(r_1) - \int_{r_1}^{\overline{x}_1} F_2(x_1) - F_1(x_1)dx_1 \tag{68}$$

Since $r_1 > r_2$, $\overline{x}_1 (1 - F_1(\overline{x}_1)) + (r_1 - r_2)F_1(r_1) > 0$. So, there exists $\epsilon > 0$ s.t. (68) and hence the right-hand side of (64) is positive when $|F_2(x) - F_1(x)| < \epsilon$.

Next, let us provide an upper bound for the right-hand side of (65) and show that this upper bound is negative under the conditions of the Theorem. First, we have:

$$\int_0^{\overline{x}_1} q_1(x_1)dx_1 = \int_{r_1}^{\overline{x}_1} F_2(\hat{x}_1(x_2))dx_1 = \int_0^{\overline{x}_2} (\overline{x}_1 - \max\{\widehat{x}_{12}(x_2), r_1\})\, dF_2(x_2) =$$
$$\overline{x}_1 F_2(\overline{x}_2) - r_1 F_2(r_2) - \int_{r_2}^{\overline{x}_2} \widehat{x}_{12}(x_2)dF_2(x_2) \geq \overline{x}_1 F_2(\overline{x}_2) - r_1 F_2(r_2) - \overline{x}_1(F_2(\overline{x}_2) - F_2(r_2)), \tag{69}$$

where the first equality has been established above, the second equality is obtained by changing the order of integration, the inequality holds because $\hat{x}_1(x_2) \leq \overline{x}_1$ for all $x_2 \in [0, \overline{x}_2]$, and the last equality is obtained by integrating by parts.

Combining (69) with $\int_0^{\overline{x}_2} q_2(x_2) = \int_{r_2}^{\overline{x}_2} F_1(\hat{x}_1(x_2))dx_2 \leq F_1(\overline{x}_1)(\overline{x}_2 - r_2)$ yields the following upper bound for the right-hand side of (65):

$$- \overline{x}_2(1 - F_1(\overline{x}_1)) + r_1 F_2(r_2) + \overline{x}_1(F_2(\overline{x}_2) - F_2(r_2)) - F_1(\overline{x}_1) r_2 \tag{70}$$

From (54) and (56) it is easy to see that there exist constants $K_1 > 0$ and $K_2 > 0$ s.t. $|\overline{x}_1 - \overline{x}_2| < \epsilon K_1$ and $|\overline{r}_2 - \overline{r}_1| < \epsilon K_2$ if $|m_2 - m_1| < \epsilon$ and $|F_2(x) - F_1(x)| < \epsilon$ for all $x \in [0, 1]$. So when $\epsilon > 0$ is sufficiently small then (70), and hence the right-hand side of (65) are negative.

So, when the right-hand side of (64) is positive and the right-hand side of (65) is negative, both (64) and (65) holds when $|m_1 - m_2| < \psi$ when is sufficiently small. So, setting $\eta = \min\{\epsilon, \psi\}$ concludes the proof of Claim 5.

Finally, note that by a direct extension of the method used in the symmetric case, the optimal asymmetric mechanism is a solution to the dual optimization problem. Since the latter is convex, the optimal mechanism is unique and is characterized by the first-order conditions which by Theorem 2 are represented by the budget constraints. So the feasible solution to these first-order condition must also be unique. Hence, the threshold pair $(\overline{x}_1, \overline{x}_2)$ solving the budget constraints and satisfying the feasibility condition must be unique. *Q.E.D.*

11 Online Appendix (Not for Publication) Top Auction and Budget-Handicap Auction with Three Bidders, under Uniform Type Distribution.

