Optimal Screening with Costly Misrepresentation.

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Abstract

We study mechanism design in environments where misrepresenting private information is costly. Specifically, a privately informed agent has to take several signalling actions, send several messages or undergo a number of tests in which it is costly for her to misrepresent her type. We derive the optimal mechanism for this environment. A surprising property of the optimal mechanism is the absence of exclusion. Particularly, in the monopoly screening setting every consumer type whose valuation for the good exceeds the marginal cost of production is allocated a positive quantity. We also establish conditions under which the set of implementable allocation profiles increases in the number of messages/signals while the overall cost of signalling diminishes. In the limit, as the number of messages becomes very large, the principal can elicit the agent's private information at a very small cost. Our results explain why employers often prefer to screen applicants via multiple rounds of interviews rather than via menus of contracts, and why the welfare losses due to unproductive signalling ("rat race") may not be too large.

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1 Introduction.

In this paper we study mechanism design and screening in settings where agents incur some cost of misrepresenting their private information. These costs may exist for several reasons. First, an individual may find it costly to misrepresent the truth for psychological or ethical reasons. Such individual may experience stress or discomfort from lying.[1]. Second, misrepresenting the truth may require costly actions, such as acquiring skills and/or technology for manufacturing evidence, and taking effort to conceal one's information or hide evidence that

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[1]Behavioral psychologists have extensively studied the physical symptoms associated with the emotional discomfort, such as " blushing," "feeling wrong," that people experience when lying. See, for example Ekman (1973)

reveals the true state of the world. For example, in order to obtain a supplier contract, or to qualify for a loan, win a grant or a promotion, a firm or an individual may need to be perceived as highly productive, successful and/or creditworthy. This goal may be attained by exhibiting "evidence" exaggerating prior performance and concealing the risk of default or non-performance. Yet, the production of such favorable but inaccurate evidence would normally require expending cost and effort.

Our model is similar to the one studied in the screening literature, but differs in one very important aspect. We consider a principal-agent model in which the agent has private type information that affects both her own and the principal's utility from a social decision (which may involve consumption allocation, or an activity such as production, etc.). In addition, the agent can send a number of signals, or messages (the terms 'signal' and 'message' are used interchangeably in the sequel) about her type. In contrast to much of the screening literature, we assume that the agent's cost of sending a particular message increases in the magnitude of type misrepresentation, and that the agent can send multiple costly messages or signals.

The motivation behind this key aspect of our approach is fairly natural and intuitive. As one interpretation, an agent may have to undergo a number of tests assessing her ability from different angles. In this case, message i would correspond to the agent's performance on test i. On every test, an agent can attain her "natural" level of performance corresponding to her type without incurring additional effort or disutility, while attaining a different level of performance requires costly effort or disutility.

Similarly, a message may correspond to an outcome of an inspection or an audit undertaken by the principal. For example, shareholders or the parent corporation may carry out several accounting and other audits of their subsidiary. The subsidiary's managers would then have to incur the cost of hiding or embellishing the true state of the world and fudging the numbers during each audit. In the job-market context the principal (employer) may ask the candidate to undergo several interviews conducted by different interviewers. Each interviewer may use a different method to evaluate the candidate's ability. Then the candidate would have to incur an extra cost of time and effort to prepare for each test or interview so that she can perform at the level above her true ability.

Further, each message could represent a different piece of evidence presented by the agent to her examiner, or to a judge in a court of law. Producing evidence, such as records, documents, etc., is typically costly, and such cost is normally higher when the agent procures untruthful evidence. In line with this motivation, we assume that an agent's cost of producing a certain message, signal, or a piece of evidence is increasing in the degree to which this message deviates from depicting the true state of the world.

Our analysis focuses on three issues. First, we explore the effect of increasing the size of the signal space, or the number of messages, on the set of implementable allocations. Second, we derive the optimal mechanism with multiple messages and characterize its properties. Third, we present a method for choosing the optimal number of messages when the principal also incurs a fixed cost of eliciting each message. Most of our results, with the exception of the characterization of the optimal mechanism, hold for a multidimensional type space.

We show that the availability of multiple costly messages, or tests, can significantly expand the set of implementable allocations. Specifically, under fairly standard conditions, the set of implementable allocation profiles increases monotonically in the number of messages. In the limit when the number of messages becomes very large, any allocation profile becomes implementable at zero communication cost provided the agent's marginal cost of misrepresenting herself does not go to zero too fast in the number of messages. When the agent's marginal cost of misrepresenting her type is zero, it becomes necessary to induce her incur some cost and send messages misrepresenting her type to some degree. Still, under analogous conditions, we establish an approximate implementation result. It says that, with sufficiently many messages, the principal can come arbitrarily close to any decision rule and any surplus allocation, and at the same time keep misrepresentation costs arbitrarily small.

Further, we characterize the optimal screening mechanism maximizing the principal's expected profits for an arbitrary number of messages. That part of our analysis in related to the contributions by Maggi and Rodriguez-Clare (1995) and Crocker and Morgan (1998), and we comment more on this in Section 4. We establish an important qualitative property of the optimal mechanism - absence of exclusion. Specifically, when costly messages are available, then each agent-type who can generate a positive surplus is assigned a non-zero allocation in the optimal mechanism. Thus, the standard result on the optimality of exclusion in optimal screening is non-robust to the availability of costly signals.[2]

Finally, we use our characterization of the optimal mechanism to determine the optimal number of signals or messages which the principal should request when there is a fixed cost for eliciting and processing each message.

Our findings have practical relevance. In particular, they can explain why the employers in a number of industries prefer to screen and interview job-applicants very thoroughly, rather than to offer self-selecting menus of contracts or strong performance incentives to them. Indeed, the interviewing process in many professional job-markets appears to be consistent with the idea of requesting multiple messages or signals from the candidate, with each signal being somewhat different from the others. For example, in the context of a departmental visit on the academic job-market a prospective candidate meets with faculty members working in different fields. It is conceivable that each conversation provides an independent signal of the candidate's ability, because different faculty members, especially if they work in different fields, assess the candidate from different perspectives and inquire about different aspects of the candidate's knowledge and skills. Similarly, consulting firms have developed rigorous selection procedures with multiple rounds of interviews that involve solving cases, conversations with consultants, managers and partners.

Our results imply that if a job candidate has to go through sufficiently many such interviews, or other tests, then substantially misrepresenting his ability will be too costly. So, the employer will have a quite accurate estimate of the candidate's ability, and would not have to offer a powerful incentive scheme to her on the job. This is notable since in reality the incentive schemes offered to the employees are often not as strong as predicted by the contracting

[2] It is worth noting that when costly signals are absent exclusion is a robust property of optimal screening mechanisms. In particular, a profit-maximizing monopolist will choose not to sell to consumers whose willingness to pay for the good is not sufficiently higher than marginal cost, under both uniform and non-linear pricing, except for the non-generic case of perfectly inelastic consumer demand at price equal to marginal cost (which requires either that there are no consumers with valuations near marginal cost, or that the density of valuations is infinite at this level) (see Maskin and Riley (1984)). Exclusion must also occur in settings with multidimensional private information (see Armstrong (1996) and Rochet and Choné (1998)).

literature.

Further, ever since Spence's (1973) seminal contribution, economists have been concerned with potential loss of welfare due to signalling. Specifically, individuals engaged in a competitive 'rat race' may spend too much time and effort in activities which provide an informative signal about their ability. Yet, those activities may be non-productive and hence constitute a loss for the society. Our results indicate that this problem could be overcome by a careful choice of the signalling procedure by the mechanism designer. When this is done properly using multiple signals, individuals will have no incentives to engage in unproductive signalling to win the 'rat race,' as it will be too costly.

Our paper is related to two strands of literature. The first of these is the literature on costly state falsification. Lacker and Weinberg (1989), Maggi and Rodriguez-Clare (1995) and Crocker and Morgan (1998) study principal-agent models in which cost of misrepresentation is smooth and increasing in the magnitude of misrepresentation.[3] These authors point out that in such environments it is easier for the principal to elicit agents' private information, and that it is optimal for the principal to induce the agent to engage in some misrepresentation. Kartik and Tercieux (2012) study Nash implementation in a game of complete information where players may manufacture "evidence" at cost. The principal feature distinguishing our paper from this literature is that we allow the agent to send multiple costly signals or messages, and study how implementability is affected when we increase the size of the message space.

Our paper is also related to the extensive literature on signalling, since we emphasize the role of communication in determining the principal's allocation decision, although ours is a screening model and we use the analytical tools of mechanism design, rather than a signalling approach. Our modelling assumption requiring the least costly message/signal to be type dependent connects our paper to several contributions on costly signalling, in particular, Bernheim and Severinov (2003), Kartik, Ottaviani and Squintani (2007), Kartik (2009) and Seidmann and Winter (1997). In Bernheim and Severinov (2003), the type-dependence of the least costly signal is derived endogenously since the signalling action has intrinsic utility. Kartik (2009) studies a signalling model with costly misrepresentation. The equilibrium in his model converges to the most informative cheap talk equilibrium as the cost of misrepresentation becomes very small, while it converges to truthtelling when the cost of lying becomes very large. One of key differences between these signalling papers and ours is that in our paper the mechanism designer plays an active role in constructing the optimal communication scheme consisting of multiple messages. The mechanism designer does so in such a way that, although each type incurs some misrepresentation cost, in equilibrium those costs are small for each type, yet imitating another type becomes very costly for everyone.

Seidmann and Winter (1997) study signalling when the set of feasible signals depends on the sender's type. They derive conditions that ensure that full revelation occurs in equilibrium. Our paper shows that type-dependency of the least cost signal has a very strong effect when the number of signals is allowed to grow. In fact, while the signalling literature has considered

[3] Green and Laffont (1986) and Alger and Ma (2003) consider environments in which the falsification cost is binary, taking on the value zero or infinity, and show that mechanisms in which the agent does not always send truthful signals can attain outcomes that are not implementable via mechanisms with truthful signals. Deneckere and Severinov (2001) study implementability when the set of messages that an agent can send in a mechanism is type-dependent private information.

models with multiple signals (Rochet and Quinzii (1985), Engers (1987)), it has not posed the question of how the equilibrium is affected by increasing the dimensionality of the signal space. Our paper is also related to Sher and Vohra (2014) who consider a seller-buyer bargaining model in which the seller may request multiple pieces of type-dependent evidence from the buyer. In contrast to our paper, the "cost" of a piece of evidence to a particular buyer is either zero or infinity. A general model of mechanism design with evidence is studied by Deneckere and Severinov (2008)

The rest of the paper is organized as follows. In section 2 we describe the model, Section 3 presents the monotonicity and asymptotics results. Section 4 studies optimal mechanisms when the type space is one-dimensional. Section 5 endogenizes the dimension of the signal space by assuming that eliciting messages is costly to the principal. All proofs are relegated to an Appendix.

2 Model.

We consider a principal-agent model with asymmetric information. To this standard environment we add a costly communication process in which the principal sends several messages or signals to the principal.[4]

Specifically, in our model the principal controls the allocation space X. We assume that X is a compact subset of a k-dimensional Euclidean space . Thus, the principal's action is the choice of an allocation $x \in X$. Allocation x can denote a vector of production and/or consumption decisions, as well as monetary transfers. The agent privately observes the outcome of a random variable θ (which we will refer to as the agent's type in the sequel) that affects the utilities of the principal and the agent. We assume that $\theta \in \Theta$, where Θ is a compact subset of $\mathbb{R}^l$. Thus, we explicitly allow for multidimensional private information.

When the principal selects an allocation $x \in X$ and the agent's type is given by θ, the agent obtains utility $u(\mathbf{x}, \theta)$ and the principal receives utility $w(x, \theta)$. We assume that both $u(x, \theta)$ and $w(x, \theta)$ are continuous in x over the space X, and that there exists an outcome $\underline{x}$ which gives each type of the agent a utility that does not exceed her utility from the outside option.

