Screening, Signalling and Costly Misrepresentation.

Raymond Deneckere and Sergei Severinov [*]

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Abstract

We study mechanism design in environments where misrepresenting private information is costly. Specifically, a privately informed agent has to take several signalling actions, send several messages or undergo tests in which it is costly for her to misrepresent her type. We establish conditions under which the set of implementable allocation profiles strictly increases in the number of messages while the overall communication cost diminishes. We then derive an optimal screening mechanism in such setting. A surprising property of this mechanism is the absence of exclusion. Particularly, every consumer type whose valuation for the good exceeds the marginal cost of production is allocated a positive quantity. Reexamining job-market signalling via education in our set-up, we show that welfare losses from unproductive signalling will be small if the students signal via course grades rather than duration of schooling and the number of courses is sufficiently large. Thus, our results explain why employers often prefer to screen applicants via multiple interviews rather than via menus of contracts, and why the social losses from signalling activities ("rat race") may not be significant.
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1   Introduction.

In this paper we study mechanism design and signalling in settings where agents incur some cost of misrepresenting their private information. These costs may exist for several reasons. First, an individual may find it costly to misrepresent the truth for psychological or ethical reasons. Such individual may experience stress or discomfort from lying.[1]. Second, misrepresenting the truth may require costly actions, such as acquiring skills and/or technology for

[1]Behavioral psychologists have extensively studied the physical symptoms associated with the emotional discomfort, such as " blushing," "feeling wrong," that people experience when lying. See, for example Ekman (1973)

manufacturing evidence, and taking effort to conceal one's information or hide evidence that reveals the true state of the world. For example, in order to obtain a supplier contract, or to qualify for a loan, win a grant or a promotion, a firm or an individual may need to be perceived as highly productive, successful and/or creditworthy. This goal may be attained by exhibiting "evidence" exaggerating prior performance and concealing the risk of default or non-performance. Yet, the production of such favorable but inaccurate evidence would normally require expending cost and effort.

We study these issues in a setting of a principal-agent model in which the agent has private type information that affects both her own and the principal's utility from a social decision (which may involve consumption allocation, or an activity such as production, etc.). In our model, the agent can send a number of signals, or messages (we use these two terms interchangeably in the sequel) about her type. In contrast to much of the screening literature, we assume that the agent's cost of sending a particular message increases in the magnitude of type misrepresentation, and that the agent can send multiple costly messages or signals.

The motivation behind this key element of our approach is fairly natural and intuitive. As one interpretation, an agent may have to undergo a number of tests assessing her ability from different angles. In this case, message i would correspond to the agent's performance on test i. On every test, an agent can attain her "natural" level of performance corresponding to her type without incurring additional effort or disutility, while attaining a different level of performance requires costly effort or disutility.

Similarly, a message may correspond to an outcome of an inspection or an audit undertaken by the principal. For example, shareholders or the parent corporation may carry out several accounting and other audits of their subsidiary. The subsidiary's managers would then have to incur the cost of hiding or embellishing the true state of the world and fudging the numbers during each audit. Alternatively, managers may have to defend their financial reports to a variety of different stakeholders and outside experts, including the board of directors, external auditors, credit rating agencies, shareholders, creditors and employees, who each question the reports from a different angle.

In the job-market context the principal (employer) may ask the candidate to undergo several interviews conducted by different interviewers. Each interviewer may use a different method to evaluate the candidate's ability. Then the candidate would have to incur an extra cost of time and effort to prepare for each test or interview so that she can perform at the level above her true ability.

Further, each message could represent a different piece of evidence presented by the agent to her examiner, or to a judge in a court of law. Producing evidence, such as records, documents, etc., is typically costly, and such cost is normally higher when the agent procures untruthful evidence. In line with this motivation, we assume that an agent's cost of producing a certain message, signal, or a piece of evidence is increasing in the degree to which this message deviates from depicting the true state of the world.

Our analysis focuses on four issues. First, we explore the effect of increasing the size of the signal space, or the number of messages, on the set of implementable allocations. Second, we derive the optimal mechanism with multiple messages and characterize its properties. Third, we present a method for choosing the optimal number of messages when the principal also incurs a fixed cost of eliciting each message. Fourth, we reexamine the issue of job-market

signalling when the signal is not the length of education but the grades earned by a candidate. Most of our results, with the exception of the characterization of the optimal mechanism, hold for a multidimensional type space.

We show that the availability of multiple costly messages, or tests, can significantly expand the set of implementable allocations. Specifically, under fairly standard conditions, the set of implementable allocation profiles increases monotonically in the number of messages. In the limit when the number of messages becomes very large, any allocation profile becomes implementable at zero communication cost provided the agent's marginal cost of misrepresenting herself does not go to zero too fast in the number of messages.

When the agent's marginal cost of misrepresenting her type is zero, it becomes necessary for her to incur some cost and send messages misrepresenting her type to some degree in the optimal mechanism. Still we establish an approximate implementation result: with sufficiently many messages, the principal can come arbitrarily close to any decision rule and any surplus allocation, and at the same time keep misrepresentation costs arbitrarily small.

Further, we characterize the optimal screening mechanism maximizing the principal's expected profits for an arbitrary number of messages. That part of our analysis in related to the contributions by Maggi and Rodriguez-Clare (1995) and Crocker and Morgan (1998), and we comment more on this in Section 4. We establish an important qualitative property of the optimal mechanism - absence of exclusion. Specifically, when costly messages are available, then each agent-type who can generate a positive surplus is assigned a non-zero allocation in the optimal mechanism. Thus, the standard result on the optimality of exclusion in optimal screening is non-robust to the availability of costly signals.[2]

We then use our characterization of the optimal mechanism to determine the optimal number of signals or messages which the principal should request when there is a fixed cost for eliciting and processing each message.

Finally, we apply our approach to reexamine the role of education as a signal in the job-market. Ever since Spence's (1973) seminal contribution, economists have been concerned with potential loss of welfare due to such signalling. Specifically, Spence demonstrates that potential job-applicants, engaged in a competitive 'rat race,' will spend too much time and effort in education in order to provide an informative signal about their ability, even though education may not enhance their productivity (or not enhance it beyond a certain point).

In our reexamination, education remains unproductive, but it is the potential employees' grades in the courses they took, rather than the number of years of schooling, that serve as a signal to the job-market. Arguably, employers pay more attention to job-applicants' grades than to the number of years that they spent in school.[3] Naturally, more able individuals can

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[2]It is worth noting that when costly signals are absent exclusion is a robust property of optimal screening mechanisms. In particular, a profit-maximizing monopolist will choose not to sell to consumers whose willingness to pay for the good is not sufficiently higher than marginal cost, under both uniform and non-linear pricing, except for the non-generic case of perfectly inelastic consumer demand at price equal to marginal cost (which requires either that there are no consumers with valuations near marginal cost, or that the density of valuations is infinite at this level) (see Maskin and Riley (1984)). Exclusion must also occur in settings with multidimensional private information (see Armstrong (1996) and Rochet and Choné (1998)).

[3]Nowadays, the fact that grades serve as a signal to the job-market is openly accepted by public and governments in various countries. For instance, the UK Universities Minister Jo Johnson speaking on the topic of grade point average classification in the UK Universities stated that "businesses need a degree classification

earn better grades at a lower cost.

We also posit that the students enjoy education to some extent and/or that it provides some other benefit, more so for the more able types. Thus, the efficient grade level - at which the cost of education in time and effort balances the pleasure (or benefit) from it- is not the lowest grade, and this efficient grade level is increasing in the ability type. In this setting, we show that the net cost of signalling becomes very small, and each student earns a profile of grades close to the efficient one when the number of courses taken by job-applicants becomes large. Thus, our paper identifies important conditions under which the individuals do not engage in unproductive signalling, or 'rat race.'

Our findings have practical relevance. In particular, they can explain why the employers in a number of industries prefer to screen and interview job-applicants very thoroughly, rather than to offer self-selecting menus of contracts or strong performance incentives to them. Indeed, the interviewing process in many professional job-markets appears to be consistent with the idea of requesting multiple messages or signals from the candidate, with each signal being somewhat different from the others.

For example, in the context of a departmental visit on the academic job-market a prospective candidate meets with faculty members working in different fields. It is conceivable that each conversation provides an independent signal of the candidate's ability, because different faculty members, especially if they work in different fields, assess the candidate from different perspectives and inquire about different aspects of the candidate's knowledge and skills. Similar interviewing procedures are used by professional services firms in consulting, law, banking, etc. The interview in such firms involve solving cases, conversations with consultants, managers and partners.

Our results imply that if a job candidate has to go through a sufficient number of such interviews, or other tests, then substantially misrepresenting his ability will be too costly. So, the employer will have a quite accurate estimate of the candidate's ability, and would not have to offer a powerful incentive scheme on the job. This is notable since the incentive schemes used in the real world are often not as strong as predicted by the contracting literature.

In the education context, our results suggest that degree programs that involve a larger number of (shorter) courses and hence a larger number of grades provide a better signalling value to the job-market than degree programs with a small number of courses.

The intuition for our results is as follows. Suppose the agent is required to send only a single message. The allocation to be implemented may then be such that if some type θ were not to misrepresent itself by sending its least cost message, then some other type θ' would have an incentive to mimic type θ. As a consequence, costly misrepresentation by type θ will be necessary to prevent this. Consider now what happens when the agent has to send a second message, along a different dimension. Type θ' will then incur an extra cost of pretending to be type θ, *even if* type θ were to send her least cost message in this extra dimension. This is because the least cost messages of types θ and θ' are different. As a consequence, less misrepresentation for type θ along the original dimension is required when an extra message is added. Furthermore, with sufficiently many messages, it will not be profitable for type θ' to

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mimic type θ, even when type θ misrepresents herself very little or not at all. So the desired allocation will become implementable at a zero or very small message/signalling cost.

Note that this reasoning would not go through if, as in the Spence's education model, the least cost message of each agent type was the same (i.e. zero education). In fact, signalling costs would then remain invariant to the number of messages sent. The novel contribution of our paper is therefore not just to increase message cost by letting the number of messages increase, but also to simultaneously require that the least cost signal differs across agent types. This is intuitive in many applications. For example, in education signalling model it means that a higher, more able type can attain a higher grade level in a course at zero or small cost of time and effort than a lower type, and/or higher types enjoy education to a greater extent than lower types.

Note that the intuition provided above applies to deviations of a discrete size, as well as to small local deviations provided the marginal cost to misrepresentation is bounded below. When the marginal cost of misrepresentation equals zero at the least cost message, some misrepresentation will be required to rule out small local deviations no matter how many messages the agent is required to send. However, because adding extra messages increases the marginal cost of misrepresentation, less misrepresentation will be then needed along each dimension, and so asymptotically misrepresentation costs still vanish in equilibrium.

Our paper is related to two strands of literature. The first of these is the literature on costly state falsification. Lacker and Weinberg (1989), Maggi and Rodriguez-Clare (1995) and Crocker and Morgan (1998) study principal-agent models in which cost of misrepresentation is smooth and increasing in the magnitude of misrepresentation. These authors point out that in such environments it is easier for the principal to elicit agents' private information, and that it is optimal for the principal to induce the agent to engage in some misrepresentation. In related work, Kartik and Tercieux (2012) study Nash implementation in a game of complete information where players may manufacture "evidence" at cost.[4] In a legal setting, Bull (2008) studies costly evidence production and disclosure under complete information. Bull (2008) explores mechanism design when there is a cost of disclosing evidence to the designer. Bull (2009) compares inquisitorial and adversarial litigation systems when a litigant may suppress evidence by incurring a cost. Finally, following serious reporting scandals such as Enron, managerial misreporting has attracted a lot of attention in the accounting and finance literature. Models involving costly misrepresentation of earnings include Fisher and Verecchia (2000), Goldman and Slezak (2006), Guttman, Kadan and Kandel (2006) and Caskey, Nagar and Petacchi (2010). The principal difference between our paper and this literature is that we allow the agent to send multiple costly signals or messages, and study how implementability is affected when we increase the size of the message space.

Our paper is also related to the extensive literature on signalling, since we emphasize the role of communication and costly messages in determining the principal's allocation decision. Our modelling assumption requiring the least costly message/signal to be type dependent

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[4] Green and Laffont (1986) and Alger and Ma (2003) consider environments in which the falsification cost is binary, taking the value zero or infinity, and show that mechanisms in which the agent does not always send truthful signals can attain outcomes that are not implementable via mechanisms with truthful signals. **?** study implementability when the set of messages that an agent can send in a mechanism is type-dependent private information.

connects our paper to several contributions on costly signalling, in particular, Bernheim and Severinov (2003), Kartik, Ottaviani and Squintani (2007), Kartik (2009) and Seidmann and Winter (1997). In Bernheim and Severinov (2003), the type-dependence of the least costly signal is derived endogenously since the signalling action has intrinsic utility. Kartik (2009) studies a signalling model with costly misrepresentation. The equilibrium in his model converges to the most informative cheap talk equilibrium as the cost of misrepresentation becomes very small, while it converges to truthtelling when the cost of lying becomes very large. One of key differences between these signalling papers and ours is that in our paper the mechanism designer plays an active role in constructing the optimal communication scheme consisting of multiple messages. The mechanism designer does so in such a way that, although each type incurs some misrepresentation cost, in equilibrium those costs are small for each type, yet imitating another type becomes very costly for everyone.

Seidmann and Winter (1997) study signalling when the set of feasible signals depends on the sender's type. They derive conditions that ensure that full revelation occurs in equilibrium. Our paper shows that type-dependency of the least cost signal has a very strong effect when the number of signals is allowed to grow. In fact, while the signalling literature has considered models with multiple signals (Rochet and Quinzii (1985), Engers (1987)), it has not posed the question of how the equilibrium is affected by increasing the dimensionality of the signal space. Our paper is also related to Sher and Vohra (2014) who consider a seller-buyer bargaining model in which the seller may request multiple pieces of type-dependent evidence from the buyer. In contrast to our paper, in their model the "cost" of a piece of evidence to a particular buyer is either zero or infinity.