11.1 Top Auction

In the top auction, the reservation value is given by $r^t = \bar{x}^t - \frac{(\bar{x}^t)^2}{2}$. Also, $q_i(x) = x^2$ for all $x \in \left[\bar{x}^t - \frac{(\bar{x}^t)^2}{2}, \bar{x}^t\right)$, and $q_i(\bar{x}^t)$ is set to satisfy the budget constraint of bidder $i \in \{1, 2, 3\}$. Then conditions (26) and (28) simplify to:

$$\begin{aligned}
&\sum_{i=1}^{3} m_i = \bar{x}^t(1+\bar{x}^t) + \left(\bar{x}^t - \frac{(\bar{x}^t)^2}{2}\right)^3 \\
&m_1 - \frac{m_2+m_3}{2} \leq \bar{x}^t \left(1 - \bar{x}^t \frac{1+\bar{x}^t}{2}\right) \\
&m_1 - m_3 \leq \bar{x}^t \left(1 - (\bar{x}^t)^2\right) \qquad (71)
\end{aligned}$$

Top auction is optimal when the system (71) has a solution $\bar{x}^t$.

11.2 Budget-Handicap Auction with Top cluster

Since $\bar{x}_1 = \bar{x}_2$ in the top cluster, we will simplify the notation and let $\bar{x}_1$ denote the threshold of bidders 1 and 2 in the rest of this subsection. So, we have $\bar{x}_1 > \bar{x}_3$, $\gamma_1(x) = \gamma_2(x) = 2x - 2\bar{x}_1 + \bar{x}_1^2$ for $x < \bar{x}_1$, $\gamma_1(\bar{x}_1) = \gamma_2(\bar{x}_1) = \bar{x}_1^2$; $\gamma_3(x) = 2x - 2\bar{x}_3 + \bar{x}_3^2$ for $x < \bar{x}_3$, $\gamma_3(\bar{x}_3) = \bar{x}_3^2$. The bidders' reservation values are given by $r_1 = r_2 = \bar{x}_1 - \frac{\bar{x}_1^2}{2}$, $r_3 = \bar{x}_3 - \frac{\bar{x}_3^2}{2}$.

Then by Lemma 7 for $i \in \{1, 2\}$, $q_i(x) = 0$ for $x < \bar{x}_1 - \frac{\bar{x}_1^2}{2}$, $q_i(x) = x(x - \bar{x}_1 + \frac{\bar{x}_1^2}{2} + \bar{x}_3 - \frac{\bar{x}_3^2}{2})$ for $x \in (\bar{x}_1 - \frac{\bar{x}_1^2}{2}, \bar{x}_1 - \frac{\bar{x}_1^2}{2} + \frac{\bar{x}_3^2}{2}]$, and $q_i(x) = x$ for $x \in (\bar{x}_1 - \frac{\bar{x}_1^2}{2} + \frac{\bar{x}_3^2}{2}, \bar{x}_1)$. The values of $q_1(\bar{x}_1)$ and $q_2(\bar{x}_1)$ are determined by the budget constraints of bidders 1 and 2.

For bidder 3, we have $q_3(x) = 0$ for $x < \bar{x}_3 - \frac{\bar{x}_3^2}{2}$, $q_3(x) = \left(x - \bar{x}_3 + \frac{\bar{x}_3^2}{2} + \bar{x}_1 - \frac{\bar{x}_1^2}{2}\right)^2$ for $x \in (\bar{x}_3 - \frac{\bar{x}_3^2}{2}, \bar{x}_3)$, and $q_3(\bar{x}) = \left(\frac{\bar{x}_3^2}{2} + \bar{x}_1 - \frac{\bar{x}_1^2}{2}\right)^2$.

Note that while $q_3(x)$ is continuous everywhere above r_3, $q_1(x)$ and $q_2(x)$ experience two jumps. First, there is a jump at $\bar{x}_1 - \frac{\bar{x}_1^2}{2} + \frac{\bar{x}_3^2}{2}$, as bidders 1 and 2 with values above this level no longer face the competition from bidder 3 because $\gamma_1(\bar{x}_1 - \frac{\bar{x}_1^2}{2} + \frac{\bar{x}_3^2}{2}) = \gamma_3(\bar{x}_3)$. The second jump happens at the threshold $\bar{x}_1$, since $\lim_{x \to \bar{x}-} q_1(x) + q_2(x) = 2\bar{x} < 1 + \bar{x} = q_1(\bar{x}) + q_2(\bar{x})$.