Further, the principal may request the agent to send up to n messages or signals $(m_1, ..., m_n)$. For example, the principal may query the agent's ability by asking her to undergo a number of tests, interviews, or inspections. This assumption reflects that each message is characterized by some specific content, or is sent along a different dimension. We assume that the i-th message m_i belongs to a compact subset M_i of $\mathbb{R}^l$. Note that we require M^i and Θ to be of the same dimension, so that each message can contain information about all of the aspects or dimensions of the agent's type.[5]Let $\mathbf{m}^n = (m_1, ..., m_n)$ denote the vector of messages or

[4]Although we often use the term 'signal' below, we use a mechanism design or screening approach, rather than signalling, as understood by the economics literature. The term 'signal' is appropriate because it is used to refer to certain agent actions or messages which possess only informational value that has to be inferred by the principal.

[5]This involves little loss of generality, as we allow the agent to send multiple messages. For example, if each message could only reflect one dimension of the agent's type, we could always bundle them in groups of l messages.

signals sent by the agent, and let $M^n = \prod_{i=1}^n M_i$. Apart the from costly messages $\mathbf{m}^n$, we also allow the agent to send a cheap talk message τ. By the Revelation Principle, we can without loss of generality take the latter to be a type announcement, so that $\tau \in \Theta$ and the agent's message space is equal to $\Theta \times \prod_{i=1}^n M_i \equiv \Theta \times M^n$. In the sequel, we rely on mechanisms that do not use cheap talk messages. However, it is important to show that our results are robust to the addition of such messages and that the latter do not affect the scope of implementation.

An agent of type θ incurs the cost $C^n(\mathbf{m}^n, \theta)$ when she sends the vector of messages $\mathbf{m}^n$. The cost function $C^n(.) : M^n \times \Theta \mapsto R_+$ has the following properties. First, there exists a mapping $\gamma_i(.) : \Theta \mapsto M_i$ s.t. message m_i is costless for type θ if and only if $m_i = \gamma_i(\theta)$. Thus we have $\gamma_i(\theta) = \arg\min_{m_i} C^n(m_1, ..., m_i, ..., m_n, \theta)$ for all $(m_1, .., m_{i-1}, m_{i+1}, ..., m_n)$, and $C^n(m_1, ..., m_n, \theta) = 0$ if and only if $m_i = \gamma_i(\theta)$ for all $i = 1, ..., n$. This implies that $C^n(m_1, ..., m_n, \theta) > 0$ whenever there exists $i \in \{1, ..., n\}$ s.t. $m_i \neq \gamma_i(\theta)$.

The function $C^n(.)$ embodies the connection between the agent's payoff-relevant type θ and her abilities to manipulate information. Note that the message $\gamma_i(\theta)$ can be regarded as "truthful" for the agent of type θ, because this is the message that agent-type would prefer to send if she did not care about affecting the principal's action. In contrast, sending any message $m_i \neq \gamma_i(\theta)$ involves costly type misrepresentation, or costly distortion.

Each message space M_i is assumed to be sufficiently large in the following sense: there exists an open set of messages B_i and $\nu > 0$ s.t. $\gamma_i(\theta) \notin B_i$ for all $\theta \in \Theta$, and for all $\theta \in \Theta$ there exists $m_i'(\theta) \in B_i$ s.t. $C^n(m_1, .., m_i'(\theta), ..., m_n, \theta) - C^n(m_1, .., \gamma_i(\theta), ..., m_n, \theta) \geq \nu$. The only additional requirement that we will introduce below and which will be needed for our asymptotic results to hold is that the cost of sending a message $m_i \neq \gamma_i(\theta)$ does not go to zero too quickly as the number of messages increases.

The agent's cost of misrepresenting her type, i.e. sending message $m_i \neq \gamma_i(\theta)$, generally depends on the other communicated messages.[6] Such dependencies arise naturally. For example, as the agent proceeds with sending more messages to the principal, she may learn how to misrepresent her information more effectively and at a lower cost. Alternatively, additional effort spent on one test could be fatiguing for the agent, and therefore raise the cost of undergoing a subsequent test.

To summarize, when the agent with preference parameter θ sends an array of messages $(m_1, ..., m_n)$ and the principal selects an allocation x, the agent's overall payoff is given by:

$$u(x, \theta) - C^n(m_1, ..., m_n, \theta) \tag{1}$$

A mechanism that the principal offers to the agent can be represented by a mapping $g(.)$ from the agent's message space $\Theta \times M^n$ into the space of allocations X. We will say that an allocation profile $x(\cdot)$ is implementable if there is a mechanism $g(\cdot) : \Theta \times M^n \mapsto X$, and an agent's message/signal rule (strategy) $\mu^n : \Theta \to \Theta \times M^n$ and agent's cheap-talk rule (strategy) $\tau : \Theta \mapsto \Theta$ s.t. for all $\theta \in \Theta$ we have $x(\theta) = g(\tau(\theta), \mu^n(\theta))$ and such that:

$$u(x(\theta), \theta) - C^n(\mu^n(\theta), \theta) \geq \max_{\theta' \in \Theta, \mathbf{m}^n \in M^n} u(g(\theta', \mathbf{m}^n), \theta) - C^n(\mathbf{m}^n, \theta) \tag{2}$$

[6] In a more general model, the communication cost function $C^n(.)$ may also depend on a separate parameter t, so that agents with the same preference parameter θ but different t may have different communication costs.

When condition (2) holds, it is optimal for the agent of any type $\theta \in \Theta$ to accept the contract $g(.)$ and send an array of messages $(\tau(\theta), \sigma^n(\theta))$, resulting in the allocation $x(\theta)$. In the next section we study how the set of implementable allocations varies with the dimension of the signal space n.

3 The Dimension of the Signal Space

3.1 Monotonicity

We start by presenting two results which illustrate the central message of this paper that the availability of multiple costly messages, or signals, has powerful implications for implementation. Thus, our focus throughout this section is on the set of implementable allocation rules, $x(.)$, and on the expected message cost $\kappa^n(x) = \int_\Theta C^n(\mu^n(\theta), \theta) f(\theta) d\theta$ where $\mu^n(.)$ is the message strategy used to implement $x(.)$. Cheap talk messages, although allowed, do not play a role in our mechanisms as every type will be identified by the array of costly messages m^n which it sends.

We make the following additional assumptions linking the cost functions $C^n(.)$ and $C^{n+1}(.)$ for environments with n and $n+1$ messages, respectively:

$$C^{n+1}(m_1, ..., m_n, m_{n+1}, \theta) \geq C^n(m_1, ..., m_n, \theta), \text{ for all } m_{n+1} \in M_{n+1} \tag{3}$$
$$C^{n+1}(m_1, ..., m_n, \gamma_{n+1}(\theta), \theta) = C^n(m_1, ..., m_n, \theta)$$

Thus sending the extra message $\gamma_{n+1}(\theta)$ is costless for type θ, but sending any other extra message is costly.

With these assumptions, enlarging the dimension of the signal space increases the set of implementable allocation profiles, without increasing communication costs:

Lemma 1 *(Monotonicity) Suppose that under signal space M^n the decision rule $x : \Theta \to X$ is implementable via message rule $\mu^n(\theta)$, at an expected signal cost $\kappa^n(x) = \int_\Theta C^n(\mu^n(\theta), \theta) f(\theta) d\theta$. Then, under signal space M^{n+1}, x is implementable at an expected signal cost $\kappa^{n+1}(x) \leq \kappa^n(x)$.*

The proof of Lemma 1 is straightforward and is therefore omitted. Importantly, Lemma 1 leaves open the possibility that the set of implementable allocations is invariant to n, and the following example shows this can indeed happen.

Example 1 *Suppose that θ is a scalar, and that $C^n(m_1, ..., m_n, \theta) = \sum_{i=1}^n c(m_i, \theta)$, where $c(m_i, \theta) = m_i/\theta$. Then since $\sum_{i=1}^n c(m_i, \theta) = c(\sum_{i=1}^n m_i, \theta)$, any allocation that can be implemented with $n > 1$ can also be implemented with $n = 1$, by selecting $m_1(\theta) = \sum_{i=1}^n m_i^n(\theta)$.*

What goes wrong in the above example is not so much the linearity of the cost function C^n, but rather that the costless message is the same across agent types, i.e. $\gamma_i(\theta) \equiv 0$.[7] Additional messages would have an effect in this example if the agents also derived some type-dependent

[7] Note that the same conclusion would obtain if $c(m, \theta)$ were concave in m. If $c(m, \theta)$ were convex, then signalling costs can be reduced by having multiple tests.

utility from a message m_i, with higher types deriving more utility from higher messages, so that $\gamma_i(\theta)$ would be strictly increasing in θ. For example, in the context of educational signalling, higher ability types presumably enjoy studying more than low ability types. In line with this motivation, our next assumption requires costless messages to vary with agent-type:

Assumption 1 *For each n there exists $\delta_n > 0$ s.t. $C^n(\mathbf{m}^{n-1}, m_n, \theta) - C^n(\mathbf{m}^{n-1}, \gamma_n(\theta), \theta) \geq \delta_n \, ||m_n - \gamma_n(\theta)||$.*

Assumption 2 *There exists $r < \infty$ s.t. $||\gamma_i(\theta') - \gamma_i(\theta)|| \geq r\|\theta' - \theta\|$, for all $\theta', \theta \in \Theta$, all i.*

Assumption 1 ensures that the signal cost of agent-type θ gets larger as $\mathbf{m}^n$ moves further away from her costless message collection $(\gamma_1(\theta), ...\gamma_n(\theta))$. Assumption 2 says that costless messages are sufficiently sensitive to the type.

Let Ω be the set of all decision rules $x : \Theta \to X$, and let $E^n = \{x(\cdot) \in \Omega : \ x(\cdot)$ is implementable with signal space $M^n\}$. We can now state our result asserting that, under these mild assumptions, increasing the dimension of the signal space strictly increases the set of implementable allocations:

Theorem 1 *Suppose that Assumptions 1 and 2 hold. If $E^n \neq \Omega$ then $E^n \subsetneq E^{n+1}$.*

3.2 Asymptotics

Theorem 1 raises the interesting question of how the set of implementable allocations and the associated message costs change as the dimension of the signal space grows arbitrarily large. We establish two kinds of results in this case. First, when the marginal cost of small misrepresentation is non-zero and does not vanish too quickly, we show that the principal can still elicit the agent's information at zero cost. On the other hand, when the marginal cost of small type misrepresentations is zero, implementation becomes harder. Nevertheless, when the marginal cost of misrepresentation increases in the magnitude of misrepresentation at a rate that does not go to zero too fast as the number of messages becomes large, the principal can still implement almost all allocation profiles, although the agent has to incur a very small communication cost.

Assumption 3 *There exists $L < \infty$ s.t. $\|x(\theta') - x(\theta)\| \leq L\|\theta' - \theta\|$, for all $\theta', \theta \in \Theta$.*

Assumption 4 *There exists $K < \infty$ s.t. $|u(x', \theta) - u(x, \theta)| \leq K\|x' - x\|$, for all $\theta \in \Theta$.*

Assumption 3 requires the allocation to be Lipschitz continuous with Lipschitz constant L. Assumption 4 strengthens the continuity requirement on the agent's utility function to uniform continuity in x. Since $X \times \Theta$ is compact, Assumption 4 holds if $u(\cdot, \cdot)$ is C^1.

Let $\Omega^L = \{x(.) : \Theta \mapsto X : x(.)$ satisfies Assumption 3 with constant $L\}$. Our next theorem shows that all allocation profiles in Ω^L are implementable at zero cost when the marginal cost of misrepresenting the type is positive and does not go to zero too fast in the number of messages.

Theorem 2 *Suppose Assumption 4 holds, and that there exists* $\alpha \in [0,1)$ *and* $\underline{c} > 0$ *s.t.* $C^n(\mathbf{m}_1, ..., m_i, ..., m_n,\theta) - C^n(m_1, ..., \gamma_i(\theta), ..., m_n, \theta) \geq \frac{\underline{c}}{n^\alpha}||m_i - \gamma_i(\theta)||$, *for all* θ, $\theta' \in \Theta$.