The rest of the paper is organized as follows. In section 2 we describe the model, Section 3 presents the monotonicity and asymptotics results. Section 4 studies optimal mechanisms when the type space is one-dimensional. Section 5 endogenizes the dimension of the signal space by assuming that eliciting messages is costly to the principal. In section 6 we reexamine the role of education as a signal in the job-market. All proofs are relegated to an Appendix.

2   Model.

We consider a principal-agent model with asymmetric information. To this standard environment we add a costly communication process in which the agent sends several messages or signals to the principal.[5]

Specifically, the principal controls the allocation space X. We assume that X is a compact subset of a k-dimensional Euclidean space. Thus, the principal's action is a choice of an allocation $x \in X$. Allocation x can denote a vector of production and/or consumption decisions, as well as monetary transfers. The agent privately observes the outcome of a random variable θ (which we will refer to as the agent's type in the sequel) that affects the utilities of the principal and the agent. We assume that $\theta \in \Theta$, where Θ is a compact subset of $\mathbb{R}^l$. Thus, we explicitly allow for multidimensional private information. We assume that θ is distributed

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[5] We use the term 'signal' both in the mechanism design and in the signalling settings below. The term 'signal' is appropriate in the mechanism design and screening parts of our paper because it is used to refer to certain agent's actions or messages which possess only informational value that has to be inferred by the principal.

according to probability distribution function $F(.)$ which possesses a continuous density $f(.)$ and which is known to the principal.

When the principal selects an allocation $x \in X$ and the agent's type is given by θ, the agent obtains utility $u(x,\theta)$ and the principal receives utility $w(x,\theta)$. We assume that both $u(x,\theta)$ and $w(x,\theta)$ are continuous in x, and that there exists an outcome $\underline{x}$ which gives each type of the agent a utility that does not exceed her utility from the outside option.

Further, the principal may request the agent to send up to n messages or signals $(m_1,...,m_n)$. For example, the principal may query the agent's ability by asking her to undergo a number of tests, interviews, or inspections. This assumption reflects that each message is characterized by some specific content, or is sent along a different dimension. We assume that the i-th message m_i belongs to a compact subset $M_i \subseteq \mathbb{R}^l$. Note that we require M_i and Θ to be of the same dimension, so that each message can contain information about all of the aspects or dimensions of the agent's type.[6] Let $\mathbf{m}^n = (m_1,...,m_n)$ denote the vector of messages or signals sent by the agent, and let $M^n = \prod_{i=1}^n M_i$. Apart from the costly messages $\mathbf{m}^n$, we also allow the agent to send a cheap talk message τ. By the Revelation Principle, we can without loss of generality take the latter to be a type announcement, so that $\tau \in \Theta$ and the agent's message space is equal to $\Theta \times \prod_{i=1}^n M_i \equiv \Theta \times M^n$. In the sequel, we rely on mechanisms that do not use cheap talk messages. However, it is important to show that our results are robust to the addition of such messages and that the latter do not affect the scope of implementation.

An agent of type θ incurs the cost $C^n(\mathbf{m}^n,\theta)$ when she sends the vector of messages $\mathbf{m}^n$. The cost function $C^n(.) : M^n \times \Theta \mapsto R_+$ has the following properties. First, there exists a mapping $\gamma_i(.) : \Theta \mapsto M_i$ s.t. message m_i is costless for type θ if and only if $m_i = \gamma_i(\theta)$. Thus we have $\gamma_i(\theta) = \arg\min_{m_i} C^n(m_1,...,m_i,...,m_n,\theta)$ for all $(m_1,..,m_{i-1},m_{i+1},...,m_n)$, and $C^n(m_1,...,m_n,\theta) = 0$ if and only if $m_i = \gamma_i(\theta)$ for all $i = 1,...,n$. This implies that $C^n(m_1,...,m_n,\theta) > 0$ whenever there exists $i \in \{1,...,n\}$ s.t. $m_i \neq \gamma_i(\theta)$.

The function $C^n(.)$ embodies the connection between the agent's payoff-relevant type θ and her abilities to manipulate information. Note that the message $\gamma_i(\theta)$ can be regarded as "truthful" for the agent of type θ, because this is the message that agent-type would prefer to send if she did not care about affecting the principal's action. In contrast, sending any message $m_i \neq \gamma_i(\theta)$ involves costly type misrepresentation, or costly distortion.

The agent's cost of misrepresenting her type, i.e. sending message $m_i \neq \gamma_i(\theta)$, generally depends on the other communicated messages. Such dependencies arise naturally. For example, as the agent proceeds with sending more messages to the principal, she may learn how to misrepresent her information more effectively and at a lower cost. Alternatively, additional effort spent on one test could be fatiguing the agent, and therefore raise the cost of undergoing a subsequent test.

So, the only additional requirement that we will introduce below and which will be needed for our asymptotic results to hold is that the cost of sending a message $m_i \neq \gamma_i(\theta)$ does not go to zero too quickly as the number of messages increases.

To summarize, when the agent with preference parameter θ sends an array of messages

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[6] This involves little loss of generality, as we allow the agent to send multiple messages. For example, if each message could only reflect one dimension of the agent's type, we could always bundle them in groups of l messages.

$(m_1, ..., m_n)$ and the principal selects an allocation x, the agent's overall payoff is given by:

$$u(x, \theta) - C^n(m_1, ..., m_n, \theta) \tag{1}$$

A mechanism that the principal offers to the agent can be represented by a mapping $g(.)$ from the agent's message space $\Theta \times M^n$ into the space of allocations X. We will say that an allocation profile $x(\cdot)$ is implementable if there is a mechanism $g(\cdot) : \Theta \times M^n \mapsto X$, and an agent's message/signal rule (strategy) $\mu^n : \Theta \to M^n$ and agent's cheap-talk rule (strategy) $\tau : \Theta \mapsto \Theta$ s.t. for all $\theta \in \Theta$ we have $x(\theta) = g(\tau(\theta), \mu^n(\theta))$ and such that:

$$u(x(\theta), \theta) - C^n(\mu^n(\theta), \theta) \geq \max_{\theta' \in \Theta, \mathbf{m}^n \in M^n} u(g(\theta', \mathbf{m}^n), \theta) - C^n(\mathbf{m}^n, \theta) \tag{2}$$

When the incentive constraints (2) hold, it is optimal for the agent of any type $\theta \in \Theta$ to accept the contract $g(.)$ and send an array of messages $(\tau(\theta), \mu^n(\theta))$, resulting in the allocation $x(\theta)$. In the next section we study how the set of implementable allocations varies with the dimension of the signal space n.

Since the primary focus of section 3 is implementability, we do not impose individual rationality in it. However, we point out below how certain results of this section can be interpreted as full surplus extraction by the principal leaving the agent at her outside option level.

3 The Dimension of the Signal Space

3.1 Monotonicity

We start by presenting two results which illustrate the central message of this paper that the availability of multiple costly messages, or signals, has powerful implications for implementation. Thus, our focus throughout this section is on the set of implementable allocation rules, $x(.)$, and on the expected message cost $\kappa^n(x) = \int_\Theta C^n(\mu^n(\theta), \theta) f(\theta) d\theta$ where $\mu^n(.)$ is the message strategy used to implement $x(.)$. Cheap talk messages will not play a role in our mechanisms as every type will be identified by the array of costly messages m^n which it sends.

We make the following additional assumptions linking the cost functions $C^n(.)$ and $C^{n+1}(.)$ for environments with n and $n+1$ messages, respectively:

$$C^{n+1}(m_1, ..., m_n, m_{n+1}, \theta) \geq C^n(m_1, ..., m_n, \theta), \tag{3}$$

$$C^{n+1}(m_1, ..., m_n, \gamma_{n+1}(\theta), \theta) = C^n(m_1, ..., m_n, \theta) \text{ for all } (m_1, ..., m_{n+1}) \in M_1 \times ... \times M_{n+1}, \ \theta \in \Theta$$

Thus sending the extra message $\gamma_{n+1}(\theta)$ is costless for type θ, but sending any other extra message is costly.

With these assumptions, enlarging the dimension of the signal space increases the set of implementable allocation profiles, without increasing communication costs:

Lemma 1 *(Monotonicity) Suppose that under signal space M^n the decision rule $x : \Theta \to X$ is implementable via message rule $\mu^n(\theta)$, at an expected signal cost $\kappa^n(x) = \int_\Theta C^n(\mu^n(\theta), \theta) f(\theta) d\theta$. Then, under signal space M^{n+1}, x is implementable at an expected signal cost $\kappa^{n+1}(x) \leq \kappa^n(x)$.*

The proof of Lemma 1 is straightforward: the principal can always commit to ignore the $(n+1)$-st signal. In this case, the cost of implementing the decision rule x would not change, as each agent-type would simply send a costless $(n+1)$-st message. Then in the optimal mechanism the principal can do at least as well as this.

Importantly, Lemma 1 leaves open the possibility that the set of implementable allocations is invariant to n. The following example shows that this can indeed happen.

Example 1 *Suppose that θ is a scalar, and that $C^n(m_1, ..., m_n, \theta) = \sum_{i=1}^n c(m_i, \theta)$, where $c(m_i, \theta) = m_i/\theta$. Then since $\sum_{i=1}^n c(m_i, \theta) = c(\sum_{i=1}^n m_i, \theta)$, any allocation that can be implemented with $n > 1$ can also be implemented with $n = 1$, by selecting $m_1(\theta) = \sum_{i=1}^n m_i^n(\theta)$.*

What goes wrong in the above example is not the linearity of the cost function C^n, but rather that the costless message is the same across agent types, i.e. $\gamma_i(\theta) \equiv 0$.[7] Additional messages would have an effect in this example if the agents also derived some type-dependent utility from a message m_i, with higher types deriving more utility from higher messages, so that $\gamma_i(\theta)$ would be strictly increasing in θ. For example, in the context of educational signalling, higher ability types presumably enjoy studying more than low ability types. In line with this motivation, we introduce two assumptions:

Assumption 1 *For each n there exists $\delta_n > 0$ s.t. $C^n(\mathbf{m}^{n-1}, m_n, \theta) - C^n(\mathbf{m}^{n-1}, \gamma_n(\theta), \theta) \geq \delta_n ||m_n - \gamma_n(\theta)||$ for all $\mathbf{m}^{n-1} \in \prod_{j=1,...,n-1} M_j$, $m_n \in M_n$, $\theta \in \Theta$.*

Assumption 2 *There exists $r > 0$ s.t. $||\gamma_i(\theta') - \gamma_i(\theta)|| \geq r||\theta' - \theta||$, for all θ', $\theta \in \Theta$, and all i.*

Assumption 1 ensures that the signal cost of agent-type θ gets larger as $\mathbf{m}^n$ moves further away from her costless message collection $\gamma^n(\theta) = (\gamma_1(\theta), ... \gamma_n(\theta))$. It also implies that the marginal cost of misrepresentation is bounded away from zero. Assumption 2 says that costless messages are sufficiently sensitive to the type.

Let Ω be the set of all decision rules $x : \Theta \to X$, and let $E^n = \{x(\cdot) \in \Omega : x(\cdot)$ is implementable with signal space $M^n\}$. E^n is non-empty for any n because a constant allocation is always implementable. We can now state our result asserting that, under these mild assumptions, increasing the dimension of the signal space strictly increases the set of implementable allocations:

Theorem 1 *Suppose that Assumptions 1 and 2 hold. If $E^n \neq \Omega$ then $E^n \subsetneq E^{n+1}$.*

3.2   Asymptotics

Theorem 1 raises the interesting question of how the set of implementable allocations and the associated message costs change as the dimension of the signal space grows large. We establish two kinds of results in this case. First, when the marginal cost of small misrepresentation is non-zero and does not vanish too quickly, we show that the principal can still elicit the

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[7] Note that the same conclusion would obtain if $c(m, \theta)$ were concave in m. If $c(m, \theta)$ were convex, then signalling costs can be reduced by having multiple tests.

agent's information at zero cost. On the other hand, when the marginal cost of small type misrepresentations is zero, implementation becomes harder. Nevertheless, when the marginal cost of misrepresentation increases in the magnitude of misrepresentation at a rate that does not go to zero too fast as the number of messages becomes large, the principal can still implement almost all allocation profiles, although the agent has to incur a small communication cost.

Assumption 3 *There exists $L < \infty$ s.t. $\|x(\theta') - x(\theta)\| \leq L\|\theta' - \theta\|$, for all $\theta', \theta \in \Theta$.*

Assumption 4 *There exists $K < \infty$ s.t. $|u(x', \theta) - u(x, \theta)| \leq K\|x' - x\|$, for all $\theta \in \Theta$.*

Assumption 3 requires the allocation to be Lipschitz continuous with Lipschitz constant L. Assumption 4 strengthens the continuity requirement on the agent's utility function to uniform continuity in x. Since $X \times \Theta$ is compact, Assumption 4 holds if $u(\cdot, \cdot)$ is C^1.

Let $\Omega_L = \{x(.) : \Theta \mapsto X : x(.)$ satisfies Assumption 3 with constant $L\}$. Our next theorem shows that all allocation profiles in Ω_L are implementable at zero cost when the marginal cost of misrepresenting the type is positive and does not go to zero too fast in the number of messages.

Theorem 2 *Suppose that Assumptions 2-4 hold, and that there exists $\alpha \in [0, 1)$ and $\underline{c} > 0$ s.t. $C^n(m_1, ..., m_i, ..., m_n, \theta) - C^n(m_1, ..., \gamma_i(\theta), ..., m_n, \theta) \geq \frac{\underline{c}}{n^\alpha}\|m_i - \gamma_i(\theta)\|$, for all $\theta, \theta' \in \Theta$ and all $i = 1, .., n$.*

Then there exists $N < \infty$ such that $\Omega_L \subset E^n$ for all $n \geq N$, and such that any $x \in \Omega_L$ is implementable with zero communication costs.