By Theorem 2 (conditions (21)-(23)), the budget-handicap auction with a top cluster is optimal if the following system of two equations and one inequality has a solution:

$$m_3 = \bar{x}_3 q_3(\bar{x}_3) - \int_{\bar{x}_3 - \frac{\bar{x}_3^2}{2}}^{\bar{x}_3} q_3(x_3)dx_3 \tag{72}$$

$$m_1 + m_2 = (1 + \bar{x}_1) - 2\int_{\bar{x}_1 - \frac{\bar{x}_1^2}{2}}^{\bar{x}_1} q_1(x_1)dx_1 \tag{73}$$

$$m_1 - m_2 \leq \bar{x}_1(1 - \bar{x}_1).$$

Using the expressions for $q_i(x)$, $i \in \{1, 2, 3\}$ in (72) and (73) yields:

$$m_3 = \bar{x}_3 \left(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_1^2}{2}\right)^2 - \int_{\bar{x}_3 - \frac{\bar{x}_3^2}{2}}^{\bar{x}_3} \left(s - \bar{x}_3 + \bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_1^2}{2}\right)^2 ds = \bar{x}_3 \left(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_1^2}{2}\right)^2 - \frac{\left(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_1^2}{2}\right)^3}{3} + \frac{\left(\bar{x}_1 - \frac{\bar{x}_1^2}{2}\right)^3}{3} = -\frac{\bar{x}_3^6}{24} + \frac{\bar{x}_3^5}{4} + \bar{x}_3^3\left(1 - \frac{\bar{x}_3}{4}\right)\left(\bar{x}_1 - \frac{\bar{x}_1^2}{2}\right) + \left(\bar{x}_3 - \frac{\bar{x}_3^2}{2}\right)\left(\bar{x}_1 - \frac{\bar{x}_1^2}{2}\right)^2 \tag{74}$$

$$\begin{aligned} m_1 + m_2 &= \bar{x}_1(1 + \bar{x}_1) - 2\int_{\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_1^2}{2}}^{\bar{x}_1} ydy - 2\int_{\bar{x}_1 - \frac{\bar{x}_1^2}{2}}^{\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_1^2}{2}} y\left(y - \bar{x}_1 + \bar{x}_3 + \frac{\bar{x}_1^2}{2} - \frac{\bar{x}_3^2}{2}\right))dy \\ &= \bar{x}_1(1 + \bar{x}_1) + \frac{\bar{x}_3^4}{4}\left(1 - \bar{x}_3 + \frac{\bar{x}_3^2}{6}\right) - \bar{x}_1^3\left(1 - \frac{\bar{x}_1}{4}\right) + \left(\bar{x}_1 - \frac{\bar{x}_1^2}{2}\right)\bar{x}_3^2\left(1 - \frac{\bar{x}_3}{2}\right)^2 \end{aligned} \tag{75}$$

Equations (74) and (75) implicitly define $\bar{x}_1$ and $\bar{x}_3$. If the solution is such that $m_1 - m_2 \leq \bar{x}_1(1 - \bar{x}_1)$, then the optimal mechanism is a handicap auction with a "top cluster." The set of budgets for which this is true is depicted in Figure 4.

11.3 Lower cluster

Next, consider the case of the "lower cluster" with $\bar{x}_1 > \bar{x}_2 = \bar{x}_3$. To simplify the presentation, we let $\bar{x}_2$ denote the threshold of bidders 2 and 3 and drop $\bar{x}_3$ from the notation. In this case we have: $\gamma_1(x_1) = 2x - 2\bar{x}_1 + \bar{x}_2^2$ for $x_1 < \bar{x}_1$, $\gamma_1(\bar{x}_1) > \gamma_1^-(\bar{x}_1) = \frac{\bar{x}_2^2}{2}$, $\gamma_2(x) = \gamma_3(x) = 2x - 2\bar{x}_2 + \bar{x}_2^2$ for $x < \bar{x}_2$, $\gamma_2(\bar{x}_2) = \gamma_3(\bar{x}_2) = \bar{x}_2^2$. The reservation values are $r_1 = \bar{x}_1 - \frac{\bar{x}_2^2}{2}$ and $r_2 = r_3 = \bar{x}_2 - \frac{\bar{x}_2^2}{2}$.