Then there exists $N < \infty$ *such that* $\Omega^L \subset E^n$ *for all* $n \geq N$, *and such that any* $x \in \Omega^L$ *is implementable with zero communication costs.*

Intuitively, local incentive constraints hold because the marginal benefit of sending a non-truthful message $m_i \neq \gamma_i(\theta)$ is finite for type θ, while the marginal cost of doing so is bounded from below by $\frac{\underline{c}}{n^\alpha}$ when m_i is near $\gamma_i(\theta)$. Therefore, when the number of messages is sufficiently large, the cost of a misrepresentation exceeds the benefit of a better allocation. This ensures that no type is willing to send non-truthful messages, and so no communication costs are incurred.

It should be noted because C^n is minimized w.r.t. m_i at $m_i = \gamma_i(\theta)$ and because $M^{(n)}$ and Θ are compact, the condition on $C^n(.)$ imposed in Theorem 2 need only hold locally around $\gamma_i(\theta)$. In other words, there only need to exist some 'start-up' costs of misrepresentation.

The reasoning in the previous two paragraphs suggests that the set of implementable allocation profiles may be more restricted when the marginal cost of type misrepresentation is zero at the 'truthful' message $\gamma_i(\theta)$. Indeed, in this case implementation will generally require that the agent send some "non-truthful" messages and incur some misrepresentation costs. Nevertheless, we will demonstrate that by carefully constructing the communication stage of the mechanism, the principal can elicit the agent's private information at negligible cost.

For this result, we specialize our model to one with transferable utility. Thus, we partition the outcome $x = (q, t)$ into a production assignment $q \in Q$, where Q is a compact subset of $\mathbb{R}^{k-1}$, and a transfer $t \in \mathbb{R}$ from the agent to the principal. The agent's utility function is quasilinear

$$u(x, \theta) = v(q, \theta) - t.$$

We assume that $v(q, \theta)$, $C^n(\mathbf{m}^n, \theta)$ and $\gamma^n(\theta)$ are twice continuously differentiable functions.

For any function $f(\mathbf{v}, \mathbf{w}) : \mathbb{R}^j \times \mathbb{R}^k \to \mathbb{R}$, where $j, k > 1$, we denote the derivative of f w.r.t. the vector v by $D_v f(\mathbf{v}, \mathbf{w})$. Thus, $D_v f(\mathbf{v}, \mathbf{w})$ is a vector containing the j partial derivatives $\frac{\partial f}{\partial v_i}$. We also denote the derivative of the function $D_{\mathbf{v}} f(\mathbf{v}, \mathbf{w})$ w.r.t. the vector v by $D^2_{\mathbf{vv}} f(v, w)$, and similarly for $D^2_{\mathbf{vw}} f(\mathbf{v}, \mathbf{w})$. Thus $D^2_{\mathbf{vw}} f(\mathbf{v}, \mathbf{w})$ is a $j \times k$ matrix containing the cross partials$\frac{\partial^2 f}{\partial v_i \partial w_l}$. We may now state:

Assumption 5 *There exist* $\alpha \in [0,1)$ *and* $\underline{\omega}$, $\bar{\omega} > 0$ *such that for all* $(\mathbf{m}^n, \theta, \theta')$ *and all* $i = 1, ..., n$*:*
(i) $||D_\theta C^n(\mathbf{m}^n, \theta) - D_\theta C^n(\gamma^n(\theta), \theta)|| \geq \frac{\underline{\omega}}{n^\alpha}||\mathbf{m}^n - \gamma^n(\theta)||$*;*
(ii) $||D_{m_i} C^n(\mathbf{m}^n, \theta) - D_{m_i} C^n(\gamma^n(\theta), \theta)|| \leq \frac{\bar{\omega}}{n^\alpha}||m_i - \gamma_i(\theta)||$*;*
(iii) $D^2_{\mathbf{m}\theta_i} C^n(\mathbf{m}^n, \theta) D_\theta \gamma_i(\theta') \equiv A^n_i$ *is negative definite, and* $\mathbf{z}^\tau A^n_i \mathbf{z} \leq -\frac{\underline{\omega}}{n^\alpha}||\mathbf{z}||^2$ *for all* n.

To interpret Assumption 5, consider the case $l = 1$. Parts (i) and (ii) then says that the cross-partial $\frac{\partial^2 C^n}{\partial \theta \partial m_i}$ is bounded above and below, and does not converge to zero too quickly as n grows large. Part (ii) adds the single crossing requirement that $\frac{\partial^2 C^n}{\partial \theta \partial m_i}$ and $\gamma'_i(\theta)$ have opposite signs.

Theorem 3 *Suppose Assumptions 2 and 5 hold. Then for every pair of continuously differentiable functions* (q, t)*, and every* $\varepsilon > 0$*, there exist* $N < \infty$*, and for each* $n \geq N$ *a transfer function* t^n *with* $|t^n(\theta) - t(\theta)| < \varepsilon$*, such that* (q, t^n) *is implementable whenever the dimension of the signal space* n *exceeds* N*. Furthermore, the associated communication cost is less than* ε *for every type of agent.*

In the mechanism implementing the allocation profile $\{q(\theta), t^n(\theta)\}$, agent-type θ sends the costly messages $\mathbf{m}^n(\theta) \neq \gamma^n(\theta)$, and hence incurs communication cost $C^n(\mathbf{m}^n(\theta), \theta)$. To understand the significance of this, note that the first-order condition necessary for local incentive compatibility -derived by maximizing the payoff of agent-type θ in (1)- is given by:

$$D_\theta t^n(\theta) = -D_q v(q(\theta), \theta) D_\theta q(\theta) + D_{\mathbf{m}^n} C^n(\mathbf{m}^n(\theta), \theta) D_\theta \mathbf{m}^n(\theta) \tag{4}$$

If the agent were to send only costless messages, then the second term in (4) would be zero, as $D_{m_i} C^n(\gamma^n(\theta), \theta) = 0$. Thus in the absence of costly messages the first-order condition imposes a restriction on the set of implementable allocations in the form of a link between $q(\theta)$ and $t^n(\theta)$. Costly signals weaken and eventually eliminate the need for such link, and this allows to implement a larger set of allocation profiles.

In particular, when the number of messages n is small, the degree to which the link between $q(\theta)$ and $t^n(\theta)$ is weakened is limited by the magnitude of the required misrepresentation $\mathbf{m}^n(\theta)$ and by the associated communication costs, which may have to be fairly large. In contrast, with sufficiently large n, the codependency between $q(\theta)$ and $t(\theta)$ is eliminated at a very small cost. In fact, by choosing n and $\mathbf{m}^n(\theta)$ appropriately, we can ensure that (4) holds for arbitrary $(q(\theta), t(\theta))$, and the agent's communication cost $C^n(\mathbf{m}^n(\theta), \theta)$ is less than an arbitrary fixed ε.

Finally, consider the second-order conditions for implementation. Maggi and Rodriguez-Clare (1995), who characterize the optimal mechanism for $n = 1$, imposed the following restrictions: $q'(\theta) \geq 0$ and $m'(\theta) \geq 0$ to guarantee that their second-order conditions hold. In contrast, a careful inspection of our proof reveals that, with many n, the second-order conditions hold because the agent sends a large number of messages which are close to her costless message $\gamma(\theta)$. Consequently, we are able to implement nearly all allocation profiles at a small communication cost.

4 Optimal Mechanisms

The previous two theorems provide conditions under which almost all decision rules become implementable as the number of signals becomes very large, while the necessary communication cost converges to zero. However, this does not mean that asymmetric information becomes irrelevant, or that the principal can always achieve his most preferred allocation profile. There are at least two reasons for this. First, the bounds on the message cost function described in the previous section may fail to hold. More importantly, the number of available messages could be limited either exogenously or endogenously. In particular, endogenous limits on the number of signals arise most naturally if the principal incurs a fixed cost to elicit and process each signal.

In this section, we explore exogenous limits on the number of messages. Specifically, we characterize the optimal mechanism when the type space is one-dimensional, i.e. $l = 1$. We continue to assume that an allocation x consists of a monetary part t and non-monetary part q. For simplicity, we assume that $Q = \mathbb{R}_+$.[8] The agent's payoff is given by $v(q, \theta) - t$ and the principal's payoff is given by $t - h(q, \theta)$. We impose individual rationality, and normalize the agent's reservation utility to equal 0.

We make the standard assumptions that $v_q > 0$, $h_q > 0$, $v_\theta > 0$, $h_\theta \leq 0$, $v_{q\theta} > 0$, $v_{q\theta} \leq 0$, $v_{\theta\theta} \leq 0$, and $v_{qq} - h_{qq} < 0$, for all $q \in Q$ and $\theta \in \Theta$. In addition, we assume that $v_q(0, \theta) - h_q(0, \theta) > 0$ for all $\theta > \underline{\theta}$ and there exists $\bar{q} < \infty$ s.t. $v_q(q, \theta) - h_q(q, \theta) < 0$ for all $q > \bar{q}$ and $\theta > \underline{\theta}$. These two assumptions guarantee that the solution $q = q^{FB}(\theta)$ to

$$\max_{q \geq 0} \{v(q, \theta) - h(q, \theta)\}$$

exists and satisfies $q^{FB}(\theta) > 0$ for all $\theta > \underline{\theta}$. We also require that $v(q, \underline{\theta}) = 0$ for all q and $v(0, \theta) = 0$ for all θ, as well as $v_q(q, \theta) - h_q(q, \theta) > 0$, so that $q^{FB}(\theta)$ is increasing in θ. Finally, we make the following technical assumptions in deriving the optimal mechanism:

Assumption 6 *(i)* $v_q(q, \theta) - h_q(q, \theta) - \frac{1 - F(\theta)}{f(\theta)} v_{q\theta}(q, \theta)$ *is increasing in* θ*;*

(ii) $\frac{\partial C^n}{\partial m_i}(\mathbf{m}^n, \theta) - \frac{1 - F(\theta)}{f(\theta)} \frac{\partial^2 C^n}{\partial m_i \partial \theta}(\mathbf{m}^n, \theta)$ *is decreasing in* θ*;*

(iii) $C^n(\mathbf{m}^n, \theta)$ *is convex in* $\mathbf{m}^n$ *and in* θ*, and* $\frac{\partial^2 C^n}{\partial m_i \partial \theta}(\mathbf{m}^n, \theta) < 0$.

Parts (i) and (ii) of Assumption 6 require the cross-partial derivatives of the agent's virtual utility and virtual communication cost to be, respectively, positive and negative. Part (iii) requires the cost function to be convex in $\mathbf{m}^n$ and in θ, and imposes a single crossing condition.

We now proceed to derive the optimal mechanism. Specifically, the principal selects a "quantity" $q(\cdot)$, a transfer $t(.)$ and a vector of messages $\mathbf{m}^n(\cdot)$ to solve:

$$\max_{q(\theta), t(\theta), \mathbf{m}^n(\theta)} \int (t(\theta) - h(q, \theta)) f(\theta) d\theta$$

subject to incentive constraints:

$$v(q(\theta), \theta) - C^n(\mathbf{m}^n(\theta), \theta) - t(\theta) \geq v(q(\theta'), \theta) - C^n(\mathbf{m}^n(\theta'), \theta) - t(\theta'), \text{ for all } \theta \text{ and } \theta' \quad (5)$$

and the individual rationality constraint:

$$U(\theta) \equiv v(q(\theta), \theta) - C^n(\mathbf{m}^n(\theta), \theta) - t(\theta) \geq 0. \quad (6)$$

Let us first consider the benchmark case with no costly messages, i.e. $n = 0$. It is well-known that the solution to the principal's problem involves selecting an allocation $q^{SB}(\theta)$ that maximizes 'virtual' welfare

$$\Gamma(q, \theta) \equiv v(q, \theta) - h(q, \theta) - \frac{1 - F(\theta)}{f(\theta)} v_\theta(q, \theta).$$

[8] Our qualitative results extend straightforwardly to the case where Q is multidimensional.