Intuitively, local incentive constraints hold because the marginal benefit of sending a message $m_i \neq \gamma_i(\theta)$ is finite for type θ, while the marginal cost of doing so is bounded from below by $\frac{\underline{c}}{n^\alpha}$ when m_i is near $\gamma_i(\theta)$. Therefore, when the number of messages is sufficiently large, the cost of a misrepresentation exceeds the benefit of a better allocation. This ensures that no type is willing to send non-truthful messages, and so no communication costs are incurred.

It should be noted that, because C^n is minimized w.r.t. m_i at $m_i = \gamma_i(\theta)$ and because M^n and Θ are compact, the condition on $C^n(.)$ imposed in Theorem 2 need only hold locally around $\gamma_i(\theta)$. In other words, there only need to exist some 'start-up' costs of misrepresentation.

The reasoning in the previous two paragraphs suggests that the set of implementable allocation profiles may be more restricted when the marginal cost of type misrepresentation is zero at the 'truthful' message $\gamma_i(\theta)$. Indeed, in this case implementation will generally require that the agent send some "non-truthful" messages and incur some misrepresentation costs. Nevertheless, we will demonstrate that by carefully constructing the communication stage of the mechanism, the principal can elicit the agent's private information at a negligible cost.

For this result, we specialize our model to one with transferable utility. Thus, we partition the outcome $x = (q, t)$ into a production assignment $q \in Q$, where Q is a compact subset of $\mathbb{R}^{k-1}$, and a transfer $t \in \mathbb{R}$ from the agent to the principal. The principal's and the agent's utility functions are quasilinear:

$$w(x, \theta) = v(q, \theta) - t$$
$$u(x, \theta) = t - h(q, \theta).$$

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We assume that $h(q, \theta)$, $v(q, \theta)$ and $C^n(\mathbf{m}^n, \theta)$ are twice continuously differentiable functions, and $\gamma^n(\theta)$ is a twice continuously differentiable vector function.

For any function $g(\rho, \eta) : \mathbb{R}^j \times \mathbb{R}^k \to \mathbb{R}$, where $j, k > 1$, we let $D^2_{\rho\rho} g(\rho, \eta)$ denote the $j \times j$ matrix of second derivatives of g w.r.t. the vector ρ. Similarly, $D^2_{\rho\eta} g(\rho, \eta)$ is $j \times k$ matrix of cross partial derivatives $\frac{\partial^2 g}{\partial \rho_i \partial \eta_l}$. Also, for any function $d(\eta) : \mathbb{R}^k \to \mathbb{R}^k$, $D_\eta d(\eta)$ denotes a $k \times k$ matrix of first derivatives of $d(\eta)$. We may now state the following Assumption:

Assumption 5 *There exist $\alpha \in [0, 1)$ and $\underline{\omega}, \bar{\omega} > 0$ such that for all $(\mathbf{m}^n, \theta, \theta')$ and all $i = 1, ..., n$:*
(i) $|D_\theta C^n(\mathbf{m}^n, \theta) - D_\theta C^n(\gamma^n(\theta), \theta)|| \geq \frac{\omega}{n^\alpha} ||m_i - \gamma_i(\theta)||$;
(ii) $||D_{m_i} C^n(\mathbf{m}^n, \theta) - D_{m_i} C^n(\gamma^n(\theta), \theta)|| \leq \frac{\bar{\omega}}{n^\alpha} ||m_i - \gamma_i(\theta)||$;
(iii) $D^2_{m_i\theta} C^n(\mathbf{m}^n, \theta) D_\theta \gamma_i(\theta)$ is negative definite, and $\mathbf{z}^\tau D^2_{m_i\theta} C^n(\mathbf{m}^n, \theta) D_\theta \gamma_i(\theta) \mathbf{z} \leq -\frac{\omega}{n^\alpha} ||\mathbf{z}||^2$ for all n.

To interpret Assumption 5, consider the case $l = 1$. Part (i) then says that the cross-partial $\frac{\partial^2 C^n}{\partial \theta \partial m_i}$ is bounded from below, and does not converge to zero too quickly as n grows large. Part (ii) says that the second derivative $\frac{\partial^2 C^n}{\partial m_i^2}$ is bounded from above. Part (iii) adds the single crossing requirement that $\frac{\partial^2 C^n}{\partial \theta \partial m_i}$ and $\gamma_i'(\theta)$ have opposite signs.

Theorem 3 *Suppose Assumptions 2 and 5 hold. Then for every pair of twice continuously differentiable functions (q, t), and every $\varepsilon > 0$, there exist $N < \infty$, and for each $n \geq N$ a transfer function t^n with $|t^n(\theta) - t(\theta)| < \varepsilon$, such that (q, t^n) is implementable whenever the dimension of the signal space n exceeds N. Furthermore, the associated communication cost is less than ε for every type of agent.*

Note that we have not imposed the agent's individual rationality on our implementation problem. However, an immediate implication of Theorem 3 is that the principal can extract the surplus from the agent, leaving the latter at her outside option value, and implement the first-best quantity allocation q^{FB} with negligible communication costs.

An important part of the mechanism design in Theorem 3 is the construction of the message profile $\mathbf{m}^n(\theta)$. The first-order condition for local incentive compatibility is:

$$D_\theta t^n(\theta) = D_q h(q(\theta), \theta) D_\theta q(\theta) + D_{\mathbf{m}^n} C^n(\mathbf{m}^n(\theta), \theta) D_\theta \mathbf{m}^n(\theta) \qquad (4)$$

If the agent was to send only costless messages, then the second term in (4) would be zero, as $D_{m_i} C^n(\gamma^n(\theta), \theta) = 0$. Thus in the absence of costly messages the first-order condition imposes a restriction on the set of implementable allocations in the form of a link between $q(\theta)$ and $t^n(\theta)$. Costly signals weaken and eventually eliminate the need for such link, and this allows to implement a larger set of allocation profiles.

In particular, when the number of messages n is small, the degree to which the link between $q(\theta)$ and $t^n(\theta)$ is weakened is limited by the magnitude of the required misrepresentation $\mathbf{m}^n(\theta)$ and by the associated communication costs, which may have to be fairly large. In contrast,

with sufficiently large n, the codependency between $q(\theta)$ and $t(\theta)$ is eliminated at a very small cost. In fact, by choosing n and $\mathbf{m}^n(\theta)$ appropriately, we can ensure that (4) holds for arbitrary $(q(\theta), t(\theta))$, and the agent's communication cost $C^n(\mathbf{m}^n(\theta), \theta)$ is less than an arbitrary $\varepsilon > 0$.

Finally, consider the second-order conditions for implementation. Maggi and Rodriguez-Clare (1995), who characterize the optimal mechanism for $n = l = 1$, imposed the following restrictions: $q'(\theta) \geq 0$ and $m'(\theta) \geq 0$ to guarantee that their second-order conditions hold. In contrast, a careful inspection of our proof reveals that, with many n, the second-order conditions hold because the agent sends a large number of messages which are close to her costless message $\gamma(\theta)$. Consequently, we are able to implement nearly all allocation profiles at a small communication cost.

4 Optimal Mechanisms

The previous two theorems provide conditions under which almost all decision rules become implementable and the communication cost converges to zero as the number of messages becomes large. However, it is also interesting to consider what happen when the number of available messages remains limited, either exogenously or endogenously.

In this section, we characterize the optimal mechanism for one-dimensional type space, i.e. $l = 1$, and every possible number of messages n, including small n. We continue to assume that an allocation x consists of a monetary part t and non-monetary part q. For simplicity, we assume that $Q = \mathbb{R}_+$.[8] The principal's payoff is given by $v(q, \theta) - t$ and the agent's payoff is given by $t - h(q, \theta)$. We impose individual rationality, and normalize the agent's reservation utility to equal 0.

We make the standard assumptions that $v_q > 0$, $h_q > 0$, $v_\theta > 0$, $h_\theta \leq 0$, $v_{q\theta} > 0$, $h_{q\theta} \leq 0$, $v_{\theta\theta} \leq 0$, and $v_{qq} - h_{qq} < 0$, for all $q \in Q$ and $\theta \in \Theta$. In addition, we assume that $v_q(0, \theta) - h_q(0, \theta) \geq 0$ for all $\theta > \underline{\theta}$ and there exists $\bar{q} < \infty$ s.t. $v_q(q, \theta) - h_q(q, \theta) < 0$ for all $q > \bar{q}$ and $\theta > \underline{\theta}$. These two assumptions guarantee that the solution $q = q^{FB}(\theta)$ to

$$\max_{q \geq 0}\{v(q, \theta) - h(q, \theta)\}$$

exists and satisfies $q^{FB}(\theta) > 0$ for all $\theta > \underline{\theta}$. We also require that $v(q, \underline{\theta}) = 0$ for all q and $v(0, \theta) = 0$ for all θ, as well as $v_q(q, \theta) - h_q(q, \theta) > 0$, so that $q^{FB}(\theta)$ is increasing in θ.

Finally, we make the following technical assumptions:

Assumption 6 *(i)* $v_q(q, \theta) - h_q(q, \theta) - \frac{1-F(\theta)}{f(\theta)} v_{q\theta}(q, \theta)$ *is increasing in θ;*

(ii) $\frac{\partial C^n}{\partial m_i}(\mathbf{m}^n, \theta) - \frac{1-F(\theta)}{f(\theta)} \frac{\partial^2 C^n}{\partial m_i \partial \theta}(\mathbf{m}^n, \theta)$ *is decreasing in θ;*

(iii) $C^n(\mathbf{m}^n, \theta)$ *is convex in $\mathbf{m}^n$ and is concave in θ, and $\frac{\partial^2 C^n}{\partial m_i \partial \theta}(\mathbf{m}^n, \theta) < 0$.*

Parts (i) and (ii) of Assumption 6 require the cross-partial derivatives of the agent's virtual utility and virtual communication cost to be, respectively, positive and negative. Part (iii) requires the cost function to be convex in $\mathbf{m}^n$ and concave in θ, and imposes a single crossing condition.

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[8] Our qualitative results extend straightforwardly to the case where Q is multidimensional.

We now proceed to derive the optimal mechanism. Specifically, the principal selects a "quantity" $q(\cdot)$, a transfer $t(.)$ and a vector of messages $\mathbf{m}^n(\cdot)$ to solve:

$$\max_{q(\theta),t(\theta),\mathbf{m}^n(\theta)} \int (h(q,\theta) - t(\theta))f(\theta)d\theta$$

subject to incentive constraints:

$$t(\theta) - h(q(\theta),\theta) - C^n(\mathbf{m}^n(\theta),\theta) \geq t(\theta') - h(q(\theta'),\theta) - C^n(\mathbf{m}^n(\theta'),\theta), \text{ for all } \theta \text{ and } \theta' \quad (5)$$

and the individual rationality constraint:

$$U(\theta) \equiv t(\theta) - h(q(\theta),\theta) - C^n(\mathbf{m}^n(\theta),\theta) \geq 0. \quad (6)$$

Let us first consider the benchmark case with no costly messages, i.e. $n = 0$. It is well-known that under Assumption 6 (i) the solution to the principal's problem involves selecting an allocation $q^{SB}(\theta)$ that maximizes 'virtual' welfare

$$\Gamma(q,\theta) \equiv v(q,\theta) - h(q,\theta) - \frac{1 - F(\theta)}{f(\theta)}v_\theta(q,\theta).$$

Let $\theta^* \in (\underline{\theta},\bar{\theta})$ be the unique solution to $\Gamma_q(0,\theta) = 0$. Then under Assumption 6 (i), the optimal quantity when $n = 0$ is equal to zero for all $\theta \in [0,\theta^*]$ and is a unique solution to $\Gamma_q(q,\theta) = 0$ for all $\theta \in [\theta^*,1]$. Thus there is exclusion in the second-best solution: all types in the interval $[0,\theta^*]$ receive zero quantity.

Next, let us consider the problem for $n > 0$. Solving for $t(\theta)$ from (6) and substituting into the objective, and replacing the incentive constraints (5) by the envelope condition associated with the agent's utility maximization, yields the following "relaxed" problem:

$$\max_{q(\theta),\mathbf{m}^n(\theta),U(\theta)} \int \{v(q(\theta),\theta) - h(q(\theta),\theta) - C^n(\mathbf{m}^n(\theta),\theta) - U(\theta)\} f(\theta)d\theta \quad (7)$$

subject to individual rationality constraint (6) and

$$U'(\theta) = v_\theta(q(\theta),\theta) - C_\theta^n(\mathbf{m}^n(\theta),\theta) \quad (8)$$

We will verify that the solution to the relaxed problem (7) subject to (6) and (8) satisfies (5) and hence also solves the unrelaxed problem. To solve the relaxed problem, define the Hamiltonian

$$H = \{v(q,\theta) - h(q,\theta) - C^n(\mathbf{m}^n,\theta) - U\}f(\theta) + \sigma\left(v_\theta(q,\theta) - C_\theta^n(\mathbf{m}^n,\theta)\right) + \rho U \quad (9)$$

Maximizing (9) w.r.t. q and $\mathbf{m}^n$ yields the first order conditions:

$$\{v_q(q,\theta) - h_q(q,\theta)\}f(\theta) + \sigma v_{q\theta}(q,\theta) \leq 0 \ (= 0, \text{ if } q > 0) \quad (10)$$

$$\frac{\partial C^n}{\partial m_i}(\mathbf{m}^n,\theta)f(\theta) + \sigma\frac{\partial^2 C^n}{\partial m_i\partial\theta}(\mathbf{m}^n,\theta) = 0 \quad (11)$$

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The costate equation is

$$\sigma'(\theta) = f(\theta) - \rho(\theta), \tag{12}$$

Furthermore, the solution has to satisfy complementary slackness conditions

$$\rho(\theta)U(\theta) = 0, \ \rho(\theta) \geq 0, \text{ and } U(\theta) \geq 0, \tag{13}$$

Also, the following transversality conditions have to hold: $\sigma(\underline{\theta})U(\underline{\theta}) = 0$, $\sigma(\bar{\theta})U(\bar{\theta}) = 0$, $\sigma(\underline{\theta}) \leq 0$ and $\sigma(\bar{\theta}) \geq 0$

To describe the solution to the relaxed problem, let $\{\widehat{q}(\theta), \widehat{\mathbf{m}}^n(\theta), \widehat{\sigma}(\theta)\}$ be the solution to the system consisting of (10) set as equality, (11) and $v_\theta(q, \theta) - C_\theta^n(\mathbf{m}^n, \theta) = 0$ (i.e. $U' = 0$). Note that over any interval of θ on which the individual rationality constraint (6) is binding, we have $U'(\theta) = 0$, and so the solution to the relaxed problem is given by $(\widehat{q}(\theta), \widehat{\mathbf{m}}^n(\theta))$.