The probabilities of trading are given by: $q_1(x_1) = 0$ for $x_1 < \bar{x}_1 - \frac{\bar{x}_2^2}{2}$, $q_1(x_1) = (x_1 - \bar{x}_1 + \bar{x}_2)^2$ for $x_1 \in \left[\bar{x}_1 - \frac{\bar{x}_2^2}{2}, \bar{x}_1\right)$, $q_1(\bar{x}_1) = 1$. For $i \in \{2, 3\}$, $q_i(x) = 0$ for $x < \bar{x}_2 - \frac{\bar{x}_2^2}{2}$, and $q_i(x) = x\left(x - \bar{x}_2 + \bar{x}_1\right)$ for $x \in \left[\bar{x}_2 - \frac{\bar{x}_2^2}{2}, \bar{x}_2\right)$. Finally, $q_2(\bar{x}_2)$ and $q_3(\bar{x}_2)$ are determined by the budget constraints of bidders 2 and 3, correspondingly.

By Theorem 2, condition (21) must hold for bidder 1 and conditions (22) and (23) must hold for bidders 2 and 3 i.e.:

$$m_1 = \bar{x}_1 - \int_{\bar{x}_1 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_1} (s - \bar{x}_1 + \bar{x}_2)^2 \, ds = \bar{x}_1 - \frac{\bar{x}_2^3}{3} + \frac{\left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)^3}{3}$$

$$= \bar{x}_1 - \frac{\bar{x}_2^2}{6}\left(\bar{x}_2^2 + \bar{x}_2\left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right) + \left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)^2\right) = \bar{x}_1 - \frac{\bar{x}_2^4}{2}\left(1 - \frac{\bar{x}_2}{2} + \frac{\bar{x}_2^2}{12}\right) \tag{76}$$

$$m_2 + m_3 = \bar{x}_2\bar{x}_1(1+\bar{x}_2) - 2\int_{\bar{x}_2 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_2} s\,(s - \bar{x}_2 + \bar{x}_1)\, ds = \bar{x}_1\bar{x}_2(1+\bar{x}_2) - \frac{2\bar{x}_2^3}{3} + \frac{2\left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)^3}{3}$$

$$-(\bar{x}_1 - \bar{x}_2)\left(\bar{x}_2^2 - \left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)^2\right) = \bar{x}_1\bar{x}_2(1+\bar{x}_2) + \frac{\bar{x}_2^5}{4}\left(1 - \frac{\bar{x}_2}{3}\right) - \bar{x}_2^3\bar{x}_1\left(1 - \frac{\bar{x}_2}{4}\right) \tag{77}$$

$$m_2 - m_3 \le \bar{x}_2(1-\bar{x}_2)\bar{x}_1 \tag{78}$$

Equations (76) and (77) implicitly define $\bar{x}_1$ and $\bar{x}_2$. If the solution satisfies (78), the optimal mechanism is the handicap auction with the lower cluster and thresholds $\bar{x}_1$ and $\bar{x}_2 = \bar{x}_3$. The set of budgets for which this is true is depicted in Figure 5.

11.4 No Clusters.

Finally, we consider the case with no clusters i.e., $\bar{x}_1 > \bar{x}_2 > \bar{x}_3$.

In this case, $\gamma_1(x_1) = 2x - 2\bar{x}_1 + \bar{x}_2^2$ for $x_1 < \bar{x}_1$, $\gamma_1(\bar{x}_1) > \gamma_1^-(\bar{x}_1) = \frac{\bar{x}_2^2}{2}$, $\gamma_2(x) = 2x - 2\bar{x}_2 + \bar{x}_2^2$ for $x < \bar{x}_2$, $\gamma_2(\bar{x}_2) = \bar{x}_2^2$, $\gamma_3(x) = 2x - 2\bar{x}_3 + \bar{x}_3^3$ for $x < \bar{x}_3$, $\gamma_3(\bar{x}_3) = \bar{x}_3^2$. The reservation values are $r_1 = \bar{x}_1 - \frac{\bar{x}_2^2}{2}$, $r_2 = \bar{x}_2 - \frac{\bar{x}_2^2}{2}$, and $r_3 = \bar{x}_3 - \frac{\bar{x}_3^2}{2}$.