Invoking Assumption 6 (i) and letting $\theta^* \in (\underline{\theta}, \bar{\theta})$ be the unique solution to $\Gamma_q(0,\theta) = 0$, we obtain that the optimal quantity when $n = 0$ is given by $q^{SB}(\theta) = 0$ for all $\theta \in [0, \theta^*]$, and by the unique solution to $\Gamma_q(q,\theta) = 0$ for all $\theta \in [\theta^*, 1]$. Thus there is exclusion in the second-best solution: all types in the interval $[0, \theta^*]$ receive zero quantity.

Next, let us consider the problem for $n > 0$. Solving for $t(\theta)$ from (6) and substituting into the objective, and replacing the incentive constraints (5) by the envelope condition associated with the agent's utility maximization, yields the following "relaxed" problem:

$$\max_{q(\theta),\mathbf{m}^n(\theta),U(\theta)} \int \{v(q(\theta),\theta) - h(q(\theta),\theta) - C^n(\mathbf{m}^n(\theta),\theta) - U(\theta)\} f(\theta)d\theta \tag{7}$$

subject to individual rationality constraint (6) and

$$U'(\theta) = v_\theta(q(\theta),\theta) - C^n_\theta(\mathbf{m}^n(\theta),\theta) \tag{8}$$

We will verify that the solution to the relaxed problem (7) subject to (6) and (8) satisfies (5) and hence also solves the unrelaxed problem. To solve the relaxed problem, define the Hamiltonian

$$H = \{v(q,\theta) - h(q,\theta) - C^n(\mathbf{m}^n,\theta) - U\}f(\theta) + \sigma\{v_\theta(q,\theta) - C^n_\theta(\mathbf{m}^n,\theta)\} + \rho U \tag{9}$$

Maximizing H w.r.t. $q \geq 0$ and $\mathbf{m}^n$ yields the first order conditions:

$$\{v_q(q,\theta) - h_q(q,\theta)\}f(\theta) + \sigma v_{q\theta}(q,\theta) \leq 0 \; (= 0, \text{ if } q > 0) \tag{10}$$

$$\frac{\partial C^n}{\partial m_i}(\mathbf{m}^n,\theta)f(\theta) + \sigma\frac{\partial^2 C^n}{\partial m_i \partial\theta}(\mathbf{m}^n,\theta) = 0 \tag{11}$$

The costate equation is

$$\sigma'(\theta) = f(\theta) - \rho(\theta), \tag{12}$$

Furthermore, the solution has to satisfy complementary slackness conditions

$$\rho(\theta)U(\theta) = 0, \; \rho(\theta) \geq 0, \text{ and } U(\theta) \geq 0, \tag{13}$$

Also, the following transversality conditions have to hold: $\sigma(\underline{\theta})U(\underline{\theta}) = 0$, $\sigma(\bar{\theta})U(\bar{\theta}) = 0$, $\sigma(\underline{\theta}) \leq 0$ and $\sigma(\bar{\theta}) \geq 0$

To describe the solution to the relaxed problem, lets us denote the global maximizers of the Hamiltonian (9) by $q_*(\sigma,\theta)$ and $m^n_*(\sigma,\theta)$.

Let $\{\widehat{q}(\theta), \widehat{\mathbf{m}}^n(\theta), \widehat{\sigma}(\theta)\}$ be the solution to the system of $(n+2)$ equations consisting of $q = q_*(\sigma,\theta)$, $\mathbf{m}^n = \mathbf{m}^n_*(\sigma,\theta)$, and the envelope condition (8) set to zero, i.e. $U'(\theta) = 0$. Note that over any interval of θ on which the individual rationality constraint (6) is binding, we have $U'(\theta) = 0$, and so the solution to the relaxed problem is given by $(\widehat{q}(\theta), \widehat{\mathbf{m}}^n(\theta))$.

Also, let $\{\widetilde{q}(\theta), \widetilde{\mathbf{m}}^n(\theta)\}$ be the solution to the relaxed problem in which the individual rationality constraint is not imposed. The costate equation and the transversality condition at $\theta = \bar{\theta}$ then require $\widetilde{\sigma}(\theta) = -(1 - F(\theta))$. Thus $\widetilde{q}(\theta) = q_*(\widetilde{\sigma}(\theta),\theta)$ and $\widetilde{\mathbf{m}}^n(\theta) = \mathbf{m}^n_*(\widetilde{\sigma}(\theta),\theta)$.

Our next Theorem shows that the solution to the full (unrelaxed) problem has the following properties. There exists $\widehat{\theta} \in [\underline{\theta}, \bar{\theta})$ such that on the interval $[\underline{\theta}, \widehat{\theta}]$ the individual rationality constraint (6) is binding and the solution is given by $(\widehat{q}(\theta), \widehat{\mathbf{m}}^n(\theta))$. On the interval $(\widehat{\theta}, \overline{\theta}]$, the individual rationality constraint is not binding, $U(\theta)$ is strictly increasing, and the solution is given by $(\widetilde{q}(\theta), \widetilde{\mathbf{m}}^n(\theta))$.

Theorem 4 *Suppose Assumption (6) holds. Then there exists $\widehat{\theta} \in (\underline{\theta}, \bar{\theta})$ such that the optimal mechanism is given by:*

$$(q(\theta), \mathbf{m}^n(\theta)) = \begin{cases} (\widehat{q}(\theta), \widehat{\mathbf{m}}^n(\theta)) & \text{if } \theta \in [\underline{\theta}, \widehat{\theta}], \\ (\widetilde{q}(\theta), \widetilde{\mathbf{m}}^n(\theta)) & \text{if } \theta \in [\widehat{\theta}, \bar{\theta}]. \end{cases} \tag{14}$$

The agent's utility $U(\cdot)$ satisfies: $U(\theta) = 0$ for all $\theta \in [\underline{\theta}, \widehat{\theta}]$, and $U(\theta) > 0$, $U'(\theta) > 0$ for all $\theta \in (\widehat{\theta}, \bar{\theta}]$.

Theorem 4 is related to Proposition 1 in Maggi and Rodriguez-Clare (1995) which characterizes the optimal mechanism with a single costly message. As in their model, a non-trivial set of types $[\underline{\theta}, \widehat{\theta}]$ are held at the reservation utility level in our optimal mechanism. So this property is robust to the number of messages. However, there are important differences between our results. First, in our model the agent sends multiple signals, and we focus on exploring how the number of signals affects the optimal mechanism (see Lemma 3). Second, we prove that the optimal mechanism exhibits no exclusion when costly signals are available (Lemma 2). This property is important, because it underscores that the optimality of exclusion in the standard case without costly signals is rather fragile. Third, our strategy of proof is different from Maggi and Rodriguez-Clare (1995), and we are able to establish our Theorem 4 and Lemmas 2 and 3 on its basis (see below) under more general conditions. In particular, unlike Maggi and Rodriguez-Clare (1995), we do not assume that the cost of a signal m depends only on the difference $(m - \theta)$ and do not require the Hamiltonian to be concave and the hazard rate to be monotone.

In large part, the value of Theorem 4 derives from the fact that it allows us to establish Lemmas 2 and 3 characterizing the properties of the optimal mechanism.

Lemma 2 *In the optimal mechanism, $q(\theta) > 0$ for all $\theta > \underline{\theta}$.*

Thus, according to Lemma 2, no type who can generate a positive surplus is excluded in the optimal mechanism as long as signal costs are positive, no matter how small.

The intuition for the absence of exclusion with costly signals is as follows. Any agent-type that generates a positive surplus in the first-best is potentially profitable to the principal. The reason some agent types are excluded in the second-best is that if the principal were to give them a positive quantity, he would also have to raise the surplus of all agents with higher valuations. But when signal costs are positive, this is no longer necessary: the principal can prevent imitation by requiring agent-types that now receive a positive quantity to send costly signals. By single crossing, higher agent types will incur lower signal costs, eliminating their incentive to imitate.

Finally, we characterize the nature of the solution as the number of costly messages, n, increases.

Lemma 3 *(i) Suppose that $C^n(m_1, ..., m_n, \theta) = \sum_{i=1}^n c_i(m_i, \theta)$. Then $\widehat{q}(\theta)$, $\widehat{\mathbf{m}}^n(\theta)$ and the cut-off $\widehat{\theta}$ are increasing in n.*

(ii) Suppose that there exist $\underline{v}$ and $\bar{v}$ such that $\underline{v} \leq \frac{\partial^2 c}{\partial m_i^2} \leq \bar{v}$ and $\underline{v} \leq |\frac{\partial^2 c}{\partial \theta \partial m_i}| \leq \bar{v}$. Then as $n \to \infty$ we have: $\widehat{\theta}(n) \to \bar{\theta}$, $\widehat{q}(\theta) \to q^{FB}(\theta)$, $\widehat{\mathbf{m}}^n(\theta) \to \gamma^n(\theta)$, and $C^n(\widehat{\mathbf{m}}^n(\theta), \theta) \to 0$.

In the next section, we use these results to derive the optimal number of messages which the agent should be required to send.

5 Endogenous Signal Space

In this section, we study the optimal number of messages in a mechanism, when the principal incurs a fixed cost F to elicit a each message, i.e. increase the size of the signal space by 1. For example, the principal may have to incur such cost for developing and administering a test, and/or processing the information received form the agent.

To simplify matters, we will treat n as a continuous variable. Let $W(n)$ denote the principal's expected surplus, gross of any fixed costs, in the optimal mechanism when the dimension of the signal space is n. The principal then selects n to maximize $W(n) - nF$. We assume that the agent's signalling costs are additively separable across messages, i.e.

$$C^n(m_1, m..., m_n, \theta) = \sum_{i=1}^{n} c(m_i, \theta),$$

and that

$$\frac{c_{m\theta}}{c_m}(m, \theta) \text{ is increasing in } m.$$

The latter assumption ensures that the solution to (11) is unique, and hence independent of i. Henceforth, we shall therefore omit the subscript of the message m_i. The optimal number of messages n^* is characterized in the following Lemma.

Lemma 4 *Suppose that $c_{m\theta}(m, \theta)^2 - c_{mm\theta}(m, \theta)c_m(m, \theta) > 0$. Then $W(n)$ is a strictly concave function, and the marginal benefit of an additional message is given by*

$$\frac{dW(n)}{dn} = \int_{\underline{\theta}}^{\bar{\theta}} \left(\frac{c_\theta(m(\theta), \theta)c_m(m(\theta), \theta)}{c_{m\theta}(m(\theta), \theta)} - c(m(\theta), \theta) \right) f(\theta)d\theta \tag{15}$$

To help us interpret the solution, let us define

$$\varepsilon = \min_{(m,\theta)} \frac{c_\theta(m, \theta)c_m(m, \theta)}{c_{m\theta}(m, \theta)c(m, \theta)} - 1, \qquad \eta = \max_{(m,\theta)} \frac{c_\theta(m, \theta)c_m(m, \theta)}{c_{m\theta}(m, \theta)c(m, \theta)} - 1$$

Then we have

Lemma 5 *Suppose that $c(m, \theta)$ is such that $\varepsilon > 0$. Then*

$$\varepsilon \int_{\underline{\theta}}^{\bar{\theta}} c(m(\theta), \theta) f(\theta)d\theta \leq \frac{dW(n)}{dn} \leq \eta \int_{\underline{\theta}}^{\bar{\theta}} c(m(\theta), \theta) f(\theta)d\theta$$

Under the conditions of this Lemma, the principal's total cost of eliciting signals, nF, will therefore lie between ε and η percent of the agent's expected total signal cost. For example, if $c(.)$ is quadratic i.e., $c(m, \theta) = (m - \theta)^2$, then $\varepsilon = \eta = 1$ so the principal and agent will spend the same amount on sending and receiving the signals.