Next, let $\{\widetilde{q}(\theta), \widetilde{\mathbf{m}}^n(\theta)\}$ be the solution to the relaxed problem in which the individual rationality constraint is not imposed, or simply put $\rho = 0$, on the interval $[\theta, \bar{\theta}]$. The costate equation and the transversality condition at $\theta = \bar{\theta}$ then require $\sigma(\theta) = -(1 - F(\theta))$. Thus, $\widetilde{q}(\theta)$ and $\widetilde{\mathbf{m}}^n(\theta)$ solve (10) set as equality, and (11) with $\sigma(\theta) = -(1 - F(\theta))$.

Our next Theorem shows that the solution to the full (unrelaxed) problem has the following properties. On a lower interval of types the individual rationality constraint (6) is binding and the solution is given by $(\widehat{q}(\theta), \widehat{\mathbf{m}}^n(\theta))$. On a higher interval of types, the individual rationality constraint is not binding, $U(\theta)$ is strictly increasing, and the solution is given by $(\widetilde{q}(\theta), \widetilde{\mathbf{m}}^n(\theta))$.

Theorem 4 *Suppose Assumption (6) holds. Then there exists $\widehat{\theta} \in (\underline{\theta}, \bar{\theta})$ such that the optimal mechanism is given by:*

$$(q(\theta), \mathbf{m}^n(\theta)) = \begin{cases} (\widehat{q}(\theta), \widehat{\mathbf{m}}^n(\theta)) & \text{if } \theta \in [\underline{\theta}, \widehat{\theta}], \\ (\widetilde{q}(\theta), \widetilde{\mathbf{m}}^n(\theta)) & \text{if } \theta \in [\widehat{\theta}, \bar{\theta}]. \end{cases} \tag{14}$$

The agent's utility $U(\cdot)$ satisfies: $U(\theta) = 0$ for all $\theta \in [\underline{\theta}, \widehat{\theta}]$, and $U'(\theta) > 0$ for all $\theta \in (\widehat{\theta}, \bar{\theta}]$.

Theorem 4 is related to Proposition 1 in Maggi and Rodriguez-Clare (1995) which characterizes the optimal mechanism with a single costly message. As in their model, a non-trivial set of types $[\underline{\theta}, \widehat{\theta}]$ are held at the reservation utility level in our optimal mechanism. So this property is robust to the number of messages. However, there are important differences between our results. First, in our model the agent sends multiple signals, and we focus on exploring how the number of signals affects the optimal mechanism (see Theorem 6). Second, we prove that the optimal mechanism exhibits no exclusion when costly signals are available (Theorem 5). This property is important, because it underscores that the optimality of exclusion in the standard case without costly signals is rather fragile. Third, our strategy of proof is different from Maggi and Rodriguez-Clare (1995), and as a consequence we are able to establish our Theorem 4 and Theorem 5 and 6 relying on it under more general conditions.

In particular, unlike Maggi and Rodriguez-Clare (1995), we do not assume that the cost of a signal m depends only on the difference $(m - \theta)$ and do not require the Hamiltonian to be concave and the hazard rate to be monotone.

Theorem 5 *In the optimal mechanism, $q(\theta) > 0$ for all $\theta > \underline{\theta}$.*

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Thus, according to Theorem 5, no type who can generate a positive surplus is excluded in the optimal mechanism as long as signal costs are positive, no matter how small.

The intuition for the absence of exclusion with costly signals is as follows. Any agent-type that generates a positive surplus in the first-best is potentially profitable to the principal. The reason some agent types are excluded in the second-best is that if the principal were to give them a positive quantity, he would also have to raise the surplus of all agents with higher valuations. But when signal costs are positive, this is no longer necessary: the principal can prevent imitation by requiring agent-types that now receive a positive quantity to send costly signals. By single crossing, higher agent types will incur lower signal costs, eliminating their incentive to imitate.

Finally, we characterize the nature of the solution as the number of costly messages, n, increases.

Theorem 6 *Suppose that $C^n(m_1, ..., m_n, \theta) = \sum_{i=1}^{n} c_i(m_i, \theta)$ and there exist $\underline{v}, \bar{v} \in (0, \infty)$ such that $\underline{v} \leq \frac{\partial^2 c}{\partial m_i^2} \leq \bar{v}$ and $\underline{v} \leq |\frac{\partial^2 c}{\partial \theta \partial m_i}| \leq \bar{v}$. Then, as $n \to \infty$, in the optimal mechanism $\widehat{\theta}(n) \to \bar{\theta}$, $\widehat{q}(\theta) \to q^{FB}(\theta)$, $\widehat{\mathbf{m}}^n(\theta) \to \gamma^n(\theta)$, and $C^n(\widehat{\mathbf{m}}^n(\theta), \theta) \to 0$.*

Theorem 6 implies that, as the number of messages gets large, the quantity allocation in the optimal mechanism converges to the first-best one while the communication/signalling cost becomes negligibly small. Thus, the principal extracts almost all surplus. The smallness of communication costs should be contrasted with the results in the literature that demonstrate significant inefficiency associated with signalling, in particular, Spence (1973).

One factor that could affect these results in the presence of additional fixed cost of each message. In the next section, we derive the optimal number of messages which the agent should be required to send in the presence of such costs.

5   Endogenous Signal Space

In this section, we study the optimal number of messages in a mechanism, when the principal incurs a fixed cost G to elicit each message. For example, the principal may have to incur such cost for developing and administering a test, and/or processing the information received form the agent.

To simplify matters, we will treat n as a continuous variable. Let $W(n)$ denote the principal's expected surplus, gross of any fixed costs, in the optimal mechanism when the dimension of the signal space is n. The principal then selects n to maximize $W(n) - nG$. We assume that the agent's signalling costs are additively separable across messages, i.e.

$$C^n(m_1, m..., m_n, \theta) = \sum_{i=1}^{n} c(m_i, \theta),$$

and that

$$\frac{c_{m\theta}}{c_m}(m, \theta) \text{ is increasing in } m.$$

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The latter assumption ensures that the solution to (11) is unique, and hence independent of i. Henceforth, we shall therefore omit the subscript of the message m_i. The optimal number of messages n^* is characterized in the following Lemma.

Lemma 2 *Suppose that $c_{m\theta}(m,\theta)^2 - c_{mm\theta}(m,\theta)c_m(m,\theta) > 0$ for all (m,θ). Then $W(n)$ is a strictly concave function, and the marginal benefit of an additional message is given by*

$$\frac{dW(n)}{dn} = \int_{\underline{\theta}}^{\bar{\theta}} \left(\frac{c_\theta(m(\theta),\theta)c_m(m(\theta),\theta)}{c_{m\theta}(m(\theta),\theta)} - c(m(\theta),\theta) \right) f(\theta)d\theta \qquad (15)$$

To help us interpret the solution, note that in the optimal mechanism $\frac{dW(n)}{dn} = G$. Further, let us define:

$$\underline{K} = \min_{(m,\theta)} \frac{c_\theta(m,\theta)c_m(m,\theta)}{c_{m\theta}(m,\theta)c(m,\theta)} - 1, \qquad \overline{K} = \max_{(m,\theta)} \frac{c_\theta(m,\theta)c_m(m,\theta)}{c_{m\theta}(m,\theta)c(m,\theta)} - 1$$

Then we have

Lemma 3 *Suppose that $\overline{K} > 0$. Then, in the optimal mechanism,*

$$\underline{K} \int_{\underline{\theta}}^{\bar{\theta}} c(m(\theta),\theta)f(\theta)d\theta \leq G \leq \overline{K} \int_{\underline{\theta}}^{\bar{\theta}} c(m(\theta),\theta)f(\theta)d\theta$$

The significance of Lemma 3 lies in characterizing the relationship between the principal's total cost of setting up the signalling system, nG, and the agent's communication costs in the optimal mechanism $C^n(m,\theta)$. As the Lemma shows, the ratio of the former to the latter is between $\underline{K}$ and $\overline{K}$. For example, if $c(.)$ is quadratic i.e., $c(m,\theta) = (m-\theta)^2$, then $\underline{K} = \overline{K} = 1$. This implies that in the optimal mechanism the principal's fixed cost of setting up the messages nF and the agent's communication cost $C(m,\theta)$ will be equal.

To illustrate this result, consider the example in which welfare and signalling costs are quadratic, and the type distribution is uniform. Specifically, suppose that $v(q,\theta) = \theta q$, $h(q,\theta) = \frac{1}{2}q^2$, $c(m,\theta) = (m-\theta)^2$, and $F(\theta) = \theta$ for $\theta \in [0,1]$. Then for fixed n the solution to the principal's problem is given by:

$$\begin{aligned} \widetilde{q}(\theta) &= \widetilde{m}(\theta) = 2\theta - 1 \\ \widehat{q}(\theta) &= \widehat{m}(\theta) = \frac{2n}{2n+1}\theta \\ \widehat{\theta}(n) &= \frac{2n+1}{2(n+1)} \end{aligned}$$

Furthermore, the principal's marginal benefit of an additional message is $\frac{dW(n)}{dn} = \frac{1}{12(n+1)^2}$, and so the optimal number of signals equals $n(G) = \sqrt{1/(12G)}$.

Recall that every message generates some amount of additional welfare, because the allocation profile gets closer to the first-best when the agent has to send more messages (see Theorem 6). To illustrate the relation between the fixed cost G and the welfare generated

by an extra message, let us express the fixed cost as a fraction of the potential surplus gain $\Delta W = W^{FB} - W^{SB}$, where W^{FB} (W^{SB}) is the total welfare under the first-best (second-best) quantity allocation. Since $W^{FB} = \frac{1}{6}$ and $W^{SB} = \frac{1}{8}$, we have

n	1	2	3	4	5	6	7	8	9	10
$G/\Delta W$	50%	22.2%	12.6%	8%	5.6%	4%	3.2%	3.4%	2%	1.6%

Thus in this example the principal will elicit at least four messages if the fixed cost of eliciting an extra message does not exceed 8% of the potential welfare gain. Note that in the process, the agent will incur expected message costs of at least 32% of the potential welfare gain, thereby dissipating a substantial portion of the benefit. It should also be noted that with four signals, the allocation q is already close to the first-best, as $\widehat{q}(\theta)$ is within 11% of $q^{FB}(\theta)$.

6   Signalling and Education

In this section we apply our approach with multiple signals to reexamine the extant results on signalling via education in the job-market, and in particular the seminal contribution of Spence (1973). To this end, we recast the model of section 4 in signalling terms. In particular, consider the agent as the worker of productivity type $\theta \in [\underline{\theta}, \overline{\theta}]$. The function $h(q, \theta)$ then stands for the type-θ worker's cost of producing q units of output (equivalently, producing the good of quality q). Similarly, $v(q, \theta)$ stands for the principal's/employer's benefit from q units of output produced by the employer of type θ. The employer pays transfer (wage) t to the worker.

As in Spence (1973), we assume that the employment market is competitive, so that the worker gets the full expected surplus $E_b(\max_q v(q, \theta) - h(q, \theta))$, where b describes the employer's beliefs about the worker's type θ. The worker obtains education before entering the job-market. Education does not affect her productivity. However, it provides an observable signal to the job-market with type-specific cost associated with it, a communication cost in our terminology.

The observable characteristic of education in our model is a profile of grades $(m_1, ..., m_n)$ attained by a worker in the courses she took, with m_i standing for the grade in course i. One possibility is that the employer conditions her offer on $\frac{\sum_{i=1,...,n} m_i}{n}$, the GPA ("grade point average"), and the number n of the courses taken. In Spence (1973), the observable signal is the duration of education. Thus, we employ a slightly different approach which is arguably more intuitive. Indeed, casual evidence suggests that the potential employers pay significantly more attention to the job-candidate's grades rather than to the years of schooling. In fact, spending too much time to obtain a degree may be seen as a negative signal in the job-market.

The worker's cost of acquiring a profile of grades $m^n = (m_1, ..., m_n)$ is given by $C^n(m^n, \theta) = \sum_{i=1}^{n} c(m_i, \theta)$. Thus, the simplifying assumptions in this section are additive separability of the cost function across courses, and symmetry: the worker's cost function of a acquiring a particular grade m is the same in each course. These assumptions are made for tractability to shorten the presentation, but out results hold under a more general cost specification as well.

The grade cost function $c(m_i, \theta)$ is such that the zero cost level for type θ is $\gamma(\theta)$ which is increasing in θ and satisfies $\gamma(\theta) > 0$ for all $\theta > \underline{\theta}$. This assumption is supported by the

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intuition that a more able type, with a higher θ, has a higher "natural" grade level which she can attain at a minimal effort, or enjoys education to some extent, more so than less able types. To better understand this assumption, it may be useful to consider the cost function $c(m_i, \theta)$ as a "net" cost equal to the difference between the time/effort cost function $e(m_i, \theta)$ and the benefit/enjoyment of education function $b(m_i, \theta)$. The "natural" grade $\gamma(\theta)$ is then the highest grade at which the cost $e(m_i, \theta)$ does not exceed the benefit $b(m_i, \theta)$.