Therefore, the probabilities of trading of bidder 1 are as follows: $q_1(x) = 0$ for $x < \bar{x}_1 - \frac{\bar{x}_2^2}{2}$, $q_1(x) = (x - \bar{x}_1 + \bar{x}_2)\left(x - \bar{x}_1 + \bar{x}_3 + \frac{\bar{x}_2^2}{2} - \frac{\bar{x}_3^2}{2}\right)$ for $x \in \left[\bar{x}_1 - \frac{\bar{x}_2^2}{2}, \bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right]$, $q_1(x) = x - \bar{x}_1 + \bar{x}_2$ for $x \in \left(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}, \bar{x}_1\right)$, and $q_1(\bar{x}_1) = 1$.

For bidder 2, $q_2(x) = 0$ for $x < \bar{x}_2 - \frac{\bar{x}_2^2}{2}$, $q_2(x) = (x - \bar{x}_2 + \bar{x}_1)\left(x - \bar{x}_2 + \bar{x}_3 + \frac{\bar{x}_2^2}{2} - \frac{\bar{x}_3^2}{2}\right)$ for $x \in \left[\bar{x}_2 - \frac{\bar{x}_2^2}{2}, \bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right]$, $q_2(x) = x - \bar{x}_2 + \bar{x}_1$ for $x \in \left(\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}, \bar{x}_2\right)$, $q_2(\bar{x}_2) = \bar{x}_1$.

Finally, for bidder 3, $q_3(x) = 0$ for $x < \bar{x}_3 - \frac{\bar{x}_3^2}{2}$, $q_3(x) = \left(x - \bar{x}_3 + \bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right) \times$ $\times \left(x - \bar{x}_3 + \bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)$ for $x \in \left[\bar{x}_3 - \frac{\bar{x}_3^2}{2}, \bar{x}_3\right)$, and $q_3(\bar{x}_3) = (\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2})(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2})$.

By Theorem 2, in the "no cluster" case the necessary and sufficient conditions characterizing the optimal thresholds $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$ are the budget constraints (21) i.e., $m_i = \bar{x}_i q_i(\bar{x}_i) - \int_{r_i}^{\bar{x}_i} q_i(s)ds$ for $i = 1, 2, 3$. If the solution to this system of three equations exists and is such that $1 \ge \bar{x}_1 > \bar{x}_2 > \bar{x}_3 \ge 0$, then this configuration with no clusters is optimal.

In the rest of this subsection, we will exhibit the system of three equations $m_i = \bar{x}_i q_i(\bar{x}_i) - \int_{r_i}^{\bar{x}_i} q_i(s)ds$ for $i = 1,2,3$ explicitly using the expressions for $q_i(.)$ above and then replace it with a simpler system. First, consider $i = 1$. We have:

$$m_1 = \bar{x}_1 - \int_{\bar{x}_1 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}} (x - \bar{x}_1 + \bar{x}_2)\left(x - \bar{x}_1 + \bar{x}_3 + \frac{\bar{x}_2^2}{2} - \frac{\bar{x}_3^2}{2}\right) dx - \int_{\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}}^{\bar{x}_1} x - \bar{x}_1 + \bar{x}_2 ds =$$

$$\bar{x}_1 - \frac{\left(\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)^3}{3} + \frac{\left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)^3}{3} + \frac{\left(\bar{x}_2 - \bar{x}_3 - \frac{\bar{x}_2^2}{2} + \frac{\bar{x}_3^2}{2}\right)}{2}\left(\left(\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)^2 - \left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)^2\right)$$

$$-\frac{\bar{x}_2^2}{2} + \frac{\left(\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)^2}{2} = \bar{x}_1 + \frac{\bar{x}_3^4}{8}\left(1 - \bar{x}_3 + \frac{\bar{x}_3^2}{6}\right) - \frac{\bar{x}_2^3}{2}\left(1 - \frac{\bar{x}_2}{4}\right) + \left(\bar{x}_2 - \frac{\bar{x}_2^2}{2}\right)\frac{\bar{x}_3^2}{2}\left(1 - \frac{\bar{x}_3}{2}\right)^2 \tag{79}$$