We end by providing an example in which welfare and signalling costs are quadratic, and the type distribution is uniform. Specifically, suppose that $v(q,\theta) = \theta q$, $h(q,\theta) = \frac{1}{2}q^2$, $c(m,\theta) = (m-\theta)^2$, and $F(\theta) = \theta$ for $\theta \in [0,1]$. Then for fixed n the solution to the principal's problem is given by:

$$\begin{aligned}\widetilde{q}(\theta) &= \widetilde{m}(\theta) = 2\theta - 1\\ \widehat{q}(\theta) &= \widehat{m}(\theta) = \frac{2n}{2n+1}\theta\\ \widehat{\theta}(n) &= \frac{2n+1}{2(n+1)}\end{aligned}$$

Furthermore, the principal's marginal benefit of an additional message is $\frac{dW(n)}{dn} = \frac{1}{12(n+1)^2}$, and so the optimal number of signals equals $n(F) = \sqrt{1/(12F)}$.

Recall that every message generates some amount of additional welfare, because the allocation profile gets closer to the first-best when the agent has to send more messages (see Lemma 3). To illustrate the relation between the fixed cost F and the welfare generated by an extra message, let us express the fixed cost as a fraction of the potential surplus gain $\Delta W = W^{FB} - W^{SB}$, where W^{FB} (W^{SB}) is the total welfare under the first-best (second-best) quantity allocation. Since $W^{FB} = \frac{1}{6}$ and $W^{SB} = \frac{1}{8}$, we have

n	1	2	3	4	5	6	7	8	9	10
$F/\Delta W$	50%	22.2%	12.6%	8%	5.6%	4%	3.2%	3.4%	2%	1.6%

Thus in this example the principal will elicit at least four messages if the fixed cost of eliciting an extra message does not exceed 8% of the potential welfare gain. Note that in the process, the agent will incur expected message costs of at least 32% of the potential welfare gain, thereby dissipating a substantial portion of the benefit. It should also be noted that with four signals, the allocation q is already close to the first-best, as $\widehat{q}(\theta)$ is within 11% of $q^{FB}(\theta)$.

6 Conclusions

This paper demonstrates that in environments with misrepresentation costs, the ability of the principal to offer mechanisms in which an agent sends several messages significantly expands the set of implementable outcomes.

Our results have a number interesting implications for screening and signaling. In particular, they suggests that the problem of the dissipation of resources and effort in unproductive signalling, the so-called 'rat race,' may not be as significant as previously thought. Our paper also indicates that an optimal method of dealing with the problem of asymmetric information regarding employees' abilities may involve the design of testing and interviewing procedures, rather than on-the-job screening via incentive schemes. This can explain why incentive schemes offered in a variety of industries are not as steep and high-powered as incentive literature may suggest. Indeed, this papers indicates that an employer can obtain a good estimate of a job-candidate's ability and at a low cost, if the tests and interviews can be designed to have the following properties: (i) Each test identifies a candidate's ability accurately if the candidate does not attempt to manipulate the results of the test by expending effort; (ii) A candidate incurs some cost of effort when (s)he attempts to misrepresent her type.

In our setting, the marginal cost of a message/signal can depend on the content and number of other messages/signals sent by the agent. For example, the amount of effort that an agent of ability θ may need to exert in the n-th test to perform at a level corresponding to ability $\theta' \neq \theta$ may depend on how hard she worked to prepare for other tests and how many other tests she has taken. Our results hold when the effect of the true ability θ on the cost of sending signal $m \neq \gamma(\theta)$ does not go to zero "too quickly' in n. Intuitively, the learning process cannot be too fast so that performing at a certain level in a testing procedure involving $n+1$ tests is slightly more costly and requires a bit more effort than performing at the same level in a testing procedure consisting of n tests.

It is conceivable that there may exist fixed costs incurred either by the principal or the agent in association with each test or interview. The presence of such costs would limit the feasible number of interviews/tests from above and perfect screening may become too costly. Still, our results indicate that multi-test procedures would dominate the ones relying on one test. Furthermore, it is likely that the fixed costs would be associated with a particular test, and not a particular job-candidate. Then test-specific fixed costs will be amortized over all the job-candidates who undergo it, and therefore would create less of an obstacle for increasing the number of tests. In this case, our model predicts that larger firms who interview many applicants will put more emphasis on rigorous testing and evaluation of candidates before hiring, rather than on providing on-the-job incentives. This appears to be broadly consistent with reality.

7 Appendix

We start with the following Lemma:

Lemma 6 *An allocation profile $x : \Theta \to X$ is implementable if and only if there exists a signal rule $\mu^n : \Theta \to M^n$ s.t.*

$$u(x(\theta),\theta) - C^n(\mu^n(\theta),\theta) \leq u(x(\theta'),\theta) - C^n(\mu^n(\theta'),\theta), \textit{ for all } \theta, \theta' \tag{16}$$

Proof : Consider a mechanism $\widetilde{g} : \Theta \times M^n \to X$ defined by

$$g(\theta'', \mathbf{m}^n) = \begin{cases} x(\theta) \text{ if } \mathbf{m}^n = \mu^n(\theta), \\ \underline{x}, \quad \text{otherwise.} \end{cases}$$

Then (2) becomes (16). Thus x is implementable.

Conversely, suppose that (16) is violated for some θ and θ'. Suppose to the contrary that there exists a mechanism g that implements x. Then by selecting $\theta'' = \tau(\theta')$ and $\mathbf{m}^n = \mu^n(\theta')$ the agent of type θ would obtain utility $u(g(\tau(\theta'), \mu^n(\theta)), \theta) - C^n(\mu^n(\theta'), \theta) > u(g(\tau(\theta), \mu^n(\theta)), \theta) - C^n(\mu^n(\theta), \theta)$, contradicting that g implements x.

Proof of Theorem 1:

Endow Ω with the uniform topology, i.e. define the uniform metric d on Ω by $d(x,y) = \sup\{||x(\theta) - y(\theta)||_X : \theta \in \Theta\}$. Let $\Omega^L \subset \Omega$ be the set of allocation profiles that are Lipschitz with Lipschitz constant $L \geq 0$, i.e. $\Omega^L = \{x \in \Omega : ||x(\theta) - x(\theta')|| \leq L||\theta - \theta'||\}$. Also define $\Sigma^n = \{\mu : \Theta \to M^n\}$ to be the space of all message functions, and endow Σ^n with the product topology (the topology of pointwise convergence). Since M^n is compact, the Tychonoff theorem guarantees that Σ^n is compact in the product topology. Thus Σ^n is sequentially compact.

First, we establish that E^n is a closed subset of Ω. Indeed, let $\{x_i\}_{i=1}^{\infty}$ be a sequence in E^n s.t. $x_i \to x \in \Omega$. Then there exists a sequence of $\{\mu_i(\cdot)\} \subset \Sigma^n$ such that (16) holds for $\{x_i(\cdot), \mu_i(\cdot)\}$. Let μ be the limit of a convergent subsequence of $\{\mu_i\}$, and renumber the subsequence so that $\mu_i \to \mu$. Observe that $x_i \to x$ and $\mu_i \to \mu$ imply that $x_i(\theta) \to x(\theta)$ and $\mu_i(\theta) \to \mu(\theta)$ for each $\theta \in \Theta$. Since weak inequalities are preserved by limit operations, it follows that $\{x, \mu\}$ satisfies (16), and hence that $x \in E^n$. Thus E^n is closed.

Second, we establish that there exists a $\hat{L} < \infty$ and $y \in \Omega^{\hat{L}}$ s.t. $y \notin E^n$. Suppose to the contrary that $\Omega^L \subset E^n$ for all $L \in \mathbb{R}_+$. Choose any $z \in \Omega$. Because z is measurable, there exist sequences $\{z_i\}$ and $\{L_i\}$ with $L_i \to \infty$ and $z_i \to z$ such that $z_i \in \Omega^{L_i}$ for all i. Since $\Omega^{L_i} \subset E^n$ for all i, and since E^n is closed, it follows that $\mathbf{z} \in E^n$. This contradicts the assumption that $\Omega \setminus E^n \neq \phi$.

Third, let $x \in E^n$ solve $\min_{z \in E^n \cap \Omega^{\hat{L}}} d(y, z)$. Such an x exists, because $d(y, \cdot)$ is continuous on X, and because $E^n \cap\ \hat{L}$ is closed and non-empty. Now for any $\varepsilon > 0$ let $x^\varepsilon(\theta) = (1 - \varepsilon)x(\theta) + \varepsilon y(\theta)$. It is immediate that $x^\varepsilon \in \Omega^{\hat{L}} \setminus E^n$. Indeed, $\Omega^{\hat{L}}$ is a convex set, and $d(x, x^\varepsilon) \leq (1 - \varepsilon)d(x, y) < d(x, y)$ so $x^\varepsilon \notin E^n$.

Finally, we claim that $x^\varepsilon(\cdot) \in E^{n+1}$ when $\varepsilon > 0$ is sufficiently small, establishing that $E^{n+1} \setminus E^n \neq \phi$. To prove the claim, let $\Delta(\theta) = x(\theta) - y(\theta)$, $W = \max_{X\times\Theta} ||u_x(x, \theta)||$, $M = \max_{X\times\Theta} ||u_{xx}(x, \theta)||$. Then:

$$\begin{aligned}
& |(u(x^\varepsilon(\theta), \theta) - u(y(\theta), \theta)) - (u(x^\varepsilon(\theta'), \theta) - u(y(\theta'), \theta))| \\
= & \left| \int_0^\varepsilon e u_x(x^\varepsilon(\theta), \theta)\Delta(\theta) de - \int_0^\varepsilon e u_x(x^\varepsilon(\theta'), \theta)\Delta(\theta') de \right| \\
= & \left| \int_0^\varepsilon e \left\{ (u_x(x^\varepsilon(\theta), \theta) - u_x(x^\varepsilon(\theta'), \theta) \right\} \Delta(\theta) + u_x(x^\varepsilon(\theta'), \theta)(\Delta(\theta) - \Delta(\theta') \right\} de \right| \\
\leq & \ \varepsilon M \hat{L}\ ||\theta - \theta'||\ d(x, y)\ +\ 2\varepsilon W \hat{L}\ ||\theta - \theta'|| \qquad (17)
\end{aligned}$$

Setting $S = \hat{L}\{Md(x, y) + 2W\}$, we thus have

$$u(x^\varepsilon(\theta), \theta) - u(x\varepsilon(\theta'), \theta) \geq u(x)(\theta), \theta) - u(x(\theta'), \theta) - \varepsilon S||\theta - \theta'||. \qquad (18)$$

Since $x^\varepsilon \in E^n$ there exists $\mu^n(\cdot)$ such that

$$u(x(\theta, T), \theta) - u(x(\theta', T), \theta) \geq C^n(\mu^n(\theta), \theta) - C^n(\mu^n(\theta'), \theta). \qquad (19)$$

To complete the proof of the claim, let us show that $x(t, \theta)$ is implementable with message rule $\mu^{n+1} = \{\mu^n, \gamma_{n+1}\}$. Thus, $C^{n+1}(\mu^{n+1}(\theta''), \theta'') = C^n(\mu^n(\theta''), \theta'')$ for all $\theta'' \in \Theta$. Further,

$$C^{n+1}(\mu^n(\theta'), \gamma_{n+1}(\theta'), \theta) - C^{n+1}(\mu^n(\theta'), \gamma_{n+1}(\theta), \theta) \geq \delta_{n+1}||\gamma_{n+1}(\theta) - \gamma_{n+1}(\theta')||\ \geq \delta_{n+1} r_{n+1}||\theta - \theta'||$$

Using the above, we have:

$$C^n(\mu^n(\theta),\theta) - C^n(\mu^n(\theta'),\theta) = C^{n+1}((\mu^{n+1}(\theta)),\theta) - C^{n+1}((\mu^n(\theta'),\gamma_{n+1}(\theta)),\theta)$$
$$\geq C^{n+1}(\mu^{n+1}(\theta),\theta) - C^{n+1}(\mu^{n+1}(\theta'),\theta) + \delta_{n+1}r_{n+1}||\theta-\theta'|| \quad (20)$$

From (18),(19) and (20) it follows that for all $\theta, \theta' \in \Theta$

$$u(x(t,\theta),\theta) - u(x(t,\theta'),\theta) \geq C^{n+1}(\mu^{n+1}(\theta),\theta) - C^{n+1}(\mu^{n+1}(\theta'),\theta) + (\delta_{n+1}r_{n+1} - \varepsilon S)||\theta-\theta'|| \quad (21)$$

Therefore, upon taking $\varepsilon \leq \delta_{n+1}r_{n+1}/S$, we obtain that $\{x(t),\mu^{n+1}\}$ satisfies (16).