Since the cost function $c(m_i, \theta)$ is the same in all courses, the "zero cost level" grade $\gamma(\theta)$ is also the same across courses. It is conceivable that a worker may have greater aptitude for some disciplines (say, mathematics) compared to others (say, social sciences). So, in a richer model one would have a course- and discipline-specific cost function $c_i(., \theta)$, with $\gamma_i(\theta)$ being the corresponding zero cost grade. For ease of interpretation of the symmetric cost function, one may also think about the worker/student taking courses in one field, this being the case when she attends a professional school, including law, business, medical or an engineering school.

We further assume that $c_m(m, \theta) > 0$ for $m > \gamma_i(\theta)$, $c_{mm}(m, \theta) > 0$, $c_\theta(m, \theta) \leq 0$, $c_{m\theta}(m, \theta) < 0$.

Our goal is to characterize a separating perfect Bayesian equilibrium of this signalling model. For fixed n, such equilibrium consists of the equilibrium grade strategy $(m_1^*(\theta), ..., m_n^*(\theta))$, the employer's beliefs $b(.)$ mapping grade profiles into probability distributions over types, and job offer $(q(b), t(b))$ which the worker gets given the employer's beliefs b.

In a separating equilibrium, the beliefs on the path are $b(m^{*n}(\theta)) = \theta$. For simplicity, the off-the-path beliefs are specified to be $b(m) = \underline{\theta}$ for any off equilibrium grade profile m. The quantity allocation in a separating equilibrium must be first-best i.e., $q^*(\theta) = \arg\max_q v(q, \theta) - h(q, \theta))$, and the competitive markets assumption implies that $t(\theta) = v(q^*(\theta), \theta) - h(q^*(\theta), \theta))$.

Finally, the following incentive constraints hold for every pair of types (θ, θ') in this equilibrium:

$$U^n(\theta) \equiv v(q^*(\theta), \theta) - h(q^*(\theta), \theta)) - \sum_{i=1,...,n} c(m_i^*(\theta), \theta) \geq v(q^*(\theta'), \theta) - h(q^*(\theta'), \theta)) - \sum_{i=1,...,n} c(m_i^*(\theta'), \theta)$$
(16)

The main result of this section is the following Theorem characterizing a separating perfect Bayesian equilibrium with efficient allocation and a small signalling cost.

Theorem 7 *For every $\varepsilon > 0$ there exists $N(\varepsilon) < \infty$ such that with n grades, $n \geq N(\varepsilon)$, there exists a separating Perfect Bayesian equilibrium of our signalling model in which type θ obtains a profile of grades $(m_1^*(\theta), ..., m_n^*(\theta))$ and is hired to produce an output $q^*(\theta)$ in exchange for the payment $(v(q^*(\theta), \theta) - h(q^*(\theta), \theta))$ in the competitive labor market. Her "cost of signalling" $\sum_{i=1,...,n} c(m_i^*(\theta), \theta)$ does not exceed ε for every $\theta \in [\underline{\theta}, \overline{\theta}]$.*

According to Theorem 7, the cost of signalling becomes small as the number of courses n increases. So, each worker-type sends only "almost" costless messages i.e. gets her "natural" grades" in equilibrium i.e., $m_i^*(\theta) \to \gamma_i(\theta)$ for every θ for all i. But the cost to type θ of imitating type θ' approaches $nc(\gamma(\theta'), \theta)$ which becomes large as n grows. This is the main intuition behind the "costless" separation outcome of Theorem 7.

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Note that in our framework education is unproductive as in the traditional Spence's model, and the cost of obtaining a higher grade decreases in the worker's type. But in contrast to the traditional model, the grade that a worker can earn without incurring a cost increases in her type, reflecting greater scholastic aptitude or intellectual curiosity of more able types. The latter feature of our model is both intuitive and crucially important for the small cost of signalling in equilibrium. Thus, we identify an important condition which prevents ineffective dissipation of resources in a "rat race."

Interestingly, the empirical evidence on the magnitude of the signalling cost of education is at best mixed (Page 2010). Several more recent studies, including Chevalier et. al (2004) and Clark and Martorell (2014), indicate that this cost may be quite small. Given the large number of courses taken by modern-day students before graduating from college, our model provides a potential explanation for these findings.

7   Conclusions

This paper demonstrates that in environments with misrepresentation costs, the ability of the principal to offer mechanisms in which an agent sends several messages significantly expands the set of implementable outcomes.

Our results have a number interesting implications for screening and signaling. In particular, they suggests that the problem of the dissipation of resources and effort in unproductive signalling, the so-called 'rat race,' may not be as significant as previously thought. Our paper also indicates that an optimal method of dealing with the problem of asymmetric information regarding employees' abilities may involve the design of testing and interviewing procedures, rather than on-the-job screening via incentive schemes. This can explain why incentive schemes offered in a variety of industries are not as steep and high-powered as incentive literature may suggest. Indeed, this papers indicates that an employer can obtain a good estimate of a job-candidate's ability and at a low cost, if the tests and interviews can be designed to have the following properties: (i) Each test identifies a candidate's ability accurately if the candidate does not attempt to manipulate the results of the test by expending effort; (ii) A candidate incurs some cost of effort when (s)he attempts to misrepresent her type.

In our setting, the marginal cost of a message/signal can depend on the content and number of other messages/signals sent by the agent. For example, the amount of effort that an agent of ability θ may need to exert in the n-th test to perform at a level corresponding to ability $\theta' \neq \theta$ may depend on how hard she worked to prepare for other tests and how many other tests she has taken. Our results hold when the effect of the true ability θ on the cost of sending signal $m \neq \gamma(\theta)$ does not go to zero "too quickly' in n. Intuitively, the learning process cannot be too fast so that performing at a certain level in a testing procedure involving $n + 1$ tests is only slightly more costly and requires a bit more effort than performing at the same level in a testing procedure consisting of n tests.

It is conceivable that there may exist fixed costs incurred either by the principal or the agent in association with each test or interview. The presence of such costs would limit the feasible number of interviews/tests from above and perfect screening may become too costly. Still, our results indicate that multi-test procedures would dominate the ones relying on one

test. Furthermore, it is likely that the fixed costs would be associated with a particular test, and not a particular job-candidate. Then test-specific fixed costs will be amortized over all the job-candidates who undergo it, and therefore would create less of an obstacle for increasing the number of tests. In this case, our model predicts that larger firms who interview many applicants will put more emphasis on rigorous testing and evaluation of candidates before hiring, rather than on providing on-the-job incentives. This appears to be broadly consistent with reality.

8 Appendix

Proof of Theorem 1:

Endow Ω with the uniform topology, i.e. define the uniform metric d on Ω by $d(x,y) = \sup\{||x(\theta)-y(\theta)||_X : \theta \in \Theta\}$. Let $\Omega_L \subset \Omega$ be the set of allocation profiles that are Lipschitz continuous with Lipschitz constant $L \geq 0$, i.e. $\Omega_L = \{x \in \Omega : ||x(\theta) - x(\theta')|| \leq L||\theta - \theta'||\}$. Also define $\Sigma^n = \{\mu : \Theta \to M^n\}$ to be the space of all measurable message functions, and endow Σ^n with the product topology (the topology of pointwise convergence). Since M^n is compact, the Tychonoff theorem guarantees that Σ^n is compact in the product topology. Thus Σ^n is sequentially compact.

First, we establish that E^n is a closed subset of Ω. Indeed, let $\{x_i\}_{i=1}^{\infty}$ be a sequence in E^n s.t. $x_i \to x \in \Omega$. Then there exists a sequence of $\{\mu_i(\cdot)\} \subset \Sigma^n$ such that incentive constraints (2) hold for $\{x_i(\cdot), \mu_i(\cdot)\}$. Let μ be the limit of a convergent subsequence of $\{\mu_i\}$, and renumber the subsequence so that $\mu_i \to \mu$. Observe that $x_i \to x$ and $\mu_i \to \mu$ imply that $x_i(\theta) \to x(\theta)$ and $\mu_i(\theta) \to \mu(\theta)$ for each $\theta \in \Theta$. Since weak inequalities are preserved by limit operations, it follows that $\{x, \mu\}$ satisfies the incentive constraints (2), and hence that $x \in E^n$. Thus E^n is closed.

Second, we establish that there exists an $\hat{L} < \infty$ and $y \in \Omega_{\hat{L}}$ s.t. $y \notin E^n$. Suppose to the contrary that $\Omega_L \subset E^n$ for all $L \in \mathbb{R}_+$. Choose any $z \in \Omega$. Because z is measurable, there exist sequences $\{z_i\}$ and $\{L_i\}$ with $L_i \to \infty$ and $z_i \to z$ such that $z_i \in \Omega_{L_i}$ for all i. Since $\Omega_{L_i} \subset E^n$ for all i, and since E^n is closed, it follows that $\mathbf{z} \in E^n$. This contradicts the assumption that $\Omega \setminus E^n \neq \phi$.

Third, let $x \in E^n$ solve $\min_{z \in E^n \cap \Omega_{\hat{L}}} d(y, z)$. Such x exists because $d(y, \cdot)$ is continuous on X, and because $E^n \cap \Omega_{\hat{L}}$ is closed and non-empty. Now for any $\varepsilon > 0$ let $x^{\varepsilon}(\theta) = (1 - \varepsilon)x(\theta) + \varepsilon y(\theta)$. Since $\Omega_{\hat{L}}$ is a convex set, and $d(x^{\varepsilon}, y) = (1-\varepsilon)d(x, y) < d(x, y)$, it follows that $x^{\varepsilon} \in \Omega_{\hat{L}} \setminus E^n$.

Finally, we claim that $x^{\varepsilon}(\cdot) \in E^{n+1}$ when $\varepsilon > 0$ is sufficiently small, establishing that $E^{n+1} \setminus E^n \neq \phi$. To prove the claim, let $\Delta(\theta) = x(\theta) - y(\theta)$, $W = \max_{X \times \Theta} ||u_x(x, \theta)||$, $M = \max_{X \times \Theta} ||u_{xx}(x, \theta)||$. Then using the fundamental theorem of calculus and the definitions of

$\hat{L}$, M and W we obtain:

$$|(u(x^\varepsilon(\theta),\theta) - u(x(\theta),\theta)) - (u(x^\varepsilon(\theta'),\theta) - u(x(\theta'),\theta))|$$

$$= \left| \int_0^\varepsilon e u_x(x^e(\theta),\theta)\Delta(\theta)de - \int_0^\varepsilon e u_x(x^e(\theta'),\theta)\Delta(\theta')de \right|$$

$$= \left| \int_0^\varepsilon e \left\{ (u_x(x^e(\theta),\theta) - u_x(x^e(\theta'),\theta)\}\Delta(\theta) + u_x(x^e(\theta'),\theta)(\Delta(\theta) - \Delta(\theta')\} \right\} de \right|$$

$$\leq \quad \varepsilon M \hat{L} \, ||\theta - \theta'|| \, d(x,y) \, + 2\varepsilon W \hat{L} ||\theta - \theta'|| \tag{17}$$

Setting $S = \hat{L}\,(Md(x,y) + 2W)$, we thus have

$$u(x^\varepsilon(\theta),\theta) - u(x^\varepsilon(\theta'),\theta) \geq u(x(\theta),\theta) - u(x(\theta'),\theta) - \varepsilon S||\theta - \theta'||. \tag{18}$$

Since $x \in E^n$ there exists $\mu^n(\cdot)$ such that

$$u(x(\theta),\theta) - u(x(\theta'),\theta) \geq C^n(\mu^n(\theta),\theta) - C^n(\mu^n(\theta'),\theta). \tag{19}$$

To complete the proof of the claim, let us show that $x^\varepsilon(\theta)$ is implementable with message rule $\mu^{n+1} = \{\mu^n,\gamma_{n+1}\}$. Thus, $C^{n+1}(\mu^{n+1}(\theta''),\theta'') = C^n(\mu^n(\theta''),\theta'')$ for all $\theta'' \in \Theta$. Further, by Assumptions 1 and 2 we have:

$$C^{n+1}(\mu^n(\theta'),\gamma_{n+1}(\theta'),\theta) - C^{n+1}(\mu^n(\theta'),\gamma_{n+1}(\theta),\theta) \geq \delta_{n+1}||\gamma_{n+1}(\theta) - \gamma_{n+1}(\theta')|| \geq \delta_{n+1}r||\theta - \theta'||$$

Using the above, we have:

$$C^n(\mu^n(\theta),\theta) - C^n(\mu^n(\theta'),\theta) = C^{n+1}(\mu^{n+1}(\theta),\theta) - C^{n+1}(\mu^n(\theta'),\gamma_{n+1}(\theta),\theta)$$

$$\geq C^{n+1}(\mu^{n+1}(\theta),\theta) - C^{n+1}(\mu^{n+1}(\theta'),\theta) + \delta_{n+1}r||\theta - \theta'|| \tag{20}$$

From (18),(19) and (20) it follows that for all $\theta,\theta' \in \Theta$

$$u(x^\varepsilon(\theta),\theta) - u(x^\varepsilon(\theta'),\theta) \geq C^{n+1}(\mu^{n+1}(\theta),\theta) - C^{n+1}(\mu^{n+1}(\theta'),\theta) + (\delta_{n+1}r - \varepsilon S)||\theta - \theta'|| \tag{21}$$

Therefore, taking $\varepsilon \leq \delta_{n+1}r/S$, we obtain that $\{x^\varepsilon(\theta),\mu^{n+1}\}$ satisfies the incentive constraints in (2).