Second, using the expressions for $q_2(.)$ and $q_3(.)$ derived above, we obtain:

$$m_2 = \bar{x}_2\bar{x}_1 - \int_{\bar{x}_2 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}} (x - \bar{x}_2 + \bar{x}_1)\left(x - \bar{x}_2 + \bar{x}_3 + \frac{\bar{x}_2^2}{2} - \frac{\bar{x}_3^2}{2}\right) dx - \int_{\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}}^{\bar{x}_2} x - \bar{x}_2 + \bar{x}_1 ds \tag{80}$$

$$m_3 = \bar{x}_3(\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2})(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}) - \int_{\bar{x}_3 - \frac{\bar{x}_3^2}{2}}^{\bar{x}_3} (x - \bar{x}_3 + \bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2})(x - \bar{x}_3 + \bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2})dx \tag{81}$$

Next, we replace (80) and (81) with the equations for $m_1 - m_2$ and $m_2 - m_3$ as follows. First, subtracting (80) from (79) we obtain:

$$m_1 - m_2 = \bar{x}_1(1 - \bar{x}_2) + \int_{\bar{x}_2 - \frac{\bar{x}_2^2}{2}}^{\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}} (\bar{x}_1 - \bar{x}_2)(x - \bar{x}_2 + \bar{x}_3 + \frac{\bar{x}_2^2}{2} - \frac{\bar{x}_3^2}{2})dx + \int_{\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}}^{\bar{x}_2} \bar{x}_1 - \bar{x}_2 ds$$

$$= \bar{x}_1(1 - \bar{x}_2) + \frac{\bar{x}_1 - \bar{x}_2}{2}\left(\bar{x}_2^2 - \left(\bar{x}_3 - \frac{\bar{x}_3^2}{2}\right)^2\right). \tag{82}$$

Finally, we perform a change of variable of integration in the second term of (80) to $y = x - \bar{x}_2 + \frac{\bar{x}_2^2}{2} + \bar{x}_3 - \frac{\bar{x}_3^2}{2}$ and subtract (81) from the result to obtain:

$$m_2 - m_3 = \bar{x}_1\bar{x}_2 - \frac{\bar{x}_1^2}{2} + \frac{\left(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)^2}{2} - \bar{x}_3\left(\bar{x}_2 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)\left(\bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)$$

$$+ \int_{\bar{x}_3 - \frac{\bar{x}_3^2}{2}}^{\bar{x}_3} \left(x - \bar{x}_3 + \bar{x}_1 + \frac{\bar{x}_3^2}{2} - \frac{\bar{x}_2^2}{2}\right)\left(\bar{x}_2 - \frac{\bar{x}_2^2}{2} - \bar{x}_3 + \frac{\bar{x}_3^2}{2}\right) dx =$$

$$\bar{x}_1\bar{x}_2 + (\bar{x}_2\bar{x}_3 - \bar{x}_1(1 - \bar{x}_3))\frac{\bar{x}_2^2 - \bar{x}_3^2}{2} + \left(\frac{1}{2} - \bar{x}_3\right)\left(\frac{\bar{x}_2^2}{2} - \frac{\bar{x}_3^2}{2}\right)^2 + \frac{\bar{x}_3^2}{2}(\bar{x}_2 - \frac{\bar{x}_2^2}{2} - \bar{x}_3 + \frac{\bar{x}_3^2}{2})(\bar{x}_1 + \frac{\bar{x}_3^2}{4} - \frac{\bar{x}_2^2}{2}). \tag{83}$$

To conclude, when the solution to the system (79), (82) and (83) satisfies $\bar{x}_1 > \bar{x}_2 > \bar{x}_3$, this is the optimal mechanism. The set of budgets for this case is depicted in Figure 5.