Proof of Theorem 2.

Lemma 6 requires the following incentive constraint to hold for all θ, θ':

$$u(x(\theta'),\theta) - u(x(\theta),\theta) \leq C^n(\gamma^n(\theta'),\theta) - C^n(\gamma^n(\theta),\theta)$$

Assumptions 3 and 4 imply that this incentive constraint holds if $KL \leq rn^{1-\alpha}\underline{c}$. Thus x is implementable if $n \geq (\frac{KL}{r\underline{c}})^{\frac{1}{1-\alpha}}$. No communication costs are incurred because the agent of type θ sends a collection of messages such that $m_i = \gamma_i(\theta)$. *Q.E.D.*

Proof of Theorem 3. First, we show that for any given a pair of twice continuously differentiable functions $q : \Theta \to Q$ and $U : \Theta \to \mathbb{R}$ there exists $N < \infty$, and a sequence of transfer and signal rules $(t^n, \mathbf{m}^n)$, such that for all $n \geq N$ and all $\theta \in \Theta$:

$$U(\theta) \equiv v(q(\theta),\theta) + t^n(\theta) - C^n(\mathbf{m}^n(\theta),\theta) = \max_{\theta'\in\Theta}\{v(q(\theta'),\theta) + t(\theta') - C^n(\mathbf{m}^n(\theta'),\theta)\} \quad (22)$$

(i) First, let us construct the signal rule $\mathbf{m}^n(\theta)$.

Let $\mathbf{z} : \Theta \to \mathbb{R}^l$and for each i select $m_i(\theta) = \gamma_i(\theta) + \mathbf{z}(\theta)$. We select $\mathbf{z}(\theta)$ as follows. If (22) holds, then the envelope theorem implies that

$$D_\theta U(\theta) = D_\theta v(q(\theta),\theta) - D_\theta C^n(\mathbf{m}^n(\theta),\theta) \quad (23)$$

For each θ, (23) consists of l equations in the l unknown variables $\mathbf{z}(\theta)$. Assumption 5(i) implies that the mapping $C^n_\theta(\cdot,\theta)$ is injective, so that the solution $\mathbf{z}(\theta)$ to (23) is unique.

(ii) Next, we construct the transfer rule $t^n(\theta)$.

Given the signal rule constructed in part (i), select the transfer rule $t^n(\cdot)$ as follows:

$$t^n(\theta) = \max_{\theta'\in\Theta}\{v(q(\theta),\theta') - C^n(\mathbf{m}^n(\theta),\theta') - U(\theta)\} \quad (24)$$

By the envelope theorem, $t(\cdot)$ is a.e. differentiable, and at points of differentiability we have:

$$D_\theta t^n(\theta) = D_q v(q(\theta),\theta) D_\theta q(\theta) - D_{\mathbf{m}^n} C^n(\mathbf{m}^n(\theta),\theta) D_\theta \mathbf{m}^n(\theta). \quad (25)$$

Furthermore, since $q(.)$ and $\mathbf{m}^n(.)$ are continuously differentiable, $t(\cdot)$ is continuously differentiable everywhere, with derivative given by (25).

(iii) Let us show that $\mathbf{m}^n(\theta) \to \gamma^n(\theta)$ and $D_\theta\mathbf{m}^n(\theta) \to D_\theta\gamma^n(\theta)$, uniformly in θ.

Let $g(\theta) = D_\theta v(q(\theta), \theta) - D_\theta U(\theta)$. It follows from (23) and Assumption 5(i) that

$$||g(\theta)|| \ = ||D_\theta C^n(\mathbf{m}^n(\theta), \theta) - D_\theta C^n(\gamma^n(\theta), \theta)|| \ \geq \frac{\omega}{n^\alpha}||\mathbf{m}^n - \gamma^n(\theta)|| \geq \underline{\omega} n^{1-\alpha}||\mathbf{z}(\theta)|| \quad (26)$$

Furthermore, since g is continuous and Θ is compact, it follows from the Weierstrass Theorem that there exists a constant $\lambda > 0$ s.t. $||g(\theta)|| \leq \lambda$ for all θ. Hence (26) implies $||\mathbf{z}(\theta)|| \leq \frac{\lambda}{\underline{\omega}} n^{-(1-\alpha)} \to 0$. We conclude that $\mathbf{m}^n(\theta) \to \gamma^n(\theta)$, uniformly in θ, and therefore that $D_\theta \mathbf{m}^n(\theta) \to D_\theta \gamma^n(\theta)$, uniformly in θ.

(iv) Next, let us demonstrate the incentive compatibility of our mechanism.

Let $V(\theta'; \theta)$ be the payoff of agent-type θ who misrepresents herself as θ'

$$V(\theta'; \theta) = v(q(\theta'), \theta) - t^n(\theta') - C^n(\mathbf{m}^n(\theta'), \theta) \quad (27)$$

By (25), we have $D_{\theta'} V(\theta, \theta) = 0$ for all $\theta \in \Theta$. Therefore

$$\begin{aligned} D_{\theta'} V(\theta', \theta) &= D_q v(q(\theta'), \theta) q_\theta(\theta') - D_\theta t^n(\theta') - D_{\mathbf{m}^n} C^n(\mathbf{m}^n(\theta'), \theta) D_\theta \mathbf{m}^n(\theta') - D_{\theta'} V(\theta', \theta') \\ &= \{D_q v(q(\theta'), \theta) - D_q v(q(\theta'), \theta')\} D_\theta q(\theta') - \\ &\quad \{D_{\mathbf{m}^n} C^n(\mathbf{m}^n(\theta'), \theta) - D_{\mathbf{m}^n} C^n(\mathbf{m}^n(\theta'), \theta')\} D_\theta \mathbf{m}^n(\theta') \end{aligned}$$

Using (27), we may compute the directional derivative of $V(\cdot, \theta)$ at θ' in the direction of θ:

$$\begin{aligned} D_{\theta'} V(\theta', \theta)(\theta - \theta') &= \{D_q v(q(\theta'), \theta) - D_q v(q(\theta'), \theta')\} D_\theta q(\theta')(\theta - \theta') \\ &\quad - \{D_{\mathbf{m}^n} C^n(\mathbf{m}^n(\theta'), \theta) - D_{\mathbf{m}^n} C^n(\mathbf{m}^n(\theta'), \theta')\} D_\theta \mathbf{m}^n(\theta')(\theta - \theta') \end{aligned}$$

Let $\beta = \max_{\theta', \theta \in \Theta} ||v_{\theta q}(q(\theta'), \theta) q_\theta(\theta')||$. Then using the mean value theorem

$$D_{\theta'} V(\theta', \theta)(\theta - \theta') \geq -\beta \ || \ \theta - \theta'||^2 - (\theta - \theta')^\tau \left\{ \sum_{i=1}^n D^2_{\theta m_i} C^n(\mathbf{m}^n(\theta'), \theta_i)(D_\theta \gamma_i(\theta') + D_\theta \mathbf{z}(\theta')) \right\} (\theta - \theta')$$

where $\theta_i = \theta + \varepsilon_i(\theta' - \theta)$, for some $\varepsilon_i \in (0, 1)$. It follows from Assumption 2 and Assumption 5(iii) that $(\theta - \theta')^\tau \{\sum_{i=1}^n D_{\theta m_i} C^n(\mathbf{m}^n(\theta'), \theta_i) \ (D_\theta \gamma_i(\theta') + D_\theta \mathbf{z}(\theta'))\} (\theta - \theta') \leq -(1 - \frac{||\mathbf{z}_\theta||}{r}) \underline{\omega} n^{-\alpha} ||\theta - \theta'||^2$. Therefore

$$D_{\theta'} V(\theta', \theta)(\theta - \theta') \geq \{(1 - \frac{||\mathbf{z}_\theta||}{r}) \underline{\omega} n^{1-\alpha} - \beta \ \} ||\theta - \theta'||^2$$

Let N be such that $(1 - \frac{||\mathbf{z}_\theta||}{r}) \omega n^{1-\alpha} - \beta > 0$ for all $n \geq N$. Then for $n \geq N$, and all $\theta' \neq \theta$ the directional derivative $D_{\theta'} V(\theta', \theta)(\theta - \theta')$ is positive, implying that $\theta' = \theta$ uniquely maximizes $V(\theta', \theta)$.

(v) Next, let us show that $C^n(\mathbf{m}^n(\theta), \theta) \to 0$.

It follows from the mean value theorem that

$$C^n(\mathbf{m}^n(\theta), \theta) = D_{\mathbf{m}^n} C^n(\widetilde{\mathbf{m}}^n(\theta), \theta)(\mathbf{m}^n(\theta) - \gamma^n(\theta))$$

where $\widetilde{\mathbf{m}}^n(\theta) = \mathbf{m}^n(\theta) + \tilde{\varepsilon}(\theta)(\mathbf{m}^n(\theta) - \gamma^n(\theta))$, for some $\tilde{\varepsilon}(\theta) \in (0, 1)$. Hence from Assumption 5(ii),

$$\begin{aligned} C^n(\mathbf{m}^n(\theta), \theta) &\leq ||\sum_i D_{m_i} C^n(\widetilde{\mathbf{m}}^n(\theta), \theta)|| \, ||\mathbf{z}(\theta)|| \\ &\leq \bar{\omega} n^{1-\alpha} ||\mathbf{z}(\theta)||^2 \leq (\frac{B}{\underline{\omega}})^2 \bar{\omega} n^{-(1-\alpha)} \end{aligned} \tag{28}$$

Consequently, $C^n(\mathbf{m}^n(\theta), \theta) \to 0$, uniformly in θ.

Finally, given any pair of continuously differentiable functions $q : \Theta \to Q$ and $t : \Theta \to \mathbb{R}$, let $U(\theta) = v(q(\theta), \theta) - t(\theta)$ and select $\{t^n, \mathbf{m}^n\}$ as defined above. It then follows from (28) that $C^n(\mathbf{m}^n(\theta), \theta) \to 0$, and hence from (24) that $t^n(\theta) \to t(\theta)$, uniformly in θ. This completes the proof. *Q.E.D.*

Proof of Theorem 4: The proof will proceed through the following steps. First, we will establish the existence of a point $\widehat{\theta} \in (\underline{\theta}, \bar{\theta})$, such that $U(\theta) = 0$ if and only if $\theta \leq \widehat{\theta}$. Second, we will show that $U''(\theta) > 0$ for all $\theta \geq \widehat{\theta}$. This implies that on the interval $[\underline{\theta}, \widehat{\theta}]$ the solution is governed by $\{\widehat{q}(\theta), \widehat{\mathbf{m}}^n(\theta)\}$, and that on the interval $[\widehat{\theta}, \bar{\theta}]$ it is governed by $\{\widetilde{q}(\theta), \widetilde{\mathbf{m}}^n(\theta)\}$. To complete the proof, we will establish that the solution to our relaxed program (7) subject to (6) and (8) also satisfies the agent's second-order conditions, and hence solves the full unrelaxed program (7).