Proof of Theorem 2. Consider a mechanism $g : \Theta \times M^n \to X$ defined by

$$g(\theta',\mathbf{m}^n) = \begin{cases} x(\theta) & \text{if } \mathbf{m}^n = \mu^n(\theta), \\ \underline{x}, & \text{otherwise.} \end{cases}$$

Then the incentive constraints (2) become:

$$u(x(\theta),\theta) - C^n(\mu^n(\theta),\theta) \geq u(x(\theta'),\theta) - C^n(\mu^n(\theta'),\theta), \text{ for all } \theta,\theta'.$$

Now select $\mu^n(\theta) = \gamma^n(\theta)$. Assumptions 2- 4 imply that the above incentive constraints holds and hence $x(.)$ is implementable, if $KL \leq rn^{1-\alpha}\underline{c}$. The latter can be rewritten as $n \geq (\frac{KL}{r\underline{c}})^{\frac{1}{1-\alpha}}$.

No communication costs are incurred because the agent of type θ sends a collection of messages such that $m_i = \gamma_i(\theta)$ for all $i = 1, ..., n$. $\hspace{4cm}$ *Q.E.D.*

Proof of Theorem 3. Given any pair of twice continuously differentiable functions $q : \Theta \to Q$ and $t : \Theta \to \mathbb{R}$ let $U(\theta) = v(q(\theta), \theta) - t(\theta)$. We will show that there exists $N < \infty$ and a sequence of transfers and messages rules $(t^n, \mathbf{m}^n)$, such that for all $n \geq N$ and all $\theta \in \Theta$:

$$U(\theta) \equiv v(q(\theta), \theta) - t^n(\theta) - C^n(\mathbf{m}^n(\theta), \theta) = \max_{\theta' \in \Theta}\{v(q(\theta'), \theta) - t^n(\theta') - C^n(\mathbf{m}^n(\theta'), \theta)\} \quad (22)$$

Thus the agent's allocation differs from the allocation to be implemented in the transfer rule and the required messages, but gives her the same net utility.

(i) First, let us construct the message rule $\mathbf{m}^n(\theta)$.

Let $\mathbf{z}^n : \Theta \to \mathbb{R}^l$ and for each i set $m_i(\theta) = \gamma_i(\theta) + \mathbf{z}^n(\theta)$. We choose $\mathbf{z}^n(\theta)$ as follows. Since (22) holds, the envelope theorem implies that

$$D_\theta U(\theta) = D_\theta v(q(\theta), \theta) - D_\theta C^n(\mathbf{m}^n(\theta), \theta) \quad (23)$$

For each θ, (23) consists of l equations in the l unknown variables $\mathbf{z}^n(\theta)$. Assumption 5(iii) implies that the mapping $C^n_\theta(\cdot, \theta)$ is injective, so that the solution $\mathbf{z}^n(\theta)$ to (23) is unique. Furthermore, the twice continuous differentiability of $q(\cdot)$ and $U(\cdot)$ implies that $z^n(\cdot)$ is continuously differentiable on Θ.

(ii) Next, we construct the transfer rule $t^n(\theta)$. Given the message rule constructed in part (i), select the transfer rule $t^n(\cdot)$ as follows:

$$t^n(\theta) = \max_{\theta' \in \Theta}\{v(q(\theta), \theta') - C^n(\mathbf{m}^n(\theta), \theta') - U(\theta')\} \quad (24)$$

Since $q(.)$ and $\mathbf{m}^n(.)$ are continuously differentiable, the envelope theorem implies that $t^n(.)$ is continuously differentiable everywhere, with derivative given by the following expression:

$$D_\theta t^n(\theta) = D_q v(q(\theta), \theta) D_\theta q(\theta) - D_{\mathbf{m}^n} C^n(\mathbf{m}^n(\theta), \theta) D_\theta \mathbf{m}^n(\theta). \quad (25)$$

(iii) Let us show that $\mathbf{m}^n(\theta) \to \gamma^n(\theta)$ and $D_\theta \mathbf{m}^n(\theta) \to D_\theta \gamma^n(\theta)$, uniformly in θ.

Observe that since $C^n(\gamma^n(\theta), \theta) \equiv 0$, and since $D_m C^n(\gamma^n(\theta), \theta) = 0$, we have $D_\theta C^n(\gamma^n(\theta), \theta) = 0$. Now let $g(\theta) = D_\theta v(q(\theta), \theta) - D_\theta U(\theta)$. It then follows from (23), Assumption 5(i) and the fact that $D_\theta C^n(\gamma^n(\theta), \theta) = 0$ that we have:

$$||g(\theta)|| = ||D_\theta C^n(\mathbf{m}^n(\theta), \theta) - D_\theta C^n(\gamma^n(\theta), \theta)|| \geq \frac{\omega}{n^\alpha}||\mathbf{m}^n - \gamma^n(\theta)|| \geq \underline{\omega} n^{1-\alpha}||\mathbf{z}(\theta)|| \quad (26)$$

Furthermore, since g is continuous and Θ is compact, it follows from the Weierstrass Theorem that there exists a constant $\lambda > 0$ s.t. $||g(\theta)|| \leq \lambda$ for all θ. Hence (26) implies $||\mathbf{z}(\theta)|| \leq \frac{\lambda}{\underline{\omega}} n^{-(1-\alpha)} \to 0$. We conclude that $\mathbf{m}^n(\theta) \to \gamma^n(\theta)$, uniformly in θ, and therefore that $D_\theta \mathbf{m}^n(\theta) \to D_\theta \gamma^n(\theta)$, uniformly in θ.

(iv) Next, let us demonstrate the incentive compatibility of our mechanism. Let $V(\theta'; \theta)$ be the payoff of agent-type θ who misrepresents herself as θ':

$$V(\theta'; \theta) = v(q(\theta'), \theta) - t^n(\theta') - C^n(\mathbf{m}^n(\theta'), \theta)$$

By (25), we have $D_{\theta'}V(\theta,\theta) = 0$ for all $\theta \in \Theta$. Therefore,

$$
\begin{aligned}
D_{\theta'}V(\theta',\theta) &= D_q v(q(\theta'),\theta)q_\theta(\theta') - D_\theta t^n(\theta') - D_{\mathbf{m}^n}C^n(\mathbf{m}^n(\theta'),\theta)D_\theta \mathbf{m}^n(\theta') - D_{\theta'}V(\theta',\theta') \\
&= \{D_q v(q(\theta'),\theta) - D_q v(q(\theta'),\theta')\}D_\theta q(\theta') - \\
&\quad \{D_{\mathbf{m}^n}C^n(\mathbf{m}^n(\theta'),\theta) - D_{\mathbf{m}^n}C^n(\mathbf{m}^n(\theta'),\theta')\}D_\theta \mathbf{m}^n(\theta') \qquad (27)
\end{aligned}
$$

Using (27), we may compute the directional derivative of $V(\cdot,\theta)$ at θ' in the direction of θ:

$$
\begin{aligned}
D_{\theta'}V(\theta',\theta)(\theta - \theta') &= \{D_q v(q(\theta'),\theta) - D_q v(q(\theta'),\theta')\}D_\theta q(\theta')(\theta - \theta') \\
&\quad -\{D_{\mathbf{m}^n}C^n(\mathbf{m}^n(\theta'),\theta) - D_{\mathbf{m}^n}C^n(\mathbf{m}^n(\theta'),\theta')\}D_\theta \mathbf{m}^n(\theta')(\theta - \theta')
\end{aligned}
$$

Let $\beta = \max_{\theta',\theta \in \Theta} \|v_{\theta q}(q(\theta'),\theta)q_\theta(\theta')\|$. Then using the mean value theorem

$$
D_{\theta'}V(\theta',\theta)(\theta - \theta') \geq -\beta \, \|\theta - \theta'\|^2 - (\theta - \theta')^\tau \left\{ \sum_{i=1}^n D^2_{\theta m_i}C^n(\mathbf{m}^n(\theta'),\theta_i)(D_\theta \gamma_i(\theta') + D_\theta \mathbf{z}^n(\theta')) \right\}(\theta - \theta')
$$

where $\theta_i = \theta + \varepsilon_i(\theta' - \theta)$, for some $\varepsilon_i \in (0,1)$. It follows from Assumption 2 and Assumption 5(iii) that $(\theta - \theta')^\tau \left\{ D^2_{\theta m_i}C^n(\mathbf{m}^n(\theta'),\theta_i)(D_\theta \gamma_i(\theta') + D_\theta \mathbf{z}^n(\theta')) \right\}(\theta - \theta') \leq -(1 - \frac{\|\mathbf{z}^n_\theta\|}{r})\underline{\omega}n^{-\alpha}\|\theta - \theta'\|^2$. Therefore

$$
D_{\theta'}V(\theta',\theta)(\theta - \theta') \geq \left(\left(1 - \frac{\|\mathbf{z}^n_\theta\|}{r}\right)\underline{\omega}n^{1-\alpha} - \beta \right)\|\theta - \theta'\|^2
$$

Let N be such that $\left(1 - \frac{\|\mathbf{z}^n_\theta\|}{r}\right)\omega n^{1-\alpha} - \beta > 0$ for all $n \geq N$. Such and N exists, because by (iii) we have $\|\mathbf{z}^n_\theta\| \to 0$ uniformly in θ. Then for $n \geq N$, and all $\theta' \neq \theta$ the directional derivative $D_{\theta'}V(\theta',\theta)(\theta - \theta')$ is positive, implying that $\theta' = \theta$ uniquely maximizes $V(\theta',\theta)$.

(v) Next, let us show that $C^n(\mathbf{m}^n(\theta),\theta) \to 0$.

It follows from the mean value theorem that

$$
C^n(\mathbf{m}^n(\theta),\theta) = D_{\mathbf{m}^n}C^n(\widetilde{\mathbf{m}}^n(\theta),\theta)(\mathbf{m}^n(\theta) - \gamma^n(\theta))
$$

where $\widetilde{\mathbf{m}}^n(\theta) = \mathbf{m}^n(\theta) + \widetilde{\varepsilon}(\theta)(\mathbf{m}^n(\theta) - \gamma^n(\theta))$, for some $\widetilde{\varepsilon}(\theta) \in (0,1)$. Hence from Assumption 5 (ii) and the part (iii) above that:

$$
\begin{aligned}
C^n(\mathbf{m}^n(\theta),\theta) &\leq \|\sum_i D_{m_i}C^n(\widetilde{\mathbf{m}}^n(\theta),\theta)\| \, \|\mathbf{z}(\theta)\| \qquad (28) \\
&\leq \bar{\omega}n^{1-\alpha}\|\mathbf{z}(\theta)\|^2 \leq \left(\frac{\lambda}{\underline{\omega}}\right)^2 \bar{\omega}n^{-(1-\alpha)}
\end{aligned}
$$

Consequently, $C^n(\mathbf{m}^n(\theta),\theta) \to 0$, uniformly in θ. From (22), this also implies that $t^n(\theta) \to t(\theta)$, uniformly in θ, completing the proof. $\hspace{2cm}$ Q.E.D.

Proof of Theorem 4: The proof will proceed through the following steps. First, we will establish the existence of a point $\widehat{\theta} \in (\underline{\theta},\overline{\theta})$, such that $U(\theta) = 0$ if and only if $\theta \leq \widehat{\theta}$. Second, we will show that $U''(\theta) > 0$ for all $\theta \geq \widehat{\theta}$. This implies that on the interval $[\underline{\theta},\widehat{\theta}]$ the solution is

given by $\{\widehat{q}(\theta), \widehat{\mathbf{m}}^n(\theta)\}$, and that on the interval $[\widehat{\theta}, \bar{\theta}]$ it is given by $\{\widetilde{q}(\theta), \widetilde{\mathbf{m}}^n(\theta)\}$. To complete the proof, we will establish that the solution to our relaxed program (7) subject to (6) and (8) also satisfies the second-order conditions, and hence solves the full unrelaxed program (7).

To begin, define $\widehat{\theta} = \inf\{\theta : U(\theta) > 0\}$. First, we show that $\widehat{\theta} \in (\underline{\theta}, \bar{\theta})$. To establish that $\widehat{\theta} < \bar{\theta}$, suppose to the contrary that $\widehat{\theta} = \bar{\theta}$. Then, since $U(\theta) = 0$ for all θ, the solution to the relaxed problem is given by $\{\widehat{q}(\theta), \widehat{\mathbf{m}}^n(\theta), \widehat{\sigma}(\theta)\}$. By the transversality condition at $\bar{\theta}$, $\widehat{\sigma}(\bar{\theta}) \geq 0$. So, the first-order conditions (10) and (11) imply that $\widehat{q}(\bar{\theta}) \geq q^{FB}(\bar{\theta})$ and $\widehat{m}_i(\bar{\theta}) \geq \gamma_i(\theta)$. Since $h_{q\theta} < 0$ and $C^n_{\theta m_i} < 0$, we then have

$$U'(\bar{\theta}) = -h_\theta(\widehat{q}(\bar{\theta}), \bar{\theta}) - C^n_\theta(\widehat{\mathbf{m}}^n(\bar{\theta}), \bar{\theta}) \geq -h_\theta(q^{FB}(\bar{\theta}), \bar{\theta}) - C^n_\theta(\gamma^n(\bar{\theta}), \bar{\theta}) > 0$$

where the final inequality follows from the fact that $C^n_\theta(\gamma^n(\theta), \theta) = 0$. But $U'(\bar{\theta}) > 0$ contradicts the presumption that $\widehat{\theta} = \bar{\theta}$, i.e. that $U'(\theta) = 0$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$. So, we cannot have $\widehat{\theta} = \bar{\theta}$.