To begin, define $\widehat{\theta} = \inf\{\theta : U(\theta) > 0\}$. First, we show that $\widehat{\theta} \in (\underline{\theta}, \bar{\theta})$. To establish that $\widehat{\theta} < \bar{\theta}$, suppose to the contrary that $\widehat{\theta} = \bar{\theta}$. Then, since $U(\theta) = 0$ for all θ, the solution to the relaxed problem is given by $\{\widehat{q}(\theta), \widehat{\mathbf{m}}^n(\theta), \widehat{\sigma}(\theta)\}$. By the transversality condition at $\bar{\theta}$, $\widehat{\sigma}(\bar{\theta}) \geq 0$. So, the first-order conditions (10) and (11) imply that $\widehat{q}(\bar{\theta}) \geq q^*(\bar{\theta})$ and $\widehat{m}_i(\bar{\theta}) \geq \gamma_i(\theta)$. Since $v_{q\theta} > 0$ and $C^n_{\theta m_i} < 0$, we then have

$$U'(\bar{\theta}) = v_\theta(q(\bar{\theta}), \bar{\theta}) - C^n_\theta(\mathbf{m}^n(\bar{\theta}), \bar{\theta}) > v_\theta(q^*(\bar{\theta}), \bar{\theta}) - C^n_\theta(\gamma^n(\bar{\theta}), \bar{\theta}) > 0$$

where the final inequality follows from the fact that $C^n_\theta(\gamma^n(\theta), \theta) = 0$. But $U'(\bar{\theta}) > 0$ contradicts the presumption that $\widehat{\theta} = \bar{\theta}$, i.e. that $U'(\theta) = 0$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$. So, we cannot have $\widehat{\theta} = \bar{\theta}$.

To prove that $\widehat{\theta} > \underline{\theta}$ suppose, to the contrary, that $U(\theta) > 0$ for all $\theta > \underline{\theta}$. Then the solution to the relaxed problem is given by $\{\widetilde{q}(\theta), \widetilde{\mathbf{m}}^n(\theta)\}$. In particular, from the first-order condition (11) it follows that $\widetilde{m}_i(\theta) < \gamma_i(\theta)$ for all i. But since $\widetilde{q}(\theta) = q^{SB}(\theta) = 0$ for all $\theta \in [\underline{\theta}, \theta^*]$, for such θ we would then have

$$U'(\theta) = v_\theta(0, \theta) - C^n_\theta(\widetilde{\mathbf{m}}^n(\theta), \theta) < 0 \tag{29}$$

where the inequality holds because $v_\theta(0, \theta) = 0$, and because $\widetilde{\mathbf{m}}^n(\theta) < \gamma^n(\theta)$ implies $C^n_\theta(\widetilde{\mathbf{m}}^n(\theta), \theta) > 0$. Since $U(\underline{\theta}) = 0$, (29) then shows that individual rationality is violated on $(\underline{\theta}, \theta^*]$. This contradiction establishes that $\widehat{\theta} > \underline{\theta}$.

Next, we claim that $U(\theta) > 0$ for all $\theta > \widehat{\theta}$. Indeed, let $\theta^+ = \sup\{t : U(\theta) > 0$ for all $\theta \in (\widehat{\theta}, t)\}$, and suppose to the contrary that $\theta^+ < \bar{\theta}$. Then on the interval $(\widehat{\theta}, \theta^+)$ the solution is given by $q_*(\sigma, \theta)$ and $m^n_*(\sigma, \theta)$ where $\sigma'(\theta) = f(\theta)$.

In Lemma 7 below, we show that the Hamiltonian (9) is supermodular in $((q, \mathbf{m}^n), \theta)$ on any interval where $\rho(\theta) = 0$, and in particular on $(\widehat{\theta}, \theta^+)$. In combination with Assumption 5(iii) this implies that $q(\theta) = q_*(\sigma(\theta), \theta)$ and $\mathbf{m}^n(\theta) = \mathbf{m}^n_*(\sigma(\theta), \theta)$ which are both nondecreasing in θ on the interval $(\widehat{\theta}, \theta^+)$. Since by assumption we also have $v_{\theta\theta} - C^n_{\theta\theta} > 0$ it follows from (8) that $U'(\theta)$ is increasing $(\widehat{\theta},\, \theta^+)$. Because $U(\widehat{\theta}) = U'(\widehat{\theta}) = 0$ this shows that $U(\theta^+) > 0$, contradicting the definition of θ^+. Thus $\theta^+ = \bar{\theta}$.

Since we have shown that $U(\theta) = 0$ for $\theta \in [\underline{\theta}, \widehat{\theta}]$ and $U(\theta) > 0$ for $\theta \in (\widehat{\theta}, \bar{\theta}]$, we conclude that the solution to the relaxed problem is given by $\{\widehat{q}(\theta), \widehat{\mathbf{m}}^n(\theta), \widehat{\sigma}(\theta)\}$ for $\theta \in [\underline{\theta}, \widehat{\theta}]$, and by $\{\widetilde{q}(\theta), \widetilde{\mathbf{m}}^n(\theta)\}$ for $\theta \in [\widehat{\theta}, \bar{\theta}]$.

It remains to show that the solution (14) satisfies (5). This will be the case if the following second-order condition is satisfied:

$$v_{q\theta} q_\theta - \sum_{i=1}^{n} C^n_{m_i}(\mathbf{m}^n(\theta), \theta) m_i'(\theta) \geq 0$$

On the interval $[\underline{\theta}, \widehat{\theta})$ we have $U'(\theta) \equiv 0$, and hence

$$U''(\theta) = v_{q\theta} q_\theta + v_{\theta\theta} - (\sum_{i=1}^{n} C^n_{m_i}(\mathbf{m}^n(\theta), \theta) m_i'(\theta) + C^n_{\theta\theta}) = 0$$

Therefore

$$v_{q\theta} q_\theta - \sum_{i=1}^{n} C^n_{m_i}(\mathbf{m}^n(\theta), \theta) m_i'(\theta) = C^n_{\theta\theta} - v_{\theta\theta} \geq 0.$$

On the interval $(\widehat{\theta}, \bar{\theta}]$ we have $(q(\theta), \mathbf{m}^n(\theta)) = (\widetilde{q}(\theta), \widetilde{\mathbf{m}}^n(\theta))$, and Assumptions 6(ii)-(iv) imply that the Hamiltonian is supermodular in (q, θ) and in $(\mathbf{m}^n, \theta)$, and so $\widetilde{q}_\theta \geq 0$ and $D_\theta \widetilde{\mathbf{m}}^n \geq 0$, implying $v_{q\theta}\widetilde{q}_\theta - \sum_{i=1}^{n} C^n_{m_i}(\mathbf{m}^n(\theta), \theta) m_i'(\theta) \widetilde{m}_i' \geq 0$. *Q.E.D.*

The following auxiliary result is used in the proof of Theorem 4.

Lemma 7 *Suppose that part (i) and (ii) of Assumption 6 hold. Then on any interval on which $\rho(\theta) = 0$ the Hamiltonian (9) is supermodular in $((q, \mathbf{m}^n), \theta)$.*

Proof: First, we claim that in any solution to the optimal control problem the costate variable $\sigma(\theta)$ satisfies the inequalities $-(1 - F(\theta)) \leq \sigma(\theta) \leq 0$.

Indeed, since $q(\theta)$ is optimal, we must have $q(\theta) \leq q^{FB}(\theta)$. Because $v_{q\theta} > 0$, it then follows from the first-order condition (10) that $\sigma(\theta) \leq 0$. In conjunction with the transversality condition at $\bar{\theta}$, the latter inequality implies $\sigma(\bar{\theta}) = 0$. Furthermore, integrating the costate equation (12) yields $\sigma(\theta) = \sigma(\bar{\theta}) - \int_\theta^{\bar{\theta}} (f(\theta) - \rho(\theta)) d\theta = -(1 - F(\theta)) + \int_\theta^{\bar{\theta}} \rho(\theta) d\theta$. Since $\rho(\theta) \geq 0$ we therefore have $\sigma(\theta) \geq -(1 - F(\theta))$.

Next, let us show that on any interval on which $\rho(\theta) = 0$ the function

$$\Psi(q, \theta) \equiv v_q(q, \theta) - h_q(q, \theta) + \frac{\sigma(\theta)}{f(\theta)} v_{q\theta}(q, \theta)$$

is increasing in θ. We may compute:

$$\begin{aligned}\Psi_\theta(q,\theta) &\equiv v_{q\theta} - h_{q\theta} + (\frac{\sigma}{f})' v_{q\theta}(q,\theta) + \frac{\sigma}{f} v_{q\theta\theta} \\ &= 2v_{q\theta} - h_{q\theta} + \frac{\sigma}{f}(v_{q\theta\theta} - \frac{f'}{f} v_{q\theta})\end{aligned}$$

Since $v_{q\theta} > 0$, $h_{q\theta} < 0$, and $\sigma \leq 0$, it follows that $\Psi_\theta > 0$ if $v_{q\theta\theta} - (f'/f)v_{q\theta} \leq 0$. Meanwhile, if $(v_{q\theta\theta} - \frac{f'}{f} v_{q\theta}) > 0$, then $\sigma(\theta) \geq F(\theta) - 1$ and Assumption (5)(i) imply $\Psi_\theta \geq 2v_{q\theta} - h_{q\theta} + \frac{F(\theta)-1}{f}(v_{q\theta\theta} - (f'/f)v_{q\theta}) > 0$. A similar argument shows that for each i the function

$$\Phi_i(\mathbf{m}^n, \theta) = \frac{\partial C^n}{\partial m_i}(\mathbf{m}^n, \theta) + \frac{\sigma(\theta)}{f(\theta)} \frac{\partial^2 C^n}{\partial m_i \partial \theta}(\mathbf{m}^n, \theta)$$

is decreasing in θ. *Q.E.D.*

Proof of Lemma 2:

First, let us demonstrate that in the optimal mechanism, $\widehat{q}(\theta) > 0$ for all $\theta \in (\underline{\theta}, \widehat{\theta}]$. Suppose instead we had $\widehat{q}(\theta) = 0$ for some $\theta \in (\underline{\theta}, \widehat{\theta}]$. Let us show that this implies $\widehat{\mathbf{m}}^n(\theta) = \gamma^n(\theta)$ and $\widehat{\sigma}(\theta) = 0$. Note that $U'(\theta) = 0$ because $\theta \leq \widehat{\theta}$. Since $v_\theta(0,\theta) = 0$, equation (8) therefore yields $C^n_\theta(\widehat{\mathbf{m}}^n(\theta), \theta) = 0$. We will establish that $\widehat{\mathbf{m}}^n(\theta) = \gamma^n(\theta)$ is a solution to this equation. Indeed, since $C^n(\gamma^n(\theta), \theta) \equiv 0$, we have $\sum_{i=1}^n C^n_{m_i}(\gamma^n(\theta), \theta) + C^n_\theta(\gamma^n(\theta), \theta) = 0$. Since $C^n(\mathbf{m}^n, \theta)$ takes on a global minimum at $\mathbf{m}^n = \gamma^n(\theta)$, it follows that $C^n_{m_i}(\gamma^n(\theta), \theta) = 0$. Thus $\mathbf{m}^n(\theta) = \gamma^n(\theta)$ is a solution to the equation $C^n_\theta(\mathbf{m}^n(\theta), \theta) = 0$.

Next, by Equation (11), $\sigma \gtreqless 0$ implies that $m_i(\theta) \lesseqgtr \gamma_i(\theta)$ for all i. Because $\bar{C}_{m\theta} < 0$ we have $C^n_\theta(\widehat{\mathbf{m}}^n(\theta), \theta) \lesseqgtr 0$ as $\sigma \gtreqless 0$. Thus $C^n_\theta(\widehat{\mathbf{m}}^n(\theta), \theta) = 0$ implies $\widehat{\mathbf{m}}^n(\theta) = \gamma^n(\theta)$ and $\widehat{\sigma}(\theta) = 0$.