To prove that $\widehat{\theta} > \underline{\theta}$ suppose, to the contrary, that $U(\theta) > 0$ for all $\theta > \underline{\theta}$. Then the solution to the relaxed problem is given by $\{\widetilde{q}(\theta), \widetilde{\mathbf{m}}^n(\theta)\}$. In particular, from the first-order condition (11) it follows that $\widetilde{m}_i(\theta) < \gamma_i(\theta)$ for all i. But since $\widetilde{q}(\theta) = q^{SB}(\theta) = 0$ for all $\theta \in [\underline{\theta}, \theta^*]$, for such θ we would then have

$$U'(\theta) = -h_\theta(0, \theta) - C^n_\theta(\widetilde{\mathbf{m}}^n(\theta), \theta) < 0 \tag{29}$$

where the inequality holds because $v_\theta(0, \theta) = 0$, and because $\widetilde{\mathbf{m}}^n(\theta) < \gamma^n(\theta)$ implies $C^n_\theta(\widetilde{\mathbf{m}}^n(\theta), \theta) > 0$. Since $U(\underline{\theta}) = 0$, (29) then shows that individual rationality is violated on $(\underline{\theta}, \theta^*]$. This contradiction establishes that $\widehat{\theta} > \underline{\theta}$.

Next, let us establish that $U(\theta) > 0$ for all $\theta > \widehat{\theta}$. Indeed, let $\theta^+ = \sup\{t : U(\theta) > 0 \text{ for all } \theta \in (\widehat{\theta}, t)\}$, and suppose to the contrary that $\theta^+ < \bar{\theta}$. Since by assumption $-h_{\theta\theta} - C^n_{\theta\theta} < 0$, it follows from (8) that $U'(\theta)$ is increasing on $[\widehat{\theta}, \theta^+)$. Because $U(\widehat{\theta}) = U'(\widehat{\theta}) = 0$, it follows that $U(\theta^+) > 0$, contradicting the definition of θ^+. Thus $\theta^+ = \bar{\theta}$.

We have now shown that $U(\theta) = 0$ for $\theta \in [\underline{\theta}, \widehat{\theta}]$ and $U(\theta) > 0$ for $\theta \in (\widehat{\theta}, \bar{\theta}]$. Therefore, we may conclude that the solution to the relaxed problem is given by $\{\widehat{q}(\theta), \widehat{\mathbf{m}}^n(\theta), \widehat{\sigma}(\theta)\}$ for $\theta \in [\underline{\theta}, \widehat{\theta}]$, and by $\{\widetilde{q}(\theta), \widetilde{\mathbf{m}}^n(\theta)\}$ for $\theta \in [\widehat{\theta}, \bar{\theta}]$.

It remains to show that the solution (14) satisfies incentive constraints (5). This will be the case if the following second-order condition is satisfied:

$$-h_{q\theta}q_\theta - \sum_{i=1}^{n} C^n_{m_i}(\mathbf{m}^n(\theta), \theta)m'_i(\theta) \geq 0$$

On the interval $[\underline{\theta}, \widehat{\theta})$ we have $U'(\theta) \equiv 0$, and hence

$$U''(\theta) = -h_{q\theta}q_\theta - h_{\theta\theta} - \left(\sum_{i=1}^{n} C^n_{m_i}(\mathbf{m}^n(\theta), \theta)m'_i(\theta) + C^n_{\theta\theta}\right) = 0$$

Therefore

$$-h_{q\theta}q_\theta - \sum_{i=1}^{n} C^n_{m_i}(\mathbf{m}^n(\theta), \theta)m'_i(\theta) = C^n_{\theta\theta} + h_{\theta\theta} \geq 0.$$

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On the interval $(\widehat{\theta}, \overline{\theta}]$ we have $(q(\theta), \mathbf{m}^n(\theta)) = (\widetilde{q}(\theta), \widetilde{\mathbf{m}}^n(\theta))$, and Assumptions 6 implies that the Hamiltonian is supermodular in (q, θ) and in $(\mathbf{m}^n, \theta)$, and so $\widetilde{q}_\theta \geq 0$ and $D_\theta \widetilde{\mathbf{m}}^n \geq 0$, implying $-h_{q\theta}\widetilde{q}_\theta - \sum_{i=1}^n C_{m_i}^n(\mathbf{m}^n(\theta), \theta)\widetilde{m}_i' \geq 0$. $\hspace{2cm}$ *Q.E.D.*

The following auxiliary result is used in the proof of Theorem 4.

Lemma 4 *Suppose that part (i) and (ii) of Assumption 6 hold. Then on any interval on which $\rho(\theta) = 0$ the Hamiltonian (9) is supermodular in $(q, \mathbf{m}^n, \theta)$.*

Proof: First, we claim that in any solution to the optimal control problem the costate variable $\sigma(\theta)$ satisfies the inequalities $-(1 - F(\theta)) \leq \sigma(\theta) \leq 0$.

Indeed, since $q(\theta)$ is optimal, we must have $q(\theta) \leq q^{FB}(\theta)$. Because $v_{q\theta} > 0$, it then follows from the first-order condition (10) that $\sigma(\theta) \leq 0$. In conjunction with the transversality condition at $\overline{\theta}$, the latter inequality implies $\sigma(\overline{\theta}) = 0$. Furthermore, integrating the costate equation (12) yields $\sigma(\theta) = \sigma(\overline{\theta}) - \int_\theta^{\overline{\theta}}(f(\theta) - \rho(\theta))d\theta = -(1 - F(\theta)) + \int_\theta^{\overline{\theta}} \rho(\theta)d\theta$. Since $\rho(\theta) \geq 0$ we therefore have $\sigma(\theta) \geq -(1 - F(\theta))$.

Next, let us show that on any interval on which $\rho(\theta) = 0$ the function

$$\Psi(q, \theta) \equiv v_q(q, \theta) - h_q(q, \theta) + \frac{\sigma(\theta)}{f(\theta)}v_{q\theta}(q, \theta)$$

is increasing in θ. We may compute:

$$
\begin{aligned}
\Psi_\theta(q, \theta) &\equiv v_{q\theta} - h_{q\theta} + \left(\frac{\sigma}{f}\right)' v_{q\theta}(q, \theta) + \frac{\sigma}{f}v_{q\theta\theta} \\
&= 2v_{q\theta} - h_{q\theta} + \frac{\sigma}{f}\left(v_{q\theta\theta} - \frac{f'}{f}v_{q\theta}\right)
\end{aligned}
$$

Since $v_{q\theta} > 0$, $h_{q\theta} \leq 0$, and $\sigma \leq 0$, it follows that $\Psi_\theta > 0$ if $v_{q\theta\theta} - (f'/f)v_{q\theta} \leq 0$. In the opposite case i.e., when $(v_{q\theta\theta} - \frac{f'}{f}v_{q\theta}) > 0$, this inequality together with $\sigma(\theta) \geq F(\theta) - 1$ and Assumption (5)(i) imply $\Psi_\theta \geq 2v_{q\theta} - h_{q\theta} + \frac{F(\theta)-1}{f}(v_{q\theta\theta} - (f'/f)v_{q\theta}) > 0$. A similar argument shows that for each i the function

$$\Phi_i(\mathbf{m}^n, \theta) = \frac{\partial C^n}{\partial m_i}(\mathbf{m}^n, \theta) + \frac{\sigma(\theta)}{f(\theta)}\frac{\partial^2 C^n}{\partial m_i \partial \theta}(\mathbf{m}^n, \theta)$$

is decreasing in θ. $\hspace{2cm}$ *Q.E.D.*

Proof of Theorem 5: First, let us demonstrate that in the optimal mechanism, $\widehat{q}(\theta) > 0$ for all $\theta \in (\underline{\theta}, \widehat{\theta}]$. Suppose instead we had $\widehat{q}(\theta) = 0$ for some $\theta \in (\underline{\theta}, \widehat{\theta}]$. Let us show that this implies $\widehat{\mathbf{m}}^n(\theta) = \gamma^n(\theta)$ and $\widehat{\sigma}(\theta) = 0$. Note that $U'(\theta) = 0$ because $\theta \leq \widehat{\theta}$. Since $v_\theta(0, \theta) = 0$, equation (8) therefore yields $C_\theta^n(\widehat{\mathbf{m}}^n(\theta), \theta) = 0$. We will establish that $\widehat{\mathbf{m}}^n(\theta) = \gamma^n(\theta)$ is a solution to this equation. Indeed, since $C^n(\gamma^n(\theta), \theta) \equiv 0$, we have $\sum_{i=1}^n C_{m_i}^n(\gamma^n(\theta), \theta) + C_\theta^n(\gamma^n(\theta), \theta) = 0$. Since $C^n(\mathbf{m}^n, \theta)$ takes on a global minimum at $\mathbf{m}^n = \gamma^n(\theta)$, it follows that $C_{m_i}^n(\gamma^n(\theta), \theta) = 0$. Thus $\mathbf{m}^n(\theta) = \gamma^n(\theta)$ is a solution to the equation $C_\theta^n(\mathbf{m}^n(\theta), \theta) = 0$.

Next, by Equation (11), $\sigma \gtreqless 0$ implies that $m_i(\theta) \lesseqgtr \gamma_i(\theta)$ for all i. Because $C^n_{m\theta} < 0$ we have $C^n_\theta(\widehat{\mathbf{m}}^n(\theta), \theta) \lesseqgtr 0$ as $\sigma \gtreqless 0$. Thus $C^n_\theta(\widehat{\mathbf{m}}^n(\theta), \theta) = 0$ implies $\widehat{\mathbf{m}}^n(\theta) = \gamma^n(\theta)$ and $\widehat{\sigma}(\theta) = 0$.

But substituting $\widehat{\sigma}(\theta) = 0$ into equation (10) then yields $v_q(\widehat{q}(\theta), \theta) - h_q(\widehat{q}(\theta), \theta) = v_q(0, \theta) - h_q(0, \theta) \leq 0$, contradicting the assumption that $v_q(0, \theta) - h_q(0, \theta) > 0$. So we must have $\widehat{q}(\theta) > 0$ for all $\theta > \underline{\theta}$.

Further, by Theorem 4, $U(\theta) > 0$ on $(\widehat{\theta}, \bar{\theta}]$, so we also have $\widetilde{q}(\theta) > 0$ on this interval. *Q.E.D.*

Proof of Theorem 6: We will first prove that $\lim_{n\to\infty} \widehat{\sigma}(\theta, n) = 0$. Let $\widehat{\sigma}_\infty(\theta)$ be the limit of any convergent subsequence of $\{\widehat{\sigma}(\theta, n)\}_{n=1}^\infty$. By renumbering the indices of the subsequence, we may without loss of generality assume that $\widehat{\sigma}(\theta, n) \to \widehat{\sigma}_\infty(\theta)$. Suppose that contrary to the desired result, we had $\widehat{\sigma}_\infty(\theta) < 0$. Using the mean value Theorem, we have $\frac{\partial C^n}{\partial m_i}(\widehat{m}_i, \widehat{\mathbf{m}}^n_{-i}, \theta) - \frac{\partial C^n}{\partial m_i}(\gamma_i(\theta), \widehat{\mathbf{m}}^n_{-i}, \theta) = \frac{\partial^2 C^n}{\partial m_i^2}(\bar{m}_i, \widehat{\mathbf{m}}^n_{-i}, \theta)(\widehat{m}_i - \gamma_i(\theta))$, for some $\bar{m}_i \in (m_i, \gamma_i)$. Since $\frac{\partial C^n}{\partial m_i}(\gamma_i(\theta), \mathbf{m}^n_{-i}, \theta) = 0$, it follows from (11) that

$$(\widehat{m}_i(\theta) - \gamma_i(\theta)) = -\widehat{\sigma}(\theta, n) \frac{\frac{\partial^2 C^n}{\partial\theta\partial m_i}(\widehat{\mathbf{m}}^n(\theta), \theta)}{\frac{\partial^2 C^n}{\partial m_i^2}(\bar{m}_i, \widehat{\mathbf{m}}^n_{-i}(\theta), \theta)} \tag{30}$$

Applying the mean value Theorem once more, we also have

$$C^n_\theta(\widehat{\mathbf{m}}^n, \theta) = \sum_{i=1}^n \frac{\partial^2 C^n}{\partial\theta\partial m_i}(\bar{\mathbf{m}}^n, \theta)(\widehat{m}_i(\theta) - \gamma_i(\theta)), \tag{31}$$

where $\bar{\mathbf{m}}^n = \gamma^n(\theta) + \varepsilon(\theta)(\widehat{\mathbf{m}}^n - \gamma^n(\theta))$, for some $\varepsilon(\theta) \in (0, 1)$. Using the assumption that $\frac{\partial^2 C^n}{\partial m_i^2} \leq \bar{v}$, and $|\frac{\partial^2 C^n}{\partial\theta\partial m_i}| \geq \underline{v}$, (30) and (31) yield

$$C^n_\theta(\widehat{\mathbf{m}}^n(\theta), \theta) \geq -n\widehat{\sigma}(\theta, n)\frac{\underline{v}}{\bar{v}} \tag{32}$$

Since $\widehat{\sigma}(\theta, n) \to \widehat{\sigma}_\infty(\theta) < 0$, we therefore have $C^n_\theta(\widehat{\mathbf{m}}^n(\theta), \theta) \to \infty$. But because $v_\theta(q, \theta)$ is bounded, this contradicts (8), thereby establishing that $\lim_{n\to\infty} \widehat{\sigma}(\theta, n) = 0$.

Next, we argue that $\lim_{n\to\infty} \widehat{\theta}(n) = \bar{\theta}$. Because $\widehat{\sigma}(\theta, n) \to 0$, the first-order conditions (10) and (11) imply that $\lim_{n\to\infty} \widehat{q}(\theta, n) = q^{FB}(\theta)$ and $\lim_{n\to\infty} \widehat{\mathbf{m}}(\theta, n) = \gamma^n(\theta)$. Since $\widehat{\theta}(n)$ is the solution to the equation $\widehat{q}(\theta, n) = \widetilde{q}(\theta) = q^{SB}(\theta)$, and since $q^{FB}(\theta) > q^{SB}(\theta)$ for every $\theta \in (0, \bar{\theta})$ we must have $\lim_{n\to\infty} \widehat{\theta}(n) = \bar{\theta}$.