But substituting $\widehat{\sigma}(\theta) = 0$ into equation (10) then yields $v_q(\widehat{q}(\theta), \theta) - h_q(\widehat{q}(\theta), \theta) \leq 0$, contradicting the assumption that $v_q(0,\theta) - h_q(0,\theta) > 0$. So we must have $\widehat{q}(\theta) > 0$ for all $\theta > \underline{\theta}$.

Further, by Theorem 4, $U(\theta) > 0$ on $(\widehat{\theta}, \bar{\theta}]$, so we also have $\widetilde{q}(\theta) > 0$ on this interval. *Q.E.D.*

Proof of Lemma 3: First, we establish that $\widehat{q}(\theta) = q_*(\widehat{\sigma}(\theta), \theta)$ and $\widehat{\mathbf{m}}^n(\theta) = m^n_*(\widehat{\sigma}(\theta), \theta)$ are nondecreasing in n. To see this, note that $q_*(\sigma, \theta)$ and $\mathbf{m}^n_*(\sigma, \theta)$ are independent of n because $C^n(m_1, ..., m_n, \theta) = \sum_{i=1}^n c_i(m_i, \theta)$. Furthermore, since the Hamiltonian is supermodular in σ, it follows that $q_*(\sigma, \theta)$ and $m^n_*(\sigma, \theta)$ are nondecreasing in σ. We now claim that $\widehat{\sigma}(\theta)$ is nondecreasing in n, thereby proving that $\widehat{q}(\theta)$ and $\widehat{\mathbf{m}}^n(\theta)$ are nondecreasing in n.

To prove the claim, recall that $\widehat{\sigma}(\theta)$ is defined as the solution to

$$\Upsilon(\sigma, \theta, n) \equiv v_\theta(q_*(\sigma, \theta), \theta) - C^n_\theta(m^n_*(\sigma, \theta), \theta) = 0.$$

Since $v_\theta q > 0$ and $\frac{\partial^2 C^n}{\partial m_i \partial \theta} < 0$ we have $\Upsilon_\sigma \geq 0$. Furthermore, $\Upsilon(\sigma, \theta, n)$ is decreasing in n because $C^n(m_1, ..., m_n, \theta) = \sum_{i=1}^n c_i(m_i, \theta)$. Therefore $\widehat{\sigma}(\theta)$ is increasing in n.

Next, let us show that $\widehat{\theta}(n)$ is nondecreasing in n. Observe that $\widehat{\theta}(n)$ is the solution in θ to the equation $\widehat{q}(\theta, n) = \widetilde{q}(\theta)$. We now claim that for $\theta > \widehat{\theta}(n)$ it is the case that

$\widetilde{q}(\theta) = q_*(\sigma(\theta), \theta) > q_*(\widehat{\sigma}(\theta), \theta) = \widehat{q}(\theta, n)$. Since $\widehat{q}(\theta, n)$ is increasing in n, this then implies that $\widehat{\theta}(n)$ increases in n.

To see that the claim holds, recall that $U'(\theta) > 0$ for $\theta > \widehat{\theta}(n)$. Now because q_* and $\mathbf{m}_*^n$ are nondecreasing in σ, and because $v_{q\theta} > 0$ and $C_{m\theta} < 0$, the inequality $U'(\theta) = v_\theta(q_*(\sigma(\theta), \theta), \theta) - C_\theta^n(\mathbf{m}_*^n(\sigma(\theta), \theta)), \theta) > v_\theta(q_*(\widehat{\sigma}(\theta), \theta), \theta) - C_\theta^n(m_*^n(\widehat{\sigma}(\theta), \theta)), \theta, n) = 0$ can hold only if $\sigma(\theta) > \widehat{\sigma}(\theta)$, implying $\sigma(\theta) > \widehat{\sigma}(\theta)$, and hence $\widetilde{q}(\theta) = q_*(\sigma(\theta), \theta) > q_*(\widehat{\sigma}(\theta), \theta) = \widehat{q}(\theta, n)$.

(ii) We will first prove that $\lim_{n\to\infty} \widehat{\sigma}(\theta, n) = 0$. Let $\widehat{\sigma}_\infty(\theta)$ be the limit of any convergent subsequence of $\{\widehat{\sigma}(\theta, n)\}_{n=1}^\infty$. By renumbering the indices of the subsequence, we may without loss of generality assume that $\widehat{\sigma}(\theta, n) \to \widehat{\sigma}_\infty(\theta)$. Suppose that contrary to the desired result, we had $\widehat{\sigma}_\infty(\theta) < 0$. Using the mean value Theorem, we have $\frac{\partial C^n}{\partial m_i}(\widehat{m}_i, \widehat{\mathbf{m}}_{-i}^n, \theta) - \frac{\partial C^n}{\partial m_i}(\gamma_i(\theta), \widehat{\mathbf{m}}_{-i}^n, \theta) = \frac{\partial^2 C^n}{\partial m_i^2}(\bar{m}_i, \widehat{\mathbf{m}}_{-i}^n, \theta)(\widehat{m}_i - \gamma_i(\theta))$, for some $\bar{m}_i \in (m_i, \gamma_i)$. Since $\frac{\partial C^n}{\partial m_i}(\gamma_i(\theta), \mathbf{m}_{-i}^n, \theta) = 0$, it follows from (11) that

$$(\widehat{m}_i(\theta) - \gamma_i(\theta)) = -\widehat{\sigma}(\theta, n) \frac{\frac{\partial^2 C^n}{\partial\theta\partial m_i}(\widehat{\mathbf{m}}^n(\theta), \theta)}{\frac{\partial^2 C^n}{\partial m_i^2}(\bar{m}_i, \widehat{\mathbf{m}}_{-i}^n(\theta), \theta)} \tag{30}$$

Applying the mean value Theorem once more, we also have

$$C_\theta^n(\widehat{\mathbf{m}}^n, \theta) = \sum_{i=1}^n \frac{\partial^2 C^n}{\partial\theta\partial m_i}(\bar{\mathbf{m}}^n, \theta)(\widehat{m}_i(\theta) - \gamma_i(\theta)), \tag{31}$$

where $\bar{\mathbf{m}}^n = \gamma^n(\theta) + \varepsilon(\theta)(\widehat{\mathbf{m}}^n - \gamma^n(\theta))$, for some $\varepsilon(\theta) \in (0, 1)$. Using the assumption that $\frac{\partial^2 C^n}{\partial m_i^2} \leq \bar{v}$, and $|\frac{\partial^2 C^n}{\partial\theta\partial m_i}| \geq \underline{v}$, (30) and (31) yield

$$C_\theta^n(\widehat{\mathbf{m}}^n(\theta), \theta) \geq -n\widehat{\sigma}(\theta, n)\frac{\underline{\underline{v}}}{\bar{v}} \tag{32}$$

Since $\widehat{\sigma}(\theta, n) \to \widehat{\sigma}_\infty(\theta) < 0$, we would therefore have $C_\theta^n(\widehat{\mathbf{m}}^n(\theta), \theta) \to \infty$. But because $v_\theta(q, \theta)$ is bounded, this contradicts 8, thereby establishing that $\lim_{n\to\infty} \widehat{\sigma}(\theta, n) = 0$.

Next, we argue that $\lim_{n\to\infty} \widehat{\theta}(n) = \bar{\theta}$. Because $\widehat{\sigma}(\theta, n) \to 0$, the first-order conditions (10) and (11) imply that $\lim_{n\to\infty} \widehat{q}(\theta, n) = q^{FB}(\theta)$ and $\lim_{n\to\infty} \widehat{\mathbf{m}}(\theta, n) = \theta$. Since $\widehat{\theta}(n)$ is the solution to the equation $\widehat{q}(\theta, n) = \widetilde{q}(\theta) = q^{SB}(\theta)$, and since $q^{FB}(\theta) > q^{SB}(\theta)$ for every $\theta \in (0, \bar{\theta})$ we must have $\lim_{n\to\infty} \widehat{\theta}(n) = \bar{\theta}$.

It remains to prove that $C^n(\widehat{\mathbf{m}}^n(\theta), \theta) \to 0$. Since $C^n(\gamma^n(\theta), \theta) = 0$ and $C_\theta^n(\gamma^n(\theta), \theta) = 0$, it follows from Taylor's Theorem that

$$C^n(\widehat{\mathbf{m}}^n(\theta), \theta) = \sum_{i=1}^\infty (\widehat{m}_i - \gamma_i(\theta))^2 \frac{\partial^2 C^n}{\partial m_i^2}(\underline{m}^n, \theta) \tag{33}$$

Now (30) yields $|\widehat{m}_i - \gamma_i(\theta)| \leq |\widehat{\sigma}(\theta, n)|\frac{\bar{v}}{\underline{v}}$, so (33) implies $C^n(\widehat{\mathbf{m}}^n(\theta), \theta) \leq n\widehat{\sigma}^2(\theta, n)\frac{\bar{v}^3}{\underline{v}^2}$. Also, (31) implies $n|\widehat{\sigma}(\theta, n)| \frac{\underline{\underline{v}}}{\bar{v}} \leq |C_\theta^n(\widehat{\mathbf{m}}^n, \theta)| \leq \max_{(q,\theta)} v_\theta = k$. Thus $C^n(\widehat{\mathbf{m}}^n, \theta) \leq k\ |\widehat{\sigma}(\theta, n)|$ $\frac{\bar{v}^4}{\underline{v}^3} \to 0$. *Q.E.D.*

Proof of Lemma 4: Since the solution to the principal's problem is unique the value function W is continuously differentiable, and

$$\frac{dW(n)}{dn} = \int_{\underline{\theta}}^{\bar{\theta}} \frac{\partial H}{\partial n}(q, m, U, \sigma, n, \theta) d\theta$$

(Seiersstad and Sydsaeter, 1999, p. 217). Using

$$\frac{\partial H}{\partial n} = -c(m(\theta), \theta) f(\theta) - \sigma(\theta) c_\theta(m(\theta), \theta)$$

and substituting for $\sigma(\theta)$ from the first-order condition (11) yields

$$\frac{\partial H}{\partial n} = f(-c + \frac{c_\theta c_m}{c_{m\theta}})$$

proving (15). Furthermore,

$$\begin{aligned} \frac{d}{dn} \frac{-cc_{m\theta} + c_\theta c_m}{c_{m\theta}} &= \frac{\partial}{\partial m} \frac{-cc_{m\theta} + c_\theta c_m}{c_{m\theta}} \frac{\partial m}{\partial n} \\ &= \frac{c_\theta \{c_{m\theta}^2 - c_{mm\theta} c_m\}}{c_{m\theta}^2} \frac{\partial m}{\partial n} \end{aligned}$$

In Lemma 3 it is established that $\frac{\partial m}{\partial n} > 0$ on $[\underline{\theta}, \widehat{\theta}(n))$. Since $c_\theta < 0$ it follows from the assumption $c_{m\theta}^2 - c_{mm\theta} c_m > 0$ that $\frac{\partial H}{\partial n} > 0$ on $[\underline{\theta}, \widehat{\theta}(n))$. Furthermore, on $(\widehat{\theta}(n), \bar{\theta}]$ we have $\frac{\partial m}{\partial n} = 0$, and hence $\frac{\partial H}{\partial n} = 0$. We conclude that $W'(n)$ is strictly decreasing in n i.e., $W(n)$ is strictly concave. *Q.E.D.*

Proof of Lemma 5: By the definition of ε and η we have

$$\varepsilon c \leq \frac{c_m c_\theta}{c_{m\theta}} - c \leq \eta c$$

and so,

$$0 < \varepsilon \int_{\underline{\theta}}^{\bar{\theta}} c(m(\theta), \theta) f(\theta) d\theta \leq \frac{dW(n)}{dn} \leq \eta \int_{\underline{\theta}}^{\bar{\theta}} c(m(\theta), \theta) f(\theta) d\theta.$$

Q.E.D.

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