It remains to prove that $C^n(\widehat{\mathbf{m}}^n(\theta), \theta) \to 0$. Since $C^n(\gamma^n(\theta), \theta) = 0$ and $C^n_\theta(\gamma^n(\theta), \theta) = 0$, it follows from Taylor's Theorem that

$$C^n(\widehat{\mathbf{m}}^n(\theta), \theta) = \sum_{i=1}^\infty (\widehat{m}_i - \gamma_i(\theta))^2 \frac{\partial^2 C^n}{\partial m_i^2}(\underline{m}^n, \theta) \tag{33}$$

Now (30) yields $|\widehat{m}_i - \gamma_i(\theta)| \leq |\widehat{\sigma}(\theta, n)|\frac{\bar{v}}{\underline{v}}$, so (33) implies $C^n(\widehat{\mathbf{m}}^n(\theta), \theta) \leq n\widehat{\sigma}^2(\theta, n)\frac{\bar{v}^3}{\underline{v}^2}$. Also, (31) implies $n|\widehat{\sigma}(\theta, n)|\frac{\underline{v}}{\bar{v}} \leq |C^n_\theta(\widehat{\mathbf{m}}^n, \theta)| \leq \max_{(q,\theta)} v_\theta = k$. Thus $C^n(\widehat{\mathbf{m}}^n, \theta) \leq k|\widehat{\sigma}(\theta, n)|\frac{\bar{v}^4}{\underline{v}^3} \to 0$. *Q.E.D.*

Proof of Lemma 2: Since the solution to the principal's problem is unique the value function W is continuously differentiable, and

$$\frac{dW(n)}{dn} = \int_{\underline{\theta}}^{\bar{\theta}} \frac{\partial H}{\partial n}(q, m, U, \sigma, n, \theta) d\theta$$

(Seierstad and Sydsaeter (1999), p. 217). Using

$$\frac{\partial H}{\partial n} = -c(m(\theta), \theta) f(\theta) - \sigma(\theta) c_\theta(m(\theta), \theta)$$

and substituting for $\sigma(\theta)$ from the first-order condition (11) yields

$$\frac{\partial H}{\partial n} = f\left(-c + \frac{c_\theta c_m}{c_{m\theta}}\right)$$

proving (15). Furthermore,

$$\frac{d}{dn} \frac{-c c_{m\theta} + c_\theta c_m}{c_{m\theta}} = \frac{\partial}{\partial m} \frac{-c c_{m\theta} + c_\theta c_m}{c_{m\theta}} \frac{\partial m}{\partial n} = \frac{c_\theta \left(c_{m\theta}^2 - c_{mm\theta} c_m\right)}{c_{m\theta}^2} \frac{\partial m}{\partial n}$$

In Theorem 6 it is established that $\frac{\partial m}{\partial n} > 0$ on $[\underline{\theta}, \widehat{\theta}(n))$. Since $c_\theta < 0$ it follows from the assumption $c_{m\theta}^2 - c_{mm\theta} c_m > 0$ that $\frac{\partial H}{\partial n} > 0$ on $[\underline{\theta}, \widehat{\theta}(n))$. Furthermore, on $(\widehat{\theta}(n), \bar{\theta}]$ we have $\frac{\partial m}{\partial n} = 0$, and hence $\frac{\partial H}{\partial n} = 0$. We conclude that $W'(n)$ is strictly decreasing in n i.e., $W(n)$ is strictly concave. $\qquad$ *Q.E.D.*

Proof of Lemma 3: By the definition of $\underline{K}$ and $\overline{K}$ we have

$$\underline{K} c \leq \frac{c_m c_\theta}{c_{m\theta}} - c \leq \overline{K} c$$

and so,

$$0 < \underline{K} \int_{\underline{\theta}}^{\bar{\theta}} c(m(\theta), \theta) f(\theta) d\theta \leq \frac{dW(n)}{dn} = G \leq \overline{K} \int_{\underline{\theta}}^{\bar{\theta}} c(m(\theta), \theta) f(\theta) d\theta.$$

$\qquad$ *Q.E.D.*

Proof of Theorem 7. The existence of a separating perfect Bayesian equilibrium in which the optimal grade strategy $m_i^*(\cdot)$ is continuously differentiable, strictly increasing, and satisfies $m_i^*(\underline{\theta}) = \gamma_i(\underline{\theta})$ for all $i = 1, ..., n$ follows from the single crossing condition on the worker's objective function and can be shown by using standard monotone comparative statics arguments. Furthermore, the symmetry of our model implies that in equilibrium the worker's selected grade is the same in every course, i.e. $m_i^*(\theta) = m_j^*(\theta)$ for all $i, j \in \{1, ..., n\}$. Naturally, under competitive market assumption, we must have $q^*(\theta) = \arg\max_q v(q, \theta) - h(q, \theta))$ and $t^*(\theta) = v(q^*, \theta) - h(q^*, \theta))$.

By standard arguments, the family of incentive constraints (16) is equivalent to following condition:

$$(v_q(q^*(\theta), \theta) - h_q(q^*(\theta), \theta))q_i^{*'}(\theta) = n c_m(m^*(\theta), \theta) m_i^{*'}(\theta) \tag{34}$$

Together with the boundary condition $m_i(\underline{\theta}) = \underline{\theta}$, the equation (34) allows to solve for the equilibrium grade strategy $m^*(\theta)$.

To establish the equilibrium properties we first show that $\| m_i^* - \gamma_i \| \equiv \max_\theta |m_i^*(\theta) - \gamma_i(\theta)|$ vanishes as n grows without bound. Indeed, suppose to the contrary that for every n there existed θ_n such that $|m_i^*(\theta_n) - \gamma_i(\theta_n)| \geq \delta > 0$. Note that we have:

$$\sum_{i=1,\ldots,n} c(m_i^*(\theta_n), \theta_n) = \sum_{i=1,\ldots,n} c(m_i^*(\theta_n), \theta_n) - \sum_{i=1,\ldots,n} c(\gamma_i(\theta_n), \theta_n)$$

$$= \left| \sum_{i=1,\ldots,n} c_\theta(\widetilde{m}_i(\theta_n), \theta_n)(m_i^*(\theta_n) - \gamma_i(\theta_n)) \right| \geq n\eta |m_i^*(\theta_n) - \gamma_i(\theta_n)| \tag{35}$$

where $\widetilde{m}_i(\theta)$ is a point on the line segment connecting $m_i^*(\theta)$ to $\gamma_i(\theta)$ and $\eta = \inf_{(m,\theta)} |c_\theta(m, \theta)|$. The second equality in (35) holds by the mean value theorem, and the first inequality holds because $c_\theta < 0$.

The inequality (35) implies that, if $|m_i^*(\theta_n) - \gamma_i(\theta_n)| \geq \delta > 0$, then $\sum_{i=1,\ldots,n} c(m_i^*(\theta_n), \theta_n) \geq n\eta\delta$. So, the net payoff of type θ_n, $U^n(\theta_n)$, converges to $-\infty$ as $n \to \infty$. But this contradicts the optimization for type θ_n, as by selecting $m_i(\theta_n) = \gamma_i(\theta_n)$ this type would earn at least

$$\min_{\theta'} \left\{ v(q^*(\theta'), \theta_n)) - h(q^*(\theta'), \theta_n) \right\} = \hat{U} > -\infty, \tag{36}$$

where the inequality follows from the continuity of the functions $h(\cdot, \theta_n), v(\cdot, \theta_n)$ and $q^*(\cdot)$ and the compactness of Θ. This establishes that $m_i^*(\theta)$ converges to $\gamma_i(\theta)$, uniformly in θ.

Next, let us show that $\sum_{i=1,\ldots,n} c(m_i^*(\theta), \theta) \to 0$ as $n \to \infty$, uniformly in θ. We have:

$$\sum_{i=1,\ldots,n} c(m_i^*(\theta), \theta) = \left| \sum_{i=1,\ldots,n} c(m_i^*(\theta), \theta) - \sum_{i=1,\ldots,n} c(\gamma_i(\theta), \theta) \right| = \left| \sum_{i=1,\ldots,n} c_m(\widetilde{m}_i(\theta), \theta)(m_i^*(\theta) - \gamma_i(\theta)) \right|$$

$$\leq n\omega |m_i^*(\theta) - \gamma_i(\theta)|^2 \leq \left(\frac{\left(\hat{U} + (v(q^*(\theta), \theta) - h(q^*(\theta), \theta)) \right)}{\eta} \right)^2 \frac{\omega}{n} \to 0.$$

The second equality holds by the mean value theorem. The first inequality follows from the fact that $c_m(\gamma(\theta), \theta) = 0$, and so

$$|c_m(\widetilde{m}_i(\theta), \theta)| = |c_m(\widetilde{m}_i(\theta), \theta) - c_m(\gamma(\theta), \theta)| \leq \omega |\widetilde{m}_i(\theta) - \gamma(\theta)| \leq \omega |m_i^*(\theta) - \gamma(\theta)|.$$

The final inequality follows because (36) implies $U^n(\theta) \geq \hat{U}$, and so we have from (35) that $|m_i^*(\theta) - \gamma(\theta)| \leq \frac{1}{n} \frac{\left(\hat{U} + (v(q^*(\theta), \theta) - h(q^*(\theta), \theta)) \right)}{\eta}.$ Q.E.D.

References

Alger, I. and C.A. Ma (2003), "Moral Hazard, Insurance and Some Collusion," *Journal of Economic Behavior and Organization*, 50, pp. 225-247.

Armstrong, M. (1996), "Multiproduct Nonlinear Pricing," *Econometrica*, 64, pp. 51-75.

Bull, J. (2008), "Costly Evidence Production and the Limits of Verifiability," *The B.E. Journal of Theoretical Economics,* 8(1) (Topics), Article 18.

Bull, J. (2008), "Mechanism Design with Moderate Evidence Cost," *Contributions in Theoretical Economics,* 8 (1), Article 15.

Bull, J. (2009), "Costly Evidence and Systems of Fact-Finding," *Bulletin of Economic Research,* 61, 103-125.

Bernheim, B. D., and S. Severinov (2003), "Bequests as Signals: An Explanation for the Equal Division Puzzle," *Journal of Political Economy,* 111(4), 2003, 733-764.

Caskey, J., V. Nagar and P. Petacchi (2010), "Reporting Bias with an Audit Committee," *The Accounting Review*, 85(2), pp. 447-481.

Chevalier, A., C. Harmon, I. Walker and Y. Zhu (2004), "Does Education Raise Productivity, or Just Reflect it?" *Economic Journal,* 114(499), pp. 499-517.

Clark, D and P. Martorell (2014), "The Signalling Value of a High School Diploma," *Journal of Political Economy,* 122(2), pp. 282-318.

Crocker, K. and J. Morgan (1998), "Is Honesty the Best Policy? Curtailing Insurance Fraud through Optimal Incentive Contracts," *Journal of Political Economy*, 106, 1998, pp. 355-375.

Deneckere, R. and S. Severinov (2008), "Mechanism Design with Partial State Verifiability," *Games and Economic Behavior*, 64, 487513

Ekman, P. (1973), "Darwin and Facial Expression : A Century of Research in Review," *Academic Press*, New York.

Engers, M. (1987), "Signaling with Many Signals," *Econometrica*, 55, pp. 663-674.

Fisher, P. and R. Verecchia (2000), "Reporting Bias," *The Accounting Review*, 75(2), pp. 229-245.

Goldman, E. and S. Slezak (2006), "An Equilibrium Model of Incentive Contracts in the Presence of Information Manipulation," *Journal of Financial Economics*, 80, pp. 603-626.

Guttman, I., O. Kadan and E. Kandel (2006), "A Rational Expectations Theory of Kinks in Financial Reporting," *The Accounting Review*, 81(4), pp. 811-848.

Green, J. and J.-J. Laffont (1986), "Partially Verifiable Information and Mechanism Design," *Review of Economic Studies*, 53, pp. 447-456.

Kartik, N., M. Ottaviani and F. Squintani (2007), "Credulity, lies, and costly talk," *Journal of Economic Theory*, 134, 93–116

Kartik, N. (2009), "Strategic Communication with Lying Costs ," Review of Economic Studies, 76, 13591395.

Kartik, N. and O. Tercieux (2012), "Implementation with Evidence," *Theoretical Economics,* 7, 323-355.

Lacker, J.M. and J.A. Weinberg (1989), "Optimal Contracts Under Costly State Falsification," *Journal of Political Economy*, 97(6), pp. 1345-1363.

Lipman, B. and D. Seppi (1995), "Robust Inference in Communication Games with Partial Provability," *Journal of Economic Theory*, 66(2), pp. 370-405.

Maggi, G. and Rodriguez-Clare, A. (1995), "Costly Distortion of Information in Agency Problems," *Rand Journal of Economics*, 26, pp. 675-689.

Maskin, E. and Riley, J. (1984), "Monopoly with Incomplete Information," *Rand Journal of Economics*, 15(2), pp. 171-196.

Page, M. (2010), "Signaling in the Labor Market," in *International Encyclopedia of Education*, Third Edition, Eva Baker, Penelope Peterson and Barry McGaw, Editors, Elsevier Limited, Oxford.

Rochet, J.C. and M. Quinzii (1985), "Multidimensional Signaling," *Journal of Mathematical Economics*, 14, pp. 261-284.

Rochet, J.-C. and P. Choné (1998), "Ironing, Sweeping, and Multidimensional Screening," *Econometrica*, 66, pp. 783-826.

Seidmann, H. S. and E. Winter (1997), "Strategic Information with Verifiable Messages," *Econometrica*, 65, pp. 163-169.

Seierstad, A. and K. Sydsaeter (1999), "Optimal Control Theory with Economic Applications," North Holland.

Severinov, S. and R. Deneckere (2006), "Screening when not All Agents are Strategic: Does a Monopolist Need to Exclude?" *Rand Journal of Economics*, 37(4), pp. 816-841.

Sher, I and R. Vohra. (2014), "Price discrimination through communication," forthcoming in *Theoretical Economics*.

Spence, A.M. (1973), "Job Market Signaling," *Quarterly Journal of Economics*, 87(3), pp. 355-